EULER EQUATIONS FOR COSSERAT MEDIA

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ABSTRACT. We consider Cosserat media as $\mathbf{SO}(3)$-structures over a domain $\mathbf{D} \subset \mathbb{R}^3$. Motions of such media are given by infinitesimal automorphisms of the $\mathbf{SO}(3)$-bundle. We present Euler-type equations for such media and discuss their structure.

1. Introduction

This paper is a realization of approach ([6]) for the case of media formed by 'rigid microelements' ([4]), or Cosserat' media.

We show that the geometry hidden behind of the such type of media is a $\mathbf{SO}(3)$-structure over a spacial domain $\mathbf{D}$.

Namely, the dynamics in such media are given by vector fields on the bundle $\pi \colon \Phi \to \mathbf{D}$ of $\mathbf{SO}(3)$-frames over $\mathbf{D}$. These fields are solutions of the system of differential equations ([6]) that generalize the well known Navier-Stocks equations.

To construct these equations we need two additional geometric objects: (1) connection in the bundle π, we call it media connection, and (2) left $\mathbf{SO}(3)$-invariant metric on fibres of π. The last determines the mechanics of 'rigid microelements' and the media connection allows us to compare 'rigid microelements' at different points of $\mathbf{D}$. The third ingredient that we needed to construct the system of differential equations is the thermodynamics of media. We apply here the point of view ([7]), where thermodynamics were considered as measurement of extensive quantities.

Using these remarks and observations we present here the Euler type equations that govern the motion. To make the presentation as compact as possible we restrict ourselves by this type equations only although the difference between Euler and Navier-Stocks type equations consists only in the description of the stress tensor.

The final form of the Euler equations is given by (5.3), (5.4), (5.5).

2. Geometry of $\mathbf{SO}(3)$

Here we collect the main properties of the orthogonal group $\mathbf{SO}(3)$.

Let $(\mathbf{T}, g)$ be a an Euclidian vector space, $\dim \mathbf{T} = 3$, where g is a metric tensor. Let $A \in \mathrm{End}\,(\mathbf{T})$ be a linear operator in $\mathbf{T}$ and let $A' \in \mathrm{End}\,(\mathbf{T})$ be the g-adjoint operator, i.e $g\,(AX, Y) = g\,(X, AY)$ for all $X, Y \in \mathbf{T}$. Remind that

$$\mathbf{SO}(3) = \{A \in \mathrm{End}\,(\mathbf{T}) \mid AA' = \mathbf{1}, \det A = 1\}.$$

Geometrically, elements of the group are counterclockwise rotations $R\,(\phi, \overline{n})$ on angle ϕ about the axis through unit vector $\overline{n} \in \mathbf{T}$. One has $R\,(\phi, n) = R\,(-\phi, -n)$ and $R\,(\pi, n) = R\,(\pi, -n)$. Therefore, as a smooth manifold, $\mathbf{SO}(3)$ is diffeomorphic to the projective space $\mathbb{R}\mathbf{P}^3$.

The Lie algebra of the group,

$$\mathfrak{so}\,(3) = \{A \in \mathrm{End}\,(\mathbf{T}) \mid A + A' = \mathbf{0}\},$$

consists of skew symmetric operators.

There is the *hat isomorphism* of the Lie algebras

$$\wedge\colon (\mathbf{T}, \times) \to \mathfrak{so}\,(3),$$

where $(\mathbf{T}, \times)$ is the Lie algebra of vectors in $\mathbf{T}$ with respect to the cross product $\times$.

In an orthonormal basis (e_1, e_2, e_3) this isomorphism has the following form

$$\wedge\colon w = (w_1, w_2, w_3) \mapsto \widehat{w} = w_1 \widehat{e_1} + w_2 \widehat{e_2} + w_3 \widehat{e_3} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}.$$

2.1. Exponent and logarithm.

In the case of Lie algebra $\mathfrak{so}\,(3)$ the exponential map

$$\exp\colon \mathfrak{so}\,(3) \to \mathbf{SO}(3),$$

has the following Rodrigues' form (see, for example, [1], [3])

$$\exp\,(\phi\widehat{n}) = \mathbf{1} + \sin\,(\phi)\,\widehat{n} + (1 - \cos\,(\phi))\widehat{n}^2,$$

and $\exp\,(\phi\widehat{n}) = R\,(\phi, n)$.

This formula gives us the following description of the logarithm map

$$\ln\,(R) = \frac{\arcsin\,(\phi)}{2\phi}\,(R - R'),$$

where $R \in \mathbf{SO}(3)$ and

$$\phi = \frac{3 + \sqrt{-\mathrm{Tr}R^2}}{2}.$$

2.2. Left invariant tensors on SO(3).

The Baker–Campbell–Hausdorff formula in $\mathfrak{so}\,(3)$ has very concrete form.

Namely, let $X, Y \in \mathfrak{so}\,(3)$, then $\exp\,(X) \cdot \exp\,(Y) \in \mathbf{SO}(3)$ and therefore has the form $\exp\,(Z(X, Y))$, for some element $Z(X, Y) \in \mathfrak{so}\,(3)$.

Then (see, for example, [2]),

$$Z\,(X, Y) = \alpha\,X + \beta\,Y + \gamma\,[X, Y],$$

where

$$\alpha = \frac{a\arcsin\,(d)}{d\theta}, \quad \beta = \frac{b\arcsin\,(d)}{d\phi}, \quad \gamma = \frac{\arcsin\,(d)}{d\phi\theta},$$

a, b, c, d are defined as follows

$$a = \sin(\theta)\cos^2\left(\frac{\phi}{2}\right) - \omega\sin(\phi)\sin^2\left(\frac{\theta}{2}\right),$$

$$b = \sin(\phi)\cos^2\left(\frac{\theta}{2}\right) - \omega\sin(\theta)\sin^2\left(\frac{\phi}{2}\right)$$

$$c = \frac{1}{2}\sin(\theta)\sin(\phi) - 2\omega\sin^2\left(\frac{\theta}{2}\right)\sin^2\left(\frac{\phi}{2}\right),$$

$$d = \sqrt{a^2 + b^2 + 2\omega ab + (1 - \omega^2)c^2},$$

and

$$\theta = \sqrt{\frac{-\operatorname{Tr}X^2}{2}}, \quad \phi = \sqrt{\frac{-\operatorname{Tr}X^2}{2}}, \quad \omega = \theta^{-1}\phi^{-1}\sqrt{\frac{-\operatorname{Tr}XY}{2}}.$$

Applying these formulae for the case $X = t\widehat{n}, Y$, where n is a unit vector, we get

$$\alpha = \frac{\phi}{2}\cot\left(\frac{\phi}{2}\right) + O(t), \quad \beta = 1 + t\left(\frac{\omega}{2}\cot\left(\frac{\phi}{2}\right) + \frac{\omega}{\phi}\right) + O(t^2), \quad \gamma = \frac{1}{2} + O(t).$$

Denote by E_1, E_2, E_3 the left invariant vector fields on $\mathbf{SO}(3)$ that correspond to the basis e_1, e_2, e_3 in $\mathfrak{so}(3)$.

Then the above formulae give us the following expressions for E_1, E_2, E_3 in the canonical coordinates (x_1, x_2, x_3) of the first kind:

$$E_1 = \frac{\phi}{2}\cot\left(\frac{\phi}{2}\right)\partial_1 + \frac{1}{2}(x_2\partial_3 - x_3\partial_2) +$$

$$x_1\left(\frac{1}{2}\cot\left(\frac{\phi}{2}\right) + \frac{1}{\phi}\right)(x_1\partial_1 + x_2\partial_2 + x_3\partial_3),$$

$$E_2 = \frac{\phi}{2}\cot\left(\frac{\phi}{2}\right)\partial_2 + \frac{1}{2}(x_3\partial_1 - x_1\partial_3) +$$

$$x_2\left(\frac{1}{2}\cot\left(\frac{\phi}{2}\right) + \frac{1}{\phi}\right)(x_1\partial_1 + x_2\partial_2 + x_3\partial_3),$$

$$E_3 = \frac{\phi}{2}\cot\left(\frac{\phi}{2}\right)\partial_3 + \frac{1}{2}(x_1\partial_2 - x_2\partial_1) +$$

$$x_3\left(\frac{1}{2}\cot\left(\frac{\phi}{2}\right) + \frac{1}{\phi}\right)(x_1\partial_1 + x_2\partial_2 + x_3\partial_3),$$

where

$$\phi = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad \text{and} \quad \partial_i = \frac{\partial}{\partial x_i}.$$

Remark that basis vectors e_1, e_2, e_3 have the following commutation relations: $[e_{\sigma(1)}, e_{\sigma(2)}] = \operatorname{sign}(\sigma)\, e_{\sigma(3)}$, for any permutation σ of three letters and accordingly vector fields E_1, E_2, E_3 inherit the same commutation relations

$$[E_{\sigma(1)}, E_{\sigma(2)}] = \operatorname{sign}(\sigma)\, E_{\sigma(3)}.$$

Let us denote by Ω_1, Ω_2, $\Omega_3 \in \Omega^1(\mathbf{SO}(3))$ differential 1-forms on Lie group $\mathbf{SO}(3)$ such that $\Omega_i(E_j) = \delta_{ij}$, then

$$d\Omega_{\sigma(3)} + \operatorname{sign}(\sigma)\ \Omega_{\sigma(1)} \wedge \Omega_{\sigma(2)}.$$

Vector fields E_i and differential 1-forms Ω_i give us the bases (over $\mathbb{R}$) in the space of left invariant vector fields and correspondingly invariant differential 1-forms on **SO**(3).

Moreover, any left invariant tensor on **SO**(3) is a linear combination of tensor products E_i and Ω_j with constant coefficients.

Thus any left invariant metric g on **SO**(3) is defined by a positive self adjoint operator Λ on $\mathfrak{so}(3)$, so-called inertia tensor.

We will take basis e_1, e_2, e_3 to be eigenvectors of the operator Λ.

Thus we get:

$$g_\lambda = \frac{1}{2} \left(\lambda_1 \Omega_1^2 + \lambda_2 \Omega_2^2 + \lambda_3 \Omega_3^2 \right), \tag{2.1}$$

where λ_i, $\lambda_i > 0$, are eigenvalues of the operator and Ω_i^2 are the symmetric squares of the 1-forms.

2.3. Levi-Civita connections on SO(3). Let ∇ be the Levi-Civita connection associated with left invariant metric g_λ. We denote by ∇_i the covariant derivative along vector field E_i.

Then we have

$$\nabla_i (E_j) = \sum_k \Gamma_{ij}^k E_k,$$

where Γ_{ij}^k are the Christoffel symbols.

This connection preserves the metric and therefore

$$g \left(\nabla_i (E_j), E_k \right) + g \left(E_j, \nabla_i (E_k) \right) = 0,$$

or

$$\lambda_k \Gamma_{ij}^k + \lambda_j \Gamma_{ik}^j = 0, \tag{2.2}$$

for all $i, j, k = 1, 2, 3$.

The condition for the connection to be torsion-free gives us the following relations:

$$\nabla_{\sigma(1)} \left(E_{\sigma(2)} \right) - \nabla_{\sigma(2)} \left(E_{\sigma(1)} \right) = E_{\sigma(3)},$$

for all permutations σ, or

$$\Gamma_{\sigma(1),\sigma(2)}^k - \Gamma_{\sigma(2),\sigma(1)}^k = \delta_{k,\sigma(3)}. \tag{2.3}$$

The solution of these equations is the following

$$\begin{aligned} \Gamma_{12}^3 &= \frac{\lambda - \lambda_1}{\lambda_3}, \Gamma_{23}^1 = \frac{\lambda - \lambda_2}{\lambda_1}, \Gamma_{31}^2 = \frac{\lambda - \lambda_3}{\lambda_2}, \\ \Gamma_{21}^3 &= \Gamma_{12}^3 - 1, \Gamma_{32}^1 = \Gamma_{23}^1 - 1, \Gamma_{13}^2 = \Gamma_{31}^2 - 1, \end{aligned} \tag{2.4}$$

where $\lambda = \lambda_1 + \lambda_2 + \lambda_3$, and all other Christoffel symbols are trivial.

Thus we have the only non-trivial relations:

$$\nabla_1 (E_2) = \frac{\lambda - \lambda_1}{\lambda_3} [E_1, E_2], \nabla_2 (E_3) = \frac{\lambda - \lambda_2}{\lambda_1} [E_2, E_3], \nabla_3 (E_1) = \frac{\lambda - \lambda_3}{\lambda_2} [E_3, E_1],$$

$$\nabla_2 (E_1) = \frac{\lambda - \lambda_2}{\lambda_3} [E_2, E_1], \nabla_3 (E_2) = \frac{\lambda - \lambda_3}{\lambda_1} [E_3, E_2], \nabla_1 (E_3) = \frac{\lambda - \lambda_1}{\lambda_2} [E_1, E_3],$$

Theorem 2.1. *The Levi-Civita connection for left invariant the metric g_λ has the form:*

$$\nabla_{\sigma(1)}\left(E_{\sigma(2)}\right) = \text{sign}\left(\sigma\right)\frac{\lambda - \lambda_{\sigma(1)}}{\lambda_{\sigma(3)}}E_{\sigma(3)},$$

$$\nabla_i\left(E_i\right) = 0,$$

for all permutations $\sigma \in S_3$ and $i = 1, 2, 3$.

3. Cosserat media and SO(3)-structures

Let $\mathbf{D}$ be a domain in $\mathbb{R}^3$ considered as Riemannian manifold equipped with the standard metric g_0. Then by *Cosserat medium* we mean a medium composed with solids or having 'rigid microstructure', ([4]). We assume that on the microlevel this media is formed by rigid elements, which we represent as orthonormal frames $f_a \colon \mathbb{R}^3 \to \mathbf{T}_a B$.

Thus configuration space of these media is the principal $\mathbf{SO}(3)$-bundle $\pi \colon \Phi \to \mathbf{D}$ of orthonormal frames on $\mathbf{D}$, (see, for example, [5]). The projection π assign the centre mass a of element f_a.

3.1. Metrics and connections, associated with Cosserat medium.
The group $\mathbf{SO}(3)$ acts in the natural way on fibres of projection π and we will continue to use notations E_1, E_2, E_3 for the induced vertical vector fields on Φ.

These fields form the basis in the module of vertical vector fields on Φ and accordingly differential 1-forms $\Omega_1, \Omega_2, \Omega_3$ define the dual basis in the space of vertical differential forms.

We assume also that the media is characterized by a $\mathbf{SO}(3)$-connection in the bundle π. This connection, we call it *media connection* and denote by ∇^μ, allows us to compare microelements at different points of $\mathbf{D}$.

To define the connection, we consider a microelement as orthonormal frame $b = (b_1, b_2, b_3)$, formed by vector fields b_i on $\mathbf{D}$. Then the covariant derivatives ∇_X along vector field X is defined by the connection form $\omega \in \Omega^1\left(\mathbf{D}, \mathfrak{so}(3)\right)$, i.e. differential 1-form on $\mathbf{D}$ with values in the Lie algebra $\mathfrak{so}(3)$, and such that

$$\nabla_X^\mu\left(b\right) = \omega\left(X\right)b. \tag{3.1}$$

By using the hat morphism, we represent the form as before

$$\omega = \left\| \begin{matrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{matrix} \right\|,$$

where ω_i are differential 1-forms on $\mathbf{D}$.

Then formula (3.1) shows us that a microelement subject to rotation along vector $(\omega_1(X), \omega_2(X), \omega_3(X))$ on the angle $\phi\left(X\right) = \sqrt{\omega_1\left(X\right)^2 + \omega_2\left(X\right)^2 + \omega_3\left(X\right)^2}$, when we transport it on vector X.

Let (x_1, x_2, x_3) be the standard Euclidian coordinates on $\mathbf{D}$ and $\partial = (\partial_1, \partial_2, \partial_3)$ be the corresponding frame. Here $\partial_i = \partial/\partial x_i$.

In these coordinates we have

$$\omega_i = \sum_{j=1}^{3} \omega_{ij} d_j,$$

where $d_j = dx_j$.

The connection form ω will be the following

$$\omega = \begin{Vmatrix} 0 & -\omega_{31} & \omega_{21} \\ \omega_{31} & 0 & -\omega_{11} \\ -\omega_{21} & \omega_{11} & 0 \end{Vmatrix} d_1 + \begin{Vmatrix} 0 & -\omega_{32} & \omega_{22} \\ \omega_{32} & 0 & -\omega_{12} \\ -\omega_{22} & \omega_{12} & 0 \end{Vmatrix} d_2 + \begin{Vmatrix} 0 & -\omega_{33} & \omega_{23} \\ \omega_{33} & 0 & -\omega_{13} \\ -\omega_{23} & \omega_{13} & 0 \end{Vmatrix} d_3.$$

This connection allows us to split tangent spaces $T_b\Phi$ into direct sum

$$T_b\Phi = V_b + H_b,$$

where V_b is the vertical part with basis $E_{1,b}$, $E_{2,b}$, $E_{3,b}$, and the horizontal space H_b is formed by vectors

$$X_a - \omega_a (X_a),$$

where $a = \pi(b)$.

Remark that geometrically spaces H_b represent 'constant frames' due to (3.1) and the basis in this space is

$$\partial_1 - \omega_{11} E_1 - \omega_{21} E_2 - \omega_{31} E_3,$$
$$\partial_2 - \omega_{12} E_1 - \omega_{22} E_2 - \omega_{32} E_3,$$
$$\partial_3 - \omega_{13} E_1 - \omega_{23} E_2 - \omega_{33} E_3.$$

The horizontal distribution $H : b \in \Phi \to H_b \subset T_b\Phi$ could be also defined as the kernel of the following differential 1-forms on Φ:

$$\theta_1 = \Omega_1 + \omega_{11} d_1 + \omega_{12} d_2 + \omega_{13} d_3,$$
$$\theta_2 = \Omega_2 + \omega_{21} d_1 + \omega_{22} d_2 + \omega_{23} d_3,$$
$$\theta_3 = \Omega_2 + \omega_{31} d_1 + \omega_{32} d_2 + \omega_{33} d_3.$$

By metric associated with the left invariant metric g_λ and standard metric g_0 and media connection form ω we mean direct sum of metric g_λ on the vertical space V and the standard metric g_0 on the horizontal space H.

We also call the media *homogeneous* if the connection form ω as well as the inertia tensor Λ are constants. Moreover, for the case of homogeneous media the Euclidian coordinates (x_1, x_2, x_3) in the domain **D** are chosen in a such way that operators $\widehat{\partial_i} \in \mathfrak{so}(3)$ are eigenvectors of the inertia tensor Λ.

Summarizing we get the following

Proposition 3.1. *The metric g^μ associated with the triple (g_λ, g_0, ω) has the following form*

$$g^\mu = \frac{1}{2} \sum_{i=1}^{3} \left(\lambda_i \Omega_1^2 + d_i^2 \right).$$

Remark 3.2. Frame $(E_1, E_2, E_3, \partial_1 - \omega(\partial_1), \partial_2 - \omega(\partial_2), \partial_2 - \omega(\partial_2))$ and coframe $(\Omega_1, \Omega_2, \Omega_3, d_1, d_2, d_3)$ are dual and g^μ-orthogonal.

3.2. Levi-Civita connection on Cosserat media. Assume that the media is homogeneous and let ∇ be the Levi-Civita connection on the configuration space Φ associated with metric g^μ. Then for basic vector fields ∂_i, E_j, where $i, j = 1, 2, 3$, we have the following commutation relations

$$[\partial_i, \partial_j] = [\partial_i, E_j] = 0, \quad [E_{\sigma(1)}, E_{\sigma(2)}] = \operatorname{sign}(\sigma) E_{(3)}.$$

Moreover, it is easy to see that the equations for Christoffel symbols do not depend on the connection form ω. On the other hand, it is clear that the Levi-Civita connection ∇, for the case $\omega = 0$, is the direct sum of the trivial connection on **D** and the left invariant Levi-Chivita connection on **SO** (3). Thus we get the following result.

Theorem 3.3. *The Levi-Civita connection ∇^c on the configuration space Φ associated with metric g^μ and homogeneous media has the form, where the only non trivial covariant derivatives are*

$$\nabla_{E_{\sigma(1)}}\left(E_{\sigma(2)}\right) = \operatorname{sign}(\sigma) \frac{\lambda - \lambda_{\sigma(1)}}{\lambda_{\sigma(3)}} E_{\sigma(3)},$$

for all permutations $\sigma \in S_3$.

3.3. Deformation tensor. Dynamics we describe by π-projectable vector fields on Φ (see [6], for more details):

$$U := \sum_{i=1}^{3} \left(X_i(x)\, \partial_i + Y_i(x, y)\, E_i\right).$$

Here $x = (x_1, x_2, x_3)$ are the Euclidian coordinates on **D** and $y = (y_1, y_2, y_3)$ are the canonical coordinates on **SO** (3).

Due to the above theorem and relations (2.4) the covariant differential $\mathcal{D}$ with respect to the Levi-Civita connection ∇^c acts in the following way:

$$
\begin{aligned}
\mathcal{D}(\partial_i) &= 0, \quad i = 1, 2, 3, && (3.2)\\
\mathcal{D}(E_1) &= (\alpha_2 + 1)\Omega_3 \otimes E_2 + (\alpha_3 - 1)\ \Omega_2 \otimes E_3,\\
\mathcal{D}(E_2) &= (\alpha_1 - 1)\Omega_3 \otimes E_1 + \alpha_3\ \Omega_1 \otimes E_3,\\
\mathcal{D}(E_3) &= \alpha_1\Omega_2 \otimes E_1 + \alpha_2\Omega_1 \otimes E_2,
\end{aligned}
$$

where

$$\Gamma^1_{32} = \alpha_1 - 1, \qquad \Gamma^1_{23} = \alpha_1 = \frac{\lambda - \lambda_2}{\lambda_1}, \qquad \Gamma^2_{31} = \alpha_2 + 1,$$

$$\Gamma^2_{13} = \alpha_2 = \frac{\lambda_1}{\lambda_2}, \qquad \Gamma^3_{21} = \alpha_3 - 1, \qquad \Gamma^3_{12} = \alpha_3 = \frac{\lambda - \lambda_1}{\lambda_3}.$$

Respectively, for the dual frame $(\Omega_1, \Omega_2, \Omega_3)$, we have

$$
\begin{aligned}
\mathcal{D}(\Omega_1) &= -\alpha_1\Omega_2 \otimes \Omega_3 - (\alpha_1 - 1)\Omega_3 \otimes \Omega_2,\\
\mathcal{D}(\Omega_2) &= -\alpha_2\Omega_1 \otimes \Omega_3 - (\alpha_2 + 1)\Omega_3 \otimes \Omega_1, && (3.3)\\
\mathcal{D}(\Omega_3) &= -\alpha_3\Omega_1 \otimes \Omega_2 - (\alpha_3 - 1)\Omega_2 \otimes \Omega_1.
\end{aligned}
$$

By the *rate of deformation tensor* $\Delta\left(U\right)$ we mean tensor

$$\Delta\left(U\right) = \mathcal{D}\left(U\right) = \sum_{i,j=1}^{3} \left(\partial_i X_i\, d_j \otimes \partial_i + \partial_i Y_i\, d_j \otimes E_i + E_j\left(Y_i\right)\Omega_j \otimes E_i\right) + \sum_{i=1}^{3} Y_i \mathcal{D}\left(E_i\right).$$

4. Thermodynamics of Cosserat media

The thermodynamics of the Cosserat media is based on measurement (see, [7], [6]) of *extensive quantities: inner energy E, volume V, mass m,* and *deformation D.* The corresponding dual, or *intensive quantities* are the temperature T, preasure p, chemical potential ξ and the stress tensor σ.

The first law of thermodynamics requires that on thermodynamical states the following differential 1-form

$$dE - \left(T\,dS - p\,dV + \mathrm{Tr}(\sigma^*dD) + \xi\,dm\right) \tag{4.1}$$

should be zero.

In other words, the thermodynamical state is a maximal integral manifold L of differential form (4.1).

It is more convenient to use another but proportional differential 1-form

$$dS - T^{-1}\left(dE + p\,dV - \mathrm{Tr}(\sigma^*dD) - \xi\,dm\right). \tag{4.2}$$

Here differential forms: dE, $T\,dS$, $-p\,dV + \mathrm{Tr}(\sigma^*dD)$ and $\xi\,dm$ represent change of inner energy, heat, work and mass respectively.

Extensivity of quantities E, V, D, m, S means that their simultaneous rescaling does not change the intensives as well as the thermodynamic state.

In other words, if we represent (S, p, σ, ξ) as functions of the extensive variables (E, V, D, m), then $S = S\left(E, V, D, m\right)$ is a homogeneous function of degree one.

Let $(s, \varepsilon, \rho, \Delta)$ be the densities of (S, E, m, D).

Then substituting expression $S = Vs(\varepsilon, \rho, \Delta)$ into (4.2) we get that 1-form

$$\psi = ds - T^{-1}\left(d\varepsilon - \mathrm{Tr}\left(\sigma^*d\Delta\right) - \xi\,d\rho\right)$$

should be zero and

$$\varepsilon - Ts = \mathrm{Tr}\left(\sigma^*\Delta\right) - p + \xi\rho.$$

The last condition shows that the density of Gibbs free energy $\varepsilon - Ts$ equals sum of density of deformation $\mathrm{Tr}\left(\sigma^*\Delta\right)$ and mechanical $-p$ and chemical $\xi\,\rho$ works.

All these observations could be formulated as follows.

Let us introduce the thermodynamic phase space of Cosserat medium ([6]) as the contact space $\widetilde{\Psi} = \mathbb{R}^5 \times \mathrm{End}\left(\mathbf{T}^*\right) \times \mathrm{End}\left(\mathbf{T}\right)$, $\dim\widetilde{\Psi} = 23$ with points $(s, T, \varepsilon, \xi, \rho, \sigma, \Delta) \in \Psi$, where $\sigma \in \mathrm{End}\left(\mathbf{T}^*\right)$, $\Delta \in \mathrm{End}\left(\mathbf{T}\right)$ and equipped with contact form ψ. Then by thermodynamic states we mean Legendrian manifolds $\widetilde{L} \subset \widetilde{\Psi}$, $\dim\widetilde{L} = 11$, of the differential ψ, or respectively, after eliminating entropy from the consideration, their Lagrangian projections $L \subset \Psi$ into $\Psi = \mathbb{R}^4 \times \mathrm{End}\left(\mathbf{T}^*\right) \times \mathrm{End}\left(\mathbf{T}\right)$, where the symplectic structure on Ψ given by the differential 2-form

$$d\psi = \tau^{-2}\left(dT \wedge d\varepsilon - \xi dT \wedge d\rho - dT \wedge \mathrm{Tr}\left(\sigma^*d\Delta\right)\right) - T^{-1}\left(d\xi \wedge d\rho - \mathrm{Tr}\left(d\sigma^*d\Delta\right)\right).$$

As it was shown in ([7],[6]) we should require, in addition, that the differential symmetric form κ shall define the Riemannian structure on L.

Also, similar to ([6]) we consider only such Legendrian manifolds $\widetilde{L}$ where (T, ρ, Δ) are coordinates. In this case we will write down form $T^{-1}\psi$ as

$$d(\varepsilon - Ts) - (s \, dT + \mathrm{Tr} \, (\sigma^* d\Delta) + \xi \, d\rho) .$$

Therefore, manifold $\widetilde{L}$ is Legendrian if and only if

$$s = h_T, \quad \sigma = h_\Delta, \quad \xi = h_\rho, \tag{4.3}$$

where h is the Gibbs free energy

$$h = \varepsilon - Ts.$$

In the case of Newton-Cosserat media we have in addition $\mathbf{SO}\,(3) \times \mathbf{SO}\,(3)$ symmetry and Gibbs free energy h is a function of $\mathbf{SO}\,(3) \times \mathbf{SO}\,(3)$-invariants (see [6], for more details).

Here we consider the Euler case ([6]), when

$$h\,(T, \rho, \Delta) = p_1\,(T, \rho)\,\mathrm{Tr}\,(\Delta) + p_2 \mathrm{Tr}\,(\Delta \Pi_V) ,$$

where Π_V is the projector on the vertical part of $T\Phi$.

In this case the stress tensor σ equals

$$\sigma = p_1\,(T, \rho) + p_2\,(T, \rho)\,\Pi_V ,$$

or

$$\sigma = (p_1\,(T, \rho) + p_2\,(T, \rho))\,\left(E_1 \otimes \Omega_1 + E_2 \otimes \Omega_2 + E_3 \otimes \Omega_3\right) + \tag{4.4}$$

$$p_2\,(T, \rho)\,(\partial_1 \otimes d_1 + \partial_2 \otimes d_2 + \partial_3 \otimes d_3) + p_2 \sum_{i,j=1}^{3} \omega_{ij} E_i \otimes d_j,$$

and energy density equals

$$\varepsilon = (p_1 - Tp_{1,T})\,\mathrm{Tr}\,(\Delta) + (p_2 - Tp_{2,T})\,\mathrm{Tr}\,(\Delta \Pi_V)$$

5. Euler equations for Cosserat media

The general form of the Navier-Stocks equations and Euler equations as well, for media with inner structures have the form ([6]):

(1) Moment conservation, or Navier-Stocks equation:

$$\rho \left(\frac{\partial U}{\partial t} + \nabla_U^c\,(U) \right) = \mathrm{div}^\flat \sigma, \tag{5.1}$$

where ∇^c is the Levi-Civita connection, $\mathrm{div}^\flat \sigma$ is a vector field dual to the differential form $\mathrm{div}\sigma \in \Omega^1\,(\Phi)$, with respect to the canonical metric g.

(2) Conservation of mass:

$$\frac{\partial \rho}{\partial t} + U\,(\rho) + \mathrm{div}\,(U)\,\rho = 0. \tag{5.2}$$

(3) Conservation of energy:

$$\frac{\partial \varepsilon}{\partial t} + \varepsilon \, \mathrm{div}\,(U) - \mathrm{div}\,(\zeta \, \mathrm{grad}\,(T)) + \mathrm{Tr}\,(\sigma' \, \mathcal{D}U) = 0,$$

where ζ is the thermal conductivity.

(4) State equations:
$$\sigma = h_\Delta, \quad \varepsilon = h - T\, h_T.$$

(5) In addition, we require that vector field U preserves the bundle $\pi\colon \Phi \to \mathbf{D}$, or that U is a π-projectable vector field.

In the case of Euler equations we have relation (4.4), and using the property (see [6])
$$\operatorname{div}(X \otimes \omega) = \operatorname{div}(X)\,\omega + \nabla_X(\omega),$$
where X is a vector field and ω is a differential 1-form, we get, due to (3.3),
$$\operatorname{div}(E_i \otimes \Omega_i) = \nabla_{E_i}(\Omega_i) = 0,$$
$$\operatorname{div}(\partial_i \otimes d_i) = \nabla_{\partial_i}(d_i) = 0,$$
and
$$\operatorname{div}\sigma = \sum_{i=1}^{3}\left(E_i\,(p_1 + p_2)\,\Omega_i + (\partial_i - \sum_{j=1}^{3}\omega_{ji}E_j)\,(p_2)\ d_i \right).$$

Therefore,
$$\operatorname{div}^\flat \sigma = \sum_{i=1}^{3}\left(\lambda_i^{-1}E_i\,(p_1 + p_2)\,E_i + (\partial_i - \sum_{j=1}^{3}\omega_{ji}E_j)\,(p_2)\,\partial_i \right).$$

Assume now that vector field U has the form
$$U = \sum_{i=1}^{3}\left(X_i\,(x)\,\partial_i + Y_i\,(x,y)\,E_i \right).$$

Then,
$$\nabla_{\partial_1}(U) = \sum_{i=1}^{3}\left(\partial_1\,(X_i)\,\partial_i + \partial_1\,(Y_i)\,E_i \right),$$

$$\nabla_{\partial_2}(U) = \sum_{i=1}^{3}\left(\partial_2\,(X_i)\,\partial_i + \partial_2\,(Y_i)\,E_i \right),$$

$$\nabla_{\partial_3}(U) = \sum_{i=1}^{3}\left(\partial_3\,(X_i)\,\partial_i + \partial_3\,(Y_i)\,E_i \right),$$

$$\nabla_{E_1}(U) = \sum_{i=1}^{3}\left(E_1\,(Y_i)\,E_i + \lambda_{1i}Y_i[E_1, E_i] \right) = \sum_{i=1}^{3} E_1\,(Y_i)\,E_i + \lambda_{12}Y_2 E_3 - \lambda_{13}Y_3 E_2,$$

$$\nabla_{E_2}(U) = \sum_{i=1}^{3}\left(E_2\,(Y_i)\,E_i + \lambda_{2i}Y_i[E_2, E_i] \right) = \sum_{i=1}^{3} E_2\,(Y_i)\,E_i - \lambda_{21}Y_1 E_3 + \lambda_{23}Y_3 E_1,$$

$$\nabla_{E_3}(U) = \sum_{i=1}^{3}\left(E_3\,(Y_i)\,E_i + \lambda_{3i}Y_i[E_3, E_i] \right) = \sum_{i=1}^{3} E_3\,(Y_i)\,E_i + \lambda_{31}Y_1 E_2 - \lambda_{32}Y_2 E_1,$$

where

$$\lambda_{12} = \frac{\lambda - \lambda_1}{\lambda_3}, \quad \lambda_{13} = \frac{\lambda - \lambda_1}{\lambda_2}, \quad \lambda_{21} = \frac{\lambda - \lambda_2}{\lambda_3},$$

$$\lambda_{23} = \frac{\lambda - \lambda_2}{\lambda_1}, \quad \lambda_{31} = \frac{\lambda - \lambda_3}{\lambda_2}, \quad \lambda_{32} = \frac{\lambda - \lambda_3}{\lambda_1}.$$

Therefore,

$$\nabla_U (U) = \sum_{j,i=1}^{3} X_j \partial_j (X_i) \partial_i + \sum_{j,i=1}^{3} X_j \partial_j (Y_i) E_i + \sum_{j,i=1}^{3} Y_j E_j (Y_i) E_i +$$
$$(\lambda_{23} - \lambda_{32}) Y_2 Y_3 E_1 + (\lambda_{31} - \lambda_{13}) Y_1 Y_3 E_2 + (\lambda_{12} - \lambda_{21}) Y_1 Y_2 E_3.$$

Summarizing, we get the following system of Euler equations:

$$\rho \left(\partial_t X_i + \sum_{j=1}^{3} X_j \partial_j (X_i) \right) = \left(\partial_i - \sum_{j=1}^{3} \omega_{ji} E_j \right) (p_2), \quad i = 1, 2, 3, \tag{5.3}$$

$$\rho \left(\partial_t Y_1 + \sum_{j=1}^{3} X_j \partial_j (Y_1) + \sum_{j=1}^{3} Y_j E_j (Y_1) \right) + (\lambda_{23} - \lambda_{32}) Y_2 Y_3 = \lambda_1^{-1} E_1 (p_1 + p_2),$$

$$\rho \left(\partial_t Y_2 + \sum_{j=1}^{3} X_j \partial_j (Y_2) + \sum_{j=1}^{3} Y_j E_j (Y_2) \right) + (\lambda_{31} - \lambda_{13}) Y_1 Y_3 = \lambda_2^{-1} E_2 (p_1 + p_2),$$

$$\rho \left(\partial_t Y_3 + \sum_{j=1}^{3} X_j \partial_j (Y_3) + \sum_{j=1}^{3} Y_j E_j (Y_3) \right) + (\lambda_{12} - \lambda_{21}) Y_1 Y_2 = \lambda_3^{-1} E_3 (p_1 + p_2).$$

Remark that the first three equations in (5.3) are very close to the classical Euler equations and the second three equations are the Euler type equations on the Lie group $\mathbf{SO}(3)$.

The mass conservation equation takes the form

$$\partial_t \rho + \sum_{i=1}^{3} (X_i \partial_i \rho + Y_i E_i \rho) + \sum_{i=1}^{3} (\partial_i X_i + E_i Y_i) \rho = 0. \tag{5.4}$$

Finally, the energy conservation equation takes the form

$$\frac{\partial \varepsilon}{\partial t} + \sum_{i=1}^{3} (\partial_i X_i + E_i Y_i) \ \varepsilon + \sum_{i=1}^{3} E_i (Y_i) = \operatorname{div}(\zeta \operatorname{grad}(T)), \tag{5.5}$$

because

$$\operatorname{Tr}(\sigma^* \mathcal{D} U) = \sum_{i=1}^{3} E_i (Y_i),$$

and here

$$\varepsilon = (p_2 - T \ p_{2,T}) (E_1 (Y_1) + E_2 (Y_2) + E_3 (Y_3)) + (p_1 - T \ p_{1,T}) \operatorname{div} U.$$

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