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CONSTRUCTION OF DEFORMED STOCHASTIC BASES OF THE 2ND KIND

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Abstract. Deformations (special two-parameter families of probability measures) $\{Q_k^n, 0 \leq k \leq n < \infty\}$ and the corresponding deformed stochastic bases of the 1st and 2nd kind with discrete time were axiomatically determined by the first author in 2008. Subsequently, he and O.V. Nazarko laid the foundations of a stochastic analysis on these structures. The present work continues this topic. The main result of the paper is the theorem, which proves the formula for representing measures $\{Q_k^n, 0 \leq k < n < \infty\}$ by the measures $\{Q_i^i, 0 \leq i < \infty\}$. This construction is important for the development of the theory of deflators on deformed structures. The paper also gives the most general definition of a deformed stochastic basis of the second kind with continuous time. Some important properties of this object are given.

1. Introduction

Let $(\Omega, (\mathcal{F}_t)_{t=0}^{\infty})$ be a filtered space with continuous (or discrete) time. Denote by $\mathcal{F}_\infty$ the least σ-field containing all $\mathcal{F}_t$. If $(\Omega, \mathcal{F}, P)$ is a probability space and $\mathcal{G}$ is a sub-σ-field of the σ-field $\mathcal{F}$, then we use the notation

$$E_{\mathcal{G}}^P f = E^P[f|\mathcal{G}].$$

Consider a family $\mathbf{Q} = (Q_s^t, \mathcal{F}_t)_{\{0 \leq s < t < \infty\}}$ of probability measures Q_s^t on $\mathcal{F}_t$ and generated by them the family of operators $\mathbf{E} = (E_s^t)_{\{0 \leq s < t < \infty\}}$ of conditional expectation

$$E_s^t f := E_{\mathcal{F}_s}^{Q_s^t} f,$$

where f is a non-negative $\mathcal{F}_t$-measurable random variable (r.v.).

Definition 1.1. A triplet

$$(\Omega, \mathbf{Q}, \mathbf{E}) \tag{1.1}$$

is called deformed stochastic basis (DSB).

In what follows, we denote the absolute continuity of measures by the symbol $<<$, and the equivalence of measures by the symbol $\sim$.

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Definition 1.2. A DSB (1.1) is called deformed stochastic basis of the 2^{nd} kind (DSB2) if the following conditions are fulfilled:

1) $\forall 0 \le s < r < t < \infty \quad Q_s^r|_{\mathcal{F}_s} << Q_s^t|_{\mathcal{F}_s}$;
2) $\forall 0 \le s < r < t < \infty \quad Q_s^r << Q_r^t|_{\mathcal{F}_r}$;
3) $\forall 0 \le s < r < t < \infty$ and for any $\mathcal{F}_t$-measurable r.v. $f \ge 0$ the equality

$$E_s^t f = E_s^r E_r^t f. \tag{1.2}$$

is satisfied Q_s^r-almost surely (a.s.). If instead of 1) the stronger condition

1′) $\forall 0 \le s < r < t < \infty \quad Q_s^r|_{\mathcal{F}_s} = Q_s^t|_{\mathcal{F}_s}$

is satisfied, then such DSB2 is called regular.

Corollary 1.3. *It evidently follows from the property* 2) *of Definition 1.2 that* $\forall 0 \le s < r \le t < u < \infty$ *the relation* $Q_s^r << Q_t^u|_{\mathcal{F}_r}$ *is satisfied.*

Corollary 1.4. *From the property* 3) *of Definition 1.2 it follows that* $\forall 0 \le s = s_0 < s_1 < \dots < s_n = t < \infty$ *and for any non-negative* $\mathcal{F}_t$*-mesurable r.v.* f *the equality*

$$E_s^t f = E_s^{s_1} E_{s_1}^{s_2} \dots E_{s_{n-1}}^t f. \tag{1.3}$$

is satisfied $Q_s^{s_1}$ *-a.s.*

Proposition 1.5. *If DSB2* $(\Omega, \boldsymbol{Q}, \boldsymbol{E})$ *is regular, then*

a) *the equalities* (1.2) *and* (1.3) *are satisfied* Q_s^t *-a.s.;*
b) $\forall 0 \le s = s_0 < s_1 < \dots < s_n = t < \infty$ *and* $\forall A \in \mathcal{F}_t$

$$Q_s^t(A) = E^{Q_s^{s_1}} E_{s_1}^{s_2} \dots E_{s_{n-1}}^t I_A; \tag{1.4}$$

c) $\forall 0 \le s < r < t < \infty \quad Q_s^r = Q_s^t|_{\mathcal{F}_r}$.

Proof. The proof of *a*) is trivial. Let us prove *b*). Using (1.3), we have:

$$Q_s^t(A) = E^{Q_s^t} E_s^t(I_A) = E^{Q_s^t} E^{Q_s^{s_1}} E_{s_1}^{s_2} \dots E_{s_{n-1}}^t I_A.$$

Applying regularity condition $Q_s^{s_1}|_{\mathcal{F}_s} = Q_s^t|_{\mathcal{F}_s}$ (c.f. Definition 1.2), we obtain (1.4). The property *c*) follows from *b*). □

Proposition 1.6. *If DSB* (1.1) *satisfies the property* 2) *from Proposition 1.5 and the equalities*

$$Q_s^t(A) = E^{Q_s^r} E_r^t I_A \tag{1.5}$$

are fulfilled $\forall 0 \le s < r < t < \infty$ *and* $\forall A \in \mathcal{F}_t$, *then* (1.1) *is a regular DSB2.*

Proof. From (1.5) it follows trivially that $\forall 0 \le s < r < t < \infty \quad Q_s^r = Q_s^t|_{\mathcal{F}_r}$. Thus, the regularity is proved. Let us prove now the equalities (1.2).

It follows from (1.5) that for any $\mathcal{F}_t$-mesurable r.v. $f \ge 0$

$$E^{Q_s^t}(f) = E^{Q_s^r} E_r^t f. \tag{1.6}$$

Let $A \in \mathcal{F}_s$. We have $E^{Q_s^t}[I_A E_s^t f] = E^{Q_s^t}(I_A f)$. On the other hand, using the equality $Q_s^r = Q_s^t|_{\mathcal{F}_r}$, we obtain $E^{Q_s^t}[I_A E_s^r E_r^t f] = E^{Q_s^r}[I_A E_s^r E_r^t f] = E^{Q_s^r} E_r^t (I_A f)$. Applying (1.6) to r.v. $I_A f$, we get the equality (1.2) Q_s^t-a.s. and hence Q_s^r-a.s. □

Corollary 1.7. *Consider a sequence of probability measures* $\left(Q^s_{s-1}, \mathcal{F}_s\right)_{s=1,2,\dots}$ *with the property:*

$$Q^s_{s-1} << Q^{s+1}_s|_{\mathcal{F}_s}. \tag{1.7}$$

Then the formula

$$Q^t_s(A) = E^{Q^{s+1}_s} E^{s+2}_{s+1} \dots E^t_{t-1} I_A, \tag{1.8}$$

where $s < t$ *and* $A \in \mathcal{F}_t$*, defines a regular DSB2 with discrete time.*

2. Closable DSB2

Definition 2.1. DSB2 $(\Omega, \mathbf{Q}, \mathbf{E})$ is called closable if $\forall 0 \le t < \infty$ there exists on $\mathcal{F}_t$ a probability measure Q^t_t (with associated identity operator E^t_t on the set of non-negative $\mathcal{F}_t$-measurable r.v.) such that the property 2) of Definition 1.2 is fulfilled $\forall 0 \le s \le r \le t < \infty$ (i.e. $\forall s < t$ $Q^s_s << Q^t_s|_{\mathcal{F}_s}$ and $Q^t_s << Q^t_t$).

Remark 2.2. It is obvious that if DSB2 $(\Omega, \mathbf{Q}, \mathbf{E})$ is closable, then the properties 1) and 3) of Definition 1.2 also fulfilled $\forall 0 \le s \le r \le t < \infty$ and $\forall s < t$ the relations

$$Q^s_s << Q^t_t|_{\mathcal{F}_s}. \tag{2.1}$$

are true.

Definition 2.3. DSB2 $(\Omega, \mathbf{Q}, \mathbf{E})$ that satisfies the conditions of Definition 1.2 $\forall 0 \le s \le r \le t < \infty$ is called closed DSB2.

Proposition 2.4. *A reguar DSB2 is uniquely closable to a reguar closed DSB2.*

Proof. It is sufficient to put $Q^s_s = Q^r_s|_{\mathcal{F}_s}$, $0 \le s < r < \infty$. Since $(\Omega, \mathbf{Q}, \mathbf{E})$ is regular, this notation is correct. The rest is trivial. □

Example 2.5. Consider a regular closed DSB2 in the case $\mathcal{F}_s = \mathcal{F}, \forall s \ge 0$. By Proposition 2.4 we have $Q^s_s = Q^t_s$, and by Remark 2.2 $Q^s_s << Q^t_t$ $(0 \le s < t < \infty)$. Conversely, let $(Q^s_s)^\infty_{s=0}$ be a sequence of probabilities such that for $0 \le s < t < \infty$ $Q^s_s << Q^t_t$. For such s and t we put $Q^t_s = Q^s_s$. It is easy to see that if $\mathbf{Q} = (Q^t_s, 0 \le s \le t < \infty)$, then $(\Omega, \mathbf{Q}, \mathbf{E})$ is a regular closed DSB2.

Proposition 2.6. *Let on the filtered space under consideration a family of probabilities* $(Q^t_t, \mathcal{F}_t)^\infty_{t=0}$ *satisfyng the condition* (2.1) *be defined. Put* $\forall s < t$ $Q^t_s = Q^t_t$ *and* $E^t_s f = E^{Q^t_t}[f|\mathcal{F}_s]$*. DSB* $(\Omega, \boldsymbol{Q}, \boldsymbol{E})$*, where* $\boldsymbol{Q} = (Q^t_s, \mathcal{F}_t)_{\{0 \le s \le t < \infty\}}$*, is closed DSB2 if and only if* $\forall 0 \le s < r < t < \infty$ *the equality*

$$E^{Q^r_r}\left(E^{Q^t_t}\left[f\,|\mathcal{F}_r\right]|\mathcal{F}_s\right) = E^{Q^t_t}\left[f|\mathcal{F}_s\right]. \tag{2.2}$$

is fulfilled Q^r_r*-a.s.*

Proof. Properties 1) and 2) of Definition 1.2 are satisfied trivially. Condition 3) of this definition is equivalent to the equality (2.2). □

It is clear that if $(Q^t_t, \mathcal{F}_t)^\infty_{t=0}$ is a consistent family of probability measures, then the conditions of Proposition 2.6 are fulfilled.

Example 2.7. Realize Proposition 2.6 in the case $\mathcal{F}_s = \mathcal{F}, \forall s \geq 0$. Let $(Q_s^s)_{s=0}^{\infty}$ be a sequence of probabilities such that for $0 \leq s < t < \infty$ $Q_s^s << Q_t^t$. For such s and t we put $Q_s^t = Q_t^t$. It is easy to see that if $\mathbf{Q} = (Q_s^t, 0 \leq s \leq t < \infty)$, then $(\Omega, \mathbf{Q}, \mathbf{E})$ is a closed DSB2 but generally not regular. It is regular if and only if $Q_s^s = Q_t^t$ for all $0 \leq s < t < \infty$.

Proposition 2.8. *Restriction of DSB2 (resp., closed DSB2) with continuous time on discrete moments of time gives DSB2 (resp., closed DSB2) with discrete time.*

Proof. The proof is trivial. □

Proposition 2.9. *DSB2 with discrete time is always closable.*

Proof. For any $s = 0, 1, 2, \ldots$ we put $Q_s^s := Q_s^{s+1}|_{\mathcal{F}_s}$. Prove the satisfying the conditions of Definition 2.1. Really, if $s < t$, then using the property 1) from Definition 1.2 we get $Q_s^{s+1}|_{\mathcal{F}_s} << Q_s^t|_{\mathcal{F}_s}$ and hence $Q_s^s = Q_s^{s+1}|_{\mathcal{F}_s} << Q_s^t|_{\mathcal{F}_s}$. Further, applying the property 2) from Definition 1.2 we obtain $Q_s^t << Q_t^{t+1}|_{\mathcal{F}_t} = Q_t^t$. Other properties follow from here. □

Proposition 2.10. *DSB2 with discrete time can be extended into closed DSB2 with continuous time.*

Proof. By virtue of Proposition 2.9 we can assume that initial DSB2 with discrete time is closed. From Remark 2.2 it follows that all the properties of Definition 1.2 are satisfied $\forall 0 \leq s \leq r \leq t < \infty$, where s, r and t are natural numbers. For any real $0 \leq s \leq t < \infty$ we put $\mathcal{F}_t := \mathcal{F}_{[t]}$ and $Q_s^t := Q_{[s]}^{[t]}$, where $[t]$ is the integer part of t. Taking arbitrary numbers $0 \leq s \leq r \leq t < \infty$ and writing down the relations 1), 2) and 3) from Definition 1.2 for natural numbers $0 \leq [s] \leq [r] \leq [t] < \infty$, we obtain the satisfaction of Definition 1.2 for obtained $(\Omega, \mathbf{Q}, \mathbf{E})$, where $\mathbf{Q} = (Q_s^t, \mathcal{F}_t)_{\{0 \leq s \leq t < \infty\}}$. □

3. General auxiliery results

In this section, we formulate several easy proved general lemmas, which we shall use in the proof of main theorem of this article.

Lemma 3.1. *Let on an measurable space $(\Omega, \mathcal{F})$ probabilities Q and P be defined such that $dQ = hdP$. Let r.v.'s f and $\tilde{f}$ be measurable with respect to $\mathcal{F}$ and $f = \tilde{f}$ Q-a.s. Then $fh = \tilde{f}h$ P-a.s.*

Lemma 3.2. *Let on a probability space $(\Omega, \mathcal{F}, P)$ a r.v. $h \geq 0$ be defined P-a.s. If $\mathcal{H}$ is sub-σ-field of σ-field $\mathcal{F}$, then the imbedding $\{h > 0\} \subset \{E^P[h|\mathcal{H}] > 0\}$ is true P-a.s.*

Lemma 3.3. *Consider a filtered space $(\Omega, (\mathcal{F}_n)_{n=0}^{\infty})$ with discrete time. Let a probability Q_{n-1}^n be defined on each σ-field $\mathcal{F}_n$, $n = 1, 2, \ldots$. Denote $Q_n^n = Q_n^{n+1}|_{\mathcal{F}_n}$ and suppose that $\forall n = 1, 2, \ldots$ $dQ_{n-1}^n = h_{n-1,n}^{n,n} dQ_n^n$. Then for $k < n$ the following formula is true:*

$$dQ_k^k = \prod_{i=k+1}^{n} E_{\mathcal{F}_k}^{Q_i^{i+1}} h_{i-1,i}^{i,i} dQ_n^n|_{\mathcal{F}_k}. \tag{3.1}$$

Lemma 3.4. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, f be a r.v. on it, and $f \geq 0$ P-a.s. Let $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ ($\mathcal{H}, \mathcal{G}$ be σ-fields) and let us define on $\mathcal{G}$ a probability measure Q such that $dQ = hdP|_{\mathcal{G}}$. Then*

$$E^Q_{\mathcal{H}} E^P_{\mathcal{G}} f = E^R[f|\mathcal{H}] \quad Q\text{-a.s.},$$

where the probability measure R on $\mathcal{F}$ is defined by the equality $dR = \hat{h}dP$ and the density $\hat{h}$ is defined by the formula:

$$\hat{h} = \frac{h}{E^P[h|\mathcal{H}]}, \quad \frac{0}{0} := 1. \tag{3.2}$$

4. The main theorem

In this section we consider DSB2 only in discrete time. The main purpose of this paper is to prove the following theorem.

Theorem 4.1. *Consider a sequence of probability measures $\left(Q^k_{k-1}, \mathcal{F}_k\right)_{k=1,2,\dots}$ with the property*

$$Q^k_{k-1} << Q^{k+1}_k|_{\mathcal{F}_k}, \tag{4.1}$$

and for any $k = 0, 1, 2, \dots$ let us put $Q^k_k := Q^{k+1}_k|_{\mathcal{F}_k}$. For $k < n$ let us introduce the measures Q^n_k by the formula $dQ^n_k = h^{n,n}_{k,n} dQ^n_n$, where

$$h^{n,n}_{k,n} = \frac{\prod\limits_{i=k+1}^{n} h^{i,i}_{i-1,i}}{E^{Q^{n-1}_{n-1}}_{\mathcal{F}_k}\left(\prod\limits_{i=k+1}^{n-1} h^{i,i}_{i-1,i}\right)}, \quad \frac{0}{0} := 1. \tag{4.2}$$

Then $(\Omega, \boldsymbol{Q}, \boldsymbol{E})$, where $\boldsymbol{Q} = (Q^n_k, \mathcal{F}_n)_{\{0 \leq k \leq n < \infty\}}$, is a closed DSB2.

Proof. We divide the proof into several parts.

1) It follows from Lemmas 3.1 and 3.2 that r.v. $h^{n,n}_{k,n}$ is well defined and is non-negative Q^n_n-a.s. Show that Q^n_k is a probability measure. Using the equality $dQ^n_{n-1} = h^{n,n}_{n-1,n} dQ^n_n$ we have:

$$E^{Q^n_n} h^{n,n}_{k,n} =$$

$$= E^{Q^n_{n-1}} \left[\frac{\prod\limits_{i=k+1}^{n-1} h^{i,i}_{i-1,i}}{E^{Q^{n-1}_{n-1}}_{\mathcal{F}_k}\left(\prod\limits_{i=k+1}^{n-1} h^{i,i}_{i-1,i}\right)} \right] = E^{Q^{n-1}_{n-1}} \left[\frac{\prod\limits_{i=k+1}^{n-1} h^{i,i}_{i-1,i}}{E^{Q^{n-1}_{n-1}}_{\mathcal{F}_k}\left(\prod\limits_{i=k+1}^{n-1} h^{i,i}_{i-1,i}\right)} \right] = 1$$

(the last equality is obtained by taking inside the expectation according to the measure Q^{n-1}_{n-1} a conditional expectation according to this measure with respect to σ-field $\mathcal{F}_k$).

2) Show now that the properties 1) and 2) from Definition 1.2 are satisfied. If $k < n$, then

$$dQ^n_k|_{\mathcal{F}_k} = E^{Q^n_{n-1}}\left[h^{n,n}_{k,n-1}|\mathcal{F}_k\right] dQ^n_{n-1}|_{\mathcal{F}_k} = dQ^n_{n-1}|_{\mathcal{F}_k},$$

i.e.,

$$Q^n_k|_{\mathcal{F}_k} = Q^n_{n-1}|_{\mathcal{F}_k}. \tag{4.3}$$

Let now $k < r < n$. Applying (4.3) and (4.1), we have

$$Q_k^r|_{\mathcal{F}_k} = Q_{r-1}^r|_{\mathcal{F}_k} << Q_{n-1}^n|_{\mathcal{F}_k} = Q_k^n|_{\mathcal{F}_k},$$

and the property 1) from Definition 1.2 is established.

Oh the other hand, by the same arguments and the formula $dQ_k^n = h_{k,n-1}^{n,n} dQ_{n-1}^n$

$$Q_k^r << Q_{r-1}^r << Q_{n-1}^n|_{\mathcal{F}_r} = Q_r^n|_{\mathcal{F}_r},$$

i.e., the property 2) from Definition 1.2 is fulfilled.

3) For moments $k < r < n$ express dQ_k^r through $dQ_r^n|_{\mathcal{F}_r}$. We have $dQ_k^r = h_{k,r}^{r,r} dQ_r^r$ and $dQ_r^r = \prod_{i=r+1}^{n-1} E_{\mathcal{F}_r}^{Q_i^{i+1}} h_{i-1,i}^{i,i} dQ_{n-1}^{n-1}|_{\mathcal{F}_r}$ (c.f. Lemma 3.3). With the help of Bayes theorem (c.f. [1], p.p. 274-275) the last formula can be easily tranformed to the form:

$$dQ_r^r = E_{\mathcal{F}_r}^{Q_{n-1}^{n-1}} \left(\prod_{i=r+1}^{n-1} h_{i-1,i}^{i,i} \right) dQ_{n-1}^{n-1}|_{\mathcal{F}_r} = E_{\mathcal{F}_r}^{Q_{n-1}^{n-1}} \left(\prod_{i=r+1}^{n-1} h_{i-1,i}^{i,i} \right) dQ_{n-1}^{n}|_{\mathcal{F}_r}.$$

As a result, we get:

$$dQ_k^r = h_{k,r}^{r,r} E_{\mathcal{F}_r}^{Q_{n-1}^{n-1}} \left(\prod_{i=r+1}^{n-1} h_{i-1,i}^{i,i} \right) dQ_{n-1}^{n}|_{\mathcal{F}_r}. \tag{4.4}$$

From (4.3) it follows:

$$dQ_k^r = h_{k,r}^{r,r} E_{\mathcal{F}_r}^{Q_{n-1}^{n-1}} \left(\prod_{i=r+1}^{n-1} h_{i-1,i}^{i,i} \right) dQ_r^n|_{\mathcal{F}_r} := h dQ_r^n|_{\mathcal{F}_r}. \tag{4.5}$$

4) Finally, we show that the relation 3) of Definition 1.2 holds. In order to apply Lemma 3.4, we use the following notations: $\mathcal{H} = \mathcal{F}_k$, $\mathcal{G} = \mathcal{F}_r$, $\mathcal{F} = \mathcal{F}_n$, $Q = Q_k^r$, $P = Q_r^n$. Since $dQ_k^r = h dQ_r^n|_{\mathcal{F}_r}$ (c.f. formula (4.5)), we put $dR = \hat{h} dQ_r^n$, where $\hat{h}$ is defined by the formula (3.2). By Lemma 3.4 $E_{\mathcal{F}_k}^R = E_k^r E_r^n f$. Since $dQ_r^n = h_{r,n}^{n,n} dQ_n^n$ we have $dR = \hat{h} h_{r,n}^{n,n} dQ_n^n$. But $dQ_k^n = h_{r,n}^{n,n} dQ_n^n$. It remains to prove that $\hat{h} h_{r,n}^{n,n} = h_{k,n}^{n,n}$ Q_n^n-a.s.(thus, we shell prove teh equality $R = Q_k^n$).

It is clear (c.f. formula (4.3)) that

$$\hat{h} h_{r,n}^{n,n} = \frac{h_{k,r}^{r,r} E_{\mathcal{F}_r}^{Q_{n-1}^{n-1}} \left(\prod_{i=r+1}^{n-1} h_{i-1,i}^{i,i} \right) h_{r,n}^{n,n}}{E_{\mathcal{F}_k}^{Q_r^n} \left(h_{k,r}^{r,r} E_{\mathcal{F}_r}^{Q_{n-1}^{n-1}} \left(\prod_{i=r+1}^{n-1} h_{i-1,i}^{i,i} \right) \right)} =$$

$$= \frac{\prod_{i=k+1}^{n} h_{i-1,i}^{i,i}}{E_{\mathcal{F}_k}^{Q_{r-1}^{r-1}} \left(\prod_{i=k+1}^{r-1} h_{i-1,i}^{i,i} \right) E_{\mathcal{F}_k}^{Q_r^n} \left(h_{k,r}^{r,r} E_{\mathcal{F}_r}^{Q_{n-1}^{n-1}} \left(\prod_{i=r+1}^{n-1} h_{i-1,i}^{i,i} \right) \right)}.$$

Using (4.3) in the form $Q^n_r|_{\mathcal{F}_r} = Q^{n-1}_{n-1}|_{\mathcal{F}_r}$, let us transform the second factor in the denominator:

$$E^{Q^n_r}_{\mathcal{F}_k}\left(h^{r,r}_{k,r} E^{Q^{n-1}_{n-1}}_{\mathcal{F}_r}\left(\prod_{i=r+1}^{n-1} h^{i,i}_{i-1,i}\right)\right) = E^{Q^{n-1}_{n-1}}_{\mathcal{F}_k}\left(E^{Q^{n-1}_{n-1}}_{\mathcal{F}_r}\left(h^{r,r}_{k,r}\prod_{i=r+1}^{n-1} h^{i,i}_{i-1,i}\right)\right) =$$

$$= E^{Q^{n-1}_{n-1}}_{\mathcal{F}_k}\left(h^{r,r}_{k,r}\prod_{i=r+1}^{n-1} h^{i,i}_{i-1,i}\right).$$

Replacing n with r in the formula (4.2) and applying it, we get

$$\hat{h}h^{n,n}_{r,n} = \frac{\prod\limits_{i=k+1}^{n} h^{i,i}_{i-1,i}}{E^{Q^{r-1}_{r-1}}_{\mathcal{F}_k}\left(\prod\limits_{i=k+1}^{r-1} h^{i,i}_{i-1,i}\right) E^{Q^{n-1}_{n-1}}_{\mathcal{F}_k}\left(h^{r,r}_{k,r}\prod\limits_{i=r+1}^{n-1} h^{i,i}_{i-1,i}\right)} =$$

$$= \frac{\prod\limits_{i=k+1}^{n} h^{i,i}_{i-1,i}}{E^{Q^{n-1}_{n-1}}_{\mathcal{F}_k}\left(\prod\limits_{i=k+1}^{n-1} h^{i,i}_{i-1,i}\right)} = h^{n,n}_{k,n}.$$

Using Lemma 3.1, it is easy to prove that the resulting equality is fulfilled Q^n_n-a.s.

Theorem 4.1 is completely proved. □

5. Nonuniqueness of the representation of the operator $E^{k+1}_k E^{k+2}_{k+1}\dots E^n_{n-1}$

It follows from Theorem 4.1 that $E^n_k f = E^{k+1}_k E^{k+2}_{k+1}\dots E^n_{n-1} f$, i.e., the operator $E^{k+1}_k E^{k+2}_{k+1}\dots E^n_{n-1}$ can be representated as a conditional expectation with respect to σ-field $\mathcal{F}_k$ and to the probability measure Q^n_k, wich density with respect to measure Q^n_{n-1} is defined by the formula (4.2). In this section, we show that the representative probability measure for a given operator is not unique.

Let $(\Omega, (\mathcal{F}_n)^\infty_{n=0})$ be a filtered space with discrete time, where each σ-field $\mathcal{F}_n$ is generated by a decomposition of Ω into finite or countable many atoms. Let the family of probability measures $\left(Q^n_{n-1}, \mathcal{F}_n\right)^\infty_{n=0}$ be such that $\forall n = 0,1,2,\dots$ Q^n_{n-1} loads all atoms of σ-field $\mathcal{F}_n$. Then $\left(Q^n_{n-1}, \mathcal{F}_n\right)^\infty_{n=0}$ generates (as in Theorem 4.1) a closed DSB2.

Until the end of this section, we will work on a filtered space equipped with a filtration of type $\mathcal{F}_0 = \{\Omega, \emptyset\}$, $\mathcal{F}_n = \sigma\{A^1, A^2, \dots, A^n, B_n\}$ (we call such filtration special Haar filtration). Consider on this filtered space a family of probabilities $\mathbf{Q} = \left(Q^n_{n-1}, \mathcal{F}_n\right)^\infty_{n=0}$, satisfying the conditions given above. Denote $q^k_n = Q^n_{n-1}(A^k) > 0$ for $k = 1,2,\dots,n$ and $q_n = Q^n_{n-1}(B_n) > 0$. Our goal is to describe all representing probability measures of the operator $E^2_1 E^3_2$ and find among them the measure Q^3_1.

We carry out the necessary reasoning.

1) Let a r.v. f be measurable with respect to $\mathcal{F}_3$, that is

$$f = a_1 I_{A_1} + a_2 I_{A_2} + a_3 I_{A_3} + b_3 I_{B_3}.$$

Let us calculate:

$$E_2^3 f = a_1 I_{A_1} + a_2 I_{A_2} + \frac{1}{q_3^3 + q_3}(a_3 q_3^3 + b_3 q_3) \cdot I_{B_2},$$

$$E_1^2 E_2^3 f = a_1 I_{A_1} + \frac{1}{q_2^2 + q_2}\left[a_2 q_2^2 + \frac{q_2}{q_3^3 + q_3}(a_3 q_3^3 + b_3 q_3)\right] \cdot I_{B_1}.$$

Let now P be any probability measure defined on $\mathcal{F}_3$. Denote $P(A^k) = p^k$, $k = 1, 2, 3$ and $P(B_3) = p_3$. It is clear that $P(B_1) = p^2 + p^3 + p_3$. We have:

$$E^P[f|\mathcal{F}_1] = a_1 I_{A_1} + \frac{1}{p^2 + p^3 + p_3}\left[a_2 p^2 + a_3 p^3 + b_3 p_3\right].$$

Thus, the following equivalence is true:

$$E_1^2 E_2^3 f = E^P[f|\mathcal{F}_1] \iff$$

$$\begin{cases} \dfrac{1}{q_2^2 + q_2}\left[a_2 q_2^2 + \dfrac{q_2}{q_3^3 + q_3}(a_3 q_3^3 + b_3 q_3)\right] = \dfrac{1}{p^2 + p^3 + p_3}\left[a_2 p^2 + a_3 p^3 + b_3 p_3\right] \\ 0 < q_2^2 + q_2 < 1 \\ 0 < q_3^3 + q_3 < 1 \\ 0 < p^2 + p^3 + p_3 < 1. \end{cases}$$

Since the first equality of this equivalence must hold for any $\mathcal{F}_3$-measurable f, the resulting system must have a solution for any a_2, a_3, b_3. We will give these parameters different meanings.

a) $a_2 = 1$, $a_3 = b_3 = 0$. Then

$$\frac{1}{c_1} := \frac{q_2^2}{q_2^2 + q_2} = \frac{p^2}{p^2 + p^3 + p_3}.$$

b) $a_2 = b_3 = 0, a_3 = 1$. Then

$$\frac{1}{c_2} := \frac{q_2 q_3^3}{(q_2^2 + q_2)(q_3^3 + q_3)} = \frac{p^3}{p^2 + p^3 + p_3}.$$

c) $a_2 = a_3 = 0, b_3 = 1$. Then

$$\frac{1}{c_3} := \frac{q_2 q_3}{(q_2^2 + q_2)(q_3^3 + q_3)} = \frac{p_3}{p^2 + p^3 + p_3}.$$

Now our system can be written as:

$$\begin{cases} \dfrac{p^2}{p^2 + p^3 + p_3} = \dfrac{1}{c_1}, \ \ c_1 > 1 \\ \dfrac{p^3}{p^2 + p^3 + p_3} = \dfrac{1}{c_2}, \ \ c_2 > 1 \\ \dfrac{p_3}{p^2 + p^3 + p_3} = \dfrac{1}{c_3}, \ \ c_3 > 1 \\ p^2 + p^3 + p_3 < 1, \ p^2 > 0, \ p^3 > 0 \ p_3 > 0. \end{cases}$$

This system is equivalent to the following one:

$$\begin{cases} p^2(1-c_1)+p^3+p_3=0 \\ p^2+p^3(1-c_2)+p_3=0 \\ p^2+p^3+p_3(1-c_3)=0 \\ p^2+p^3+p_3<1,\ p^2>0,\ p^3>0\ p_3>0. \end{cases}$$

The general solution of this system is $p^2=\dfrac{t}{c_1}$, $p^3=\dfrac{t}{c_2}$, $p_3=\dfrac{t}{c_3}$, $0<t<1$. Using the equality $\dfrac{1}{c_1}+\dfrac{1}{c_2}+\dfrac{1}{c_3}=1$, we obtain $P=\left(1-t,\dfrac{t}{c_1},\dfrac{t}{c_2},\dfrac{t}{c_3}\right)$, $0<t<1$. It is a finite open interval in the space $\mathbb{R}^4$.

2) Let us calculate the measure Q_1^3. We naturally identify the measures Q_{n-1}^n with the vectors of the space $\mathbb{R}^{n+1}$. We have:

$$Q_1^2=(q_2^1,q_2^2,q_2),\quad Q_2^3=(q_3^1,q_3^2,q_3^3,q_3),\quad Q_2^2=(q_3^1,q_3^2,q_3^3+q_3).$$

Further:

$$h_{1,2}^{2,2}=\frac{dQ_1^2}{dQ_2^2}=\frac{q_2^1}{q_3^1}I_{A_1}+\frac{q_2^2}{q_3^2}I_{A_2}+\frac{q_2}{q_3^3+q_3}I_{B_2};$$

$$E^{Q_2^2}[h_{1,2}^{2,2}|\mathcal{F}_1]=\frac{q_2^1}{q_3^1}I_{A_1}+\frac{1}{q_3^2+q_3^3+q_3}\left[\frac{q_2^2}{q_3^2}\cdot q_3^2+\frac{q_2}{q_3^3+q_3}\cdot(q_3^3+q_3)\right]I_{B_1}=$$

$$=\frac{q_2^1}{q_3^1}I_{A_1}+\frac{q_2^2+q_2}{q_3^2+q_3^3+q_3}\cdot I_{B_1};$$

$$dQ_1^3=\frac{h_{1,2}^{2,2}}{E^{Q_2^2}[h_{1,2}^{2,2}|\mathcal{F}_1]}=\left(I_{A_1}+\frac{q_2^2(q_3^2+q_3^3+q_3)}{q_3^2(q_2^2+q_2)}I_{A_2}+\frac{q_2(q_3^2+q_3^3+q_3)}{(q_2^2+q_2)(q_3^3+q_3)}I_{B_2}\right)dQ_2^3;$$

$$Q_1^3=\left(q_3^1,\frac{q_2^2(q_3^2+q_3^3+q_3)}{q_2^2+q_2},\frac{q_2q_3^3(q_3^2+q_3^3+q_3)}{(q_2^2+q_2)(q_3^3+q_3)},\frac{q_2q_3(q_3^2+q_3^3+q_3)}{(q_2^2+q_2)(q_3^3+q_3)}\right).$$

Denoting $\tilde{t}=q_3^2+q_3^3+q_3$, we get $Q_1^3=\left(1-\tilde{t},\dfrac{\tilde{t}}{c_1},\dfrac{\tilde{t}}{c_2},\dfrac{\tilde{t}}{c_3}\right)$. It is clear that the measure Q_1^3 is identified with a point lying inside the interval $P=\left(1-t,\dfrac{t}{c_1},\dfrac{t}{c_2},\dfrac{t}{c_3}\right)$, $0<t<1$.

6. Conclusion

A detailed study of deformed stochastic bases is necessary not only for the further development of generalized stochastic analysis, but also for financial mathematics (deformed financial markets, deflators, etc.). Works [2]-[12] are devoted to such investigations. The constructions of deformed stochachtic bases of the first kind with discrete time are devoted to works [6] and [8]. Note that deformed stochastic bases have been little studied in continuous time. The authors hope that constructions similar to those made in Theorem 4.1 will be realized in this case as well.

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