*Global and Stochastic Analysis*
*Vol. 7 No. 2 (July-December, 2020)*
MUK PUBLICATIONS
Open Access Publisher

# SYMMETRIES AND DIFFERENTIAL INVARIANTS FOR VISCID FLOWS ON A CURVE

ANNA DUYUNOVA, VALENTIN LYCHAGIN, AND SERGEY TYCHKOV

Abstract. In this paper, flows of a viscid fluids on curves are considered. Symmetry algebras and the corresponding fields of differential invariants are found. We study their dependence on thermodynamic states of media, and provide classification of thermodynamic states.

## 1. Introduction

Consider flow of an viscid medium on an oriented Riemannian manifold $(M, g)$ in the field of a constant gravitational field. Motion of viscous media are described by the PDE system consisting of the Navier-Stokes equation, the laws of mass and energy conservation (see [2], [5] for details):

$$\begin{cases} \rho(\mathbf{u}_t + \nabla_{\mathbf{u}}\mathbf{u}) - \operatorname{div}\sigma - \mathbf{g}\rho = 0, \\ \dfrac{\partial(\rho\,\Omega_g)}{\partial t} + \mathcal{L}_{\mathbf{u}}\,(\rho\,\Omega_g) = 0, \\ \rho T\,(s_t + \nabla_{\mathbf{u}} s) - \Phi + k(\Delta_g T) = 0, \end{cases} \tag{1.1}$$

where the vector field $\mathbf{u}$ is the flow velocity, $p$, $\rho$, $s$, $T$ are the pressure, density, specific entropy, temperature of the fluid respectively, $k$ is the thermal conductivity, which is supposed to be constant, and $\mathbf{g}$ is the gravitational acceleration. The stress tensor $\sigma$ depends on two viscousites, which are alse considered constant.

In this paper, we consider the case, when $M$ is a naturally-parameterized curve in the three-dimensional Euclidean space

$$M = \{x = f(a),\, y = g(a),\, z = h(a)\}$$

In this case, vector $\mathbf{g}$ is the restriction of the vector field $(0, 0, \mathrm{g})$ on $M$, i.e.,

$$\mathbf{g} = \mathrm{g}h'\partial_a.$$

First of all, we should note that two additional relations involving thermodynamic quantites are needed to complete the system (1.1). To obtain them, we apply the method described in the paper [3] in detail. The general idea behind this method is representation of thermodynamic states with Legendrian, or Lagrangian, manifolds in a contact, or symplectic, space correspondingly.

---

*Date*: Date of Submission June 02, 2020; Date of Acceptance June 30, 2020 , Communicated by Yuri E. Gliklikh.

2010 *Mathematics Subject Classification.* Primary 35B06; Secondary 76N99.

*Key words and phrases.* Symmetry algebra, differential invariants, Navier-Stokes equation.

The research was partially supported by RFBR Grant No 18-29-10013.

So, by the Navier-Stokes system $\mathcal{E}$ we mean the equations (1.1) together with two equations of the thermodynamic state

$$L = \{\, F(p,\rho,s,T) = 0,\, G(p,\rho,s,T) = 0 \,\} \tag{1.2}$$

that meet the condition

$$[F,G] = 0 \mod \{F=0,\, G=0\},$$

where $[F,G]$ is the Poisson bracket with respect to the symplectic form

$$\Omega = ds \wedge dT + \rho^{-2} d\rho \wedge dp.$$

Moreover, the restriction of the quadratic differential form

$$\kappa = d(T^{-1}) \cdot d\epsilon - \rho^{-2} d(pT^{-1}) \cdot d\rho$$

to the manifold of thermodynamic state is negative definite, here $\epsilon$ is the specific internal energy.

The paper is organized as follows.

In Section 2 we study symmetry Lie algebras of the Navier-Stokes system $\mathcal{E}$ and their dependence on the form of the function $h(a)$. There are six different forms, besides the general case, of the function $h$ that correspond to different symmetry algebras.

In Section 3 we consider the case when the thermodynamic state admits a one-dimensional symmetry algebra and find the corresponding Lie algebras. For such thermodynamic states, we find an explicit form of Lagrangian surface in terms of two equations on the thermodynamic quantities $p$, $T$, $\rho$ and $s$.

In Section 4 we recall the notion of differential invariants and introduce Navier-Stokes and kinematic invariants. For these types we find the field of differential invariants.

A space curve can be represented as a lift of a plane curve. Connection between the function $h$ and a way of lifting curve was discussed in [4].

Most of the computations in this paper were done in Maple with the Differential Geometry package by I. Anderson and his team [1].

## 2. Symmetry Lie algebra

Using the standard techniques for calculating of symmetries we find dependence of symmetry algebra of system $\mathcal{E}$ on the function $h(a)$.

To this end, we consider a Lie algebra $\mathfrak{g}$ of point symmetries of the system (1.1).

Let $\vartheta\colon \mathfrak{g} \to \mathfrak{h}$ be the following Lie algebras homomorphism

$$\vartheta\colon X \mapsto X(\rho)\partial_\rho + X(s)\partial_s + X(p)\partial_p + X(T)\partial_T,$$

where $\mathfrak{h}$ is a Lie algebra generated by vector fields that act on the thermodynamic variables $p$, $\rho$, $s$ and $T$.

The kernel of the homomorphism $\vartheta$ is an ideal $\mathfrak{g}_\mathfrak{m} \subset \mathfrak{g}$, and we call the elements of $\mathfrak{g}_\mathfrak{m}$ *geometric symmetries*.

Let $\mathfrak{h}_\mathfrak{t}$ be a such Lie subalgebra of the algebra $\mathfrak{h}$ that preserves thermodynamic state (1.2).

Then the following theorem is true (see for details [3]).

**Theorem 2.1.** *A Lie algebra $\mathfrak{g}_{\mathfrak{sym}}$ of symmetries of the Navier-Stokes system $\mathcal{E}$ coincides with*

$$\vartheta^{-1}(\mathfrak{h}_{\mathfrak{t}}).$$

First of all, consider the general case, when $h(a)$ is an arbitrary function. Then the Lie algebra $\mathfrak{g}^{\circ}$ of point symmetries of the system (1.1) is generated by the vector fields

$$X_1 = \partial_t, \qquad X_2 = \partial_p, \qquad X_3 = \partial_s.$$

The pure thermodynamic part $\mathfrak{h}^{\circ}$ of the system symmetry algebra in this case is generated by

$$Y_1 = \partial_p, \qquad Y_2 = \partial_s.$$

The PDE system $\mathcal{E}$ has the smallest Lie algebra of point symmetries $\vartheta^{-1}(\mathfrak{h}_{\mathfrak{t}}^{\circ})$, when the function $h(a)$ is arbitrary.

The special cases of the function $h(a)$ are listed below.

**1.** $h(a) = const$

The symmetry Lie algebra $\mathfrak{g}^{1}$ of sthe system (1.1) is generated by $X_1, X_2, X_3$ and by the following vector fields

$$\begin{aligned} X_4 &= \partial_a, & X_6 &= t\,\partial_t + a\,\partial_a - p\,\partial_p - \rho\,\partial_\rho, \\ X_5 &= t\,\partial_a + \partial_u, & X_7 &= a\,\partial_a + u\,\partial_u - 2\rho\,\partial_\rho + 2T\,\partial_T. \end{aligned}$$

The Lie algebra $\mathfrak{g}^{1}$ is solvable and the sequence of derived algebras is the following

$$\mathfrak{g}^{1} = \langle X_1, X_2, \dots, X_7\rangle \supset \langle X_1, X_2, X_3, X_4, X_5\rangle \supset \langle X_4\rangle\,.$$

The pure thermodynamic part $\mathfrak{h}^{1}$ of the symmetry algebra is generated by the vector fields

$$Y_1 = \partial_p, \qquad Y_2 = \partial_s, \qquad Y_3 = p\,\partial_p + \rho\,\partial_\rho, \qquad Y_4 = \rho\,\partial_\rho - T\,\partial_T.$$

Hence, the PDE system $\mathcal{E}$ admits a Lie algebra of point symmetries $\vartheta^{-1}(\mathfrak{h}_{\mathfrak{t}}^{1})$.

**2.** $h(a) = \lambda a$, $\lambda \neq 0$

In this case the Lie algebra $\mathfrak{g}^{2}$ of point symmetries of the system (1.1) is generated by $X_1, X_2, X_3$ and by the following vector fields

$$\begin{aligned} X_4 &= \partial_a, & X_6 &= t\,\partial_t + 2a\,\partial_a + u\,\partial_u - p\,\partial_p - 3\rho\,\partial_\rho + 2T\partial_T, \\ X_5 &= t\,\partial_a + \partial_u, & X_7 &= t\,\partial_t + (\frac{\lambda g t^2}{2} + a)\,\partial_a + \lambda g t\,\partial_u - p\,\partial_p - \rho\,\partial_\rho. \end{aligned}$$

The Lie algebra $\mathfrak{g}^{2}$ is solvable and its sequence of derived algebras is

$$\mathfrak{g}^{2} = \langle X_1, X_2, \dots, X_7\rangle \supset \langle X_1, X_2, X_3, X_4, X_5\rangle \supset \langle X_4\rangle\,.$$

The pure thermodynamic part $\mathfrak{h}^{2}$ of the symmetry algebra is generated by the vector fields

$$Y_1 = \partial_p, \qquad Y_2 = \partial_s, \qquad Y_3 = p\,\partial_p + \rho\,\partial_\rho, \qquad Y_4 = \rho\,\partial_\rho - T\,\partial_T.$$

Hence, the PDE system $\mathcal{E}$ admits a Lie algebra of point symmetries $\vartheta^{-1}(\mathfrak{h}_{\mathfrak{t}}^{2})$.

**3.** $h(a) = \lambda a^2$, $\lambda \neq 0$

In this case the Lie algebra $\mathfrak{g}^3$ of point symmetries of the system (1.1) is generated by the vector fields $X_1, X_2, X_3$ and, if $\lambda < 0$, by the vector fields

$$X_4 = \sin(\sqrt{2\lambda \mathrm{g}}\, t)\, \partial_a + \sqrt{2\lambda \mathrm{g}} \cos(\sqrt{2\lambda \mathrm{g}}\, t)\, \partial_u,$$
$$X_5 = \cos(\sqrt{2\lambda \mathrm{g}}\, t)\, \partial_a - \sqrt{2\lambda \mathrm{g}} \sin(\sqrt{2\lambda \mathrm{g}}\, t)\, \partial_u,$$
$$X_6 = a\,\partial_a + u\,\partial_u - 2\rho\,\partial_\rho + 2T\partial_T$$

and, if $\lambda > 0$, by the vector fields

$$X_4 = \exp(\sqrt{-2\lambda \mathrm{g}}\, t)\, \partial_a + \sqrt{-2\lambda \mathrm{g}} \exp(\sqrt{-2\lambda \mathrm{g}}\, t)\, \partial_u,$$
$$X_5 = \exp(-\sqrt{-2\lambda \mathrm{g}}\, t)\, \partial_a - \sqrt{-2\lambda \mathrm{g}} \exp(-\sqrt{-2\lambda \mathrm{g}}\, t)\, \partial_u,$$
$$X_6 = a\,\partial_a + u\,\partial_u - 2\rho\,\partial_\rho + 2T\partial_T.$$

The Lie algebra $\mathfrak{g}^3$ is solvable and its sequence of derived algebras is

$$\mathfrak{g}^3 = \langle X_1, X_2, \ldots, X_6 \rangle \supset \langle X_2, X_3, X_4, X_5 \rangle .$$

The pure thermodynamic part $\mathfrak{h}^3$ of the symmetry algebra is generated by the vector fields

$$Y_1 = \partial_p, \qquad Y_2 = \partial_s, \qquad Y_3 = \rho\,\partial_\rho - T\,\partial_T.$$

Hence, the PDE system $\mathcal{E}$ admits a Lie algebra of point symmetries $\vartheta^{-1}(\mathfrak{h}^3_{\mathfrak{t}})$.

**4.** $h(a) = \lambda_1 a^{\lambda_2}$, $\lambda_2 \neq 0, 1, 2$

The Lie algebra $\mathfrak{g}^4$ of point symmetries of the system (1.1) is generated by the vector fields $X_1, X_2, X_3$ and by the vector field

$$X_4 = t\,\partial_t - \frac{2a}{\lambda_2 - 2}\partial_a - \frac{\lambda_2 u}{\lambda_2 - 2}\partial_u - p\,\partial_p + \frac{\lambda_2 + 2}{\lambda_2 - 2}\rho\,\partial_\rho - \frac{2\lambda_2}{\lambda_2 - 2}T\,\partial_T.$$

The Lie algebra $\mathfrak{g}^4$ is solvable and the sequence of derived algebras is the following

$$\mathfrak{g}^4 = \langle X_1, X_2, X_3, X_4 \rangle \supset \langle X_1, X_2, X_3 \rangle .$$

The pure thermodynamic part $\mathfrak{h}^4$ of the symmetry algebra is generated by the vector fields

$$Y_1 = \partial_p, \qquad Y_2 = \partial_s, \qquad Y_3 = p\,\partial_p - (\lambda_2 + 2)\rho\,\partial_\rho + 2\lambda_2 T\,\partial_T.$$

Hence, the PDE system $\mathcal{E}$ admits a Lie algebra of point symmetries $\vartheta^{-1}(\mathfrak{h}^4_{\mathfrak{t}})$.

**5.** $h(a) = \lambda_1 e^{\lambda_2 a}$, $\lambda_2 \neq 0$

In this case, the symmetry Lie algebra $\mathfrak{g}^5$ of the system (1.1) is generated by the vector fields $X_1, X_2, X_3$ and by the vector field

$$X_4 = t\,\partial_t - \frac{2}{\lambda_2}\,\partial_a - u\,\partial_u - p\,\partial_p + \rho\,\partial_\rho - 2T\partial_T.$$

The Lie algebra $\mathfrak{g}^5$ is solvable and the derived algebras are the following

$$\mathfrak{g}^5 = \langle X_1, X_2, X_3, X_4 \rangle \supset \langle X_1, X_2, X_3 \rangle .$$

The pure thermodynamic part $\mathfrak{h}^5$ of the symmetry algebra is generated by the vector fields

$$Y_1 = \partial_p, \qquad Y_2 = \partial_s, \qquad Y_3 = p\,\partial_p - \rho\,\partial_\rho + 2T\,\partial_T.$$

Hence, the PDE system $\mathcal{E}$ admits a Lie algebra of point symmetries $\vartheta^{-1}(\mathfrak{h}^5_{\mathfrak{t}})$.

**6.** $h(a) = \ln a$

The Lie algebra $\mathfrak{g}^6$ of point symmetries of the system (1.1) is generated by the vector fields $X_1, X_2, X_3$ and by the vector field

$$X_4 = t\,\partial_t + a\,\partial_a - p\,\partial_p - \rho\,\partial_\rho.$$

The Lie algebra $\mathfrak{g}^6$ is solvable and the sequence of derived algebras is the following

$$\mathfrak{g}^6 = \langle X_1, X_2, X_3, X_4 \rangle \supset \langle X_1, X_2 \rangle\,.$$

The pure thermodynamic part $\mathfrak{h}^6$ of the symmetry algebra is generated by the vector fields

$$Y_1 = \partial_p, \qquad Y_2 = \partial_s, \qquad Y_3 = p\,\partial_p + \rho\,\partial_\rho.$$

Hence, the PDE system $\mathcal{E}$ admits a Lie algebra of point symmetries $\vartheta^{-1}(\mathfrak{h}^6_{\mathfrak{t}})$.

## 3. Thermodynamic states with a one-dimensional symmetry algebra

In this section we consider the thermodynamic states, or the Lagrangian surfaces $L$, admitting a one-dimensional symmetry algebra. The cases, when thermodynamic states admit a two-dimensional symmetry algebra, can be studied in the similar way.

Let the thermodynamic state admit a one-dimensional symmetry algebra. Denote by

$$Z = \gamma_1 Y_1 + \gamma_2 Y_2 + \ldots + \gamma_k Y_k$$

a basis vector of this algebra, then the Lagrangian surface can be found from the solution of PDE (see [3] for details)

$$\begin{cases} \Omega|_L = 0, \\ (\iota_Z \Omega)|_L = 0. \end{cases}$$

This system in terms of specific energy can be written as

$$\epsilon = \epsilon(\rho, s), \quad T = \epsilon_s, \quad p = \rho^2 \epsilon_\rho.$$

Solving it, we find thermodynamic state $L$, which must also satisfy $\kappa|_L < 0$.

Straightforward computations show that, for an arbitrary function $h(a)$, there are no thermodynamic states that admit a one-dimensional symmetry algebra.

**1, 2.** $h(a) = const$, $h(a) = \lambda a$

The pure thermodynamic part of the system symmetry algebra coincides with the thermodynamic part of the 2d Navier-Stokes case. Thus, the classification of the thermodynamic states for these two cases can be found in [3]. **3.** $h(a) = \lambda a^2$, $\lambda \neq 0$

Let a basis vector of a one-dimensional symmetry algebra be

$$\gamma_1 \partial_p + \gamma_2 \partial_s + \gamma_3 (\rho\,\partial_\rho - T\,\partial_T),$$

then in the general case expressions for the pressure and temperature have the form

$$p = \frac{\gamma_2}{\gamma_3} F' - F - \frac{\gamma_1}{\gamma_3}(\ln\rho - 1), \quad T = \frac{F'}{\rho}, \quad F = F\left(s + \frac{\gamma_2}{\gamma_3}\ln\rho\right),$$

where $F$ is an arbitrary function. The condition of negative definiteness of the differential form $\kappa$ leads to the relations

$$F' > 0, \quad F'' > 0, \quad \frac{(\gamma_2 F' - \gamma_1)F''}{\gamma_3} - F'^2 > 0.$$

**4.** $h(a) = \lambda_1 a^{\lambda_2}$, $\lambda_2 \neq 0, 1, 2$

Let a basis vector of a one-dimensional symmetry algebra be

$$\gamma_1 \partial_p + \gamma_2 \partial_s + \gamma_3 (p\,\partial_p - (\lambda_2 + 2)\rho\,\partial_\rho + 2\lambda_2 T\,\partial_T),$$

then in the general case expressions for the pressure and temperature have the form

$$p = \frac{\rho^{\frac{2-\lambda_2}{\lambda_2+2}}(\gamma_2 F' - 2\lambda_2\gamma_3 F)}{\gamma_3(\lambda_2 + 2)} - \frac{\gamma_1}{\gamma_3(\lambda_2 - 2)}, \quad T = \rho^{\frac{-2\lambda_2}{\lambda_2+2}} F',$$
$$F = F\left(s + \frac{\gamma_2}{\gamma_3(\lambda_2 + 2)} \ln\rho\right),$$

where $F$ is an arbitrary function. The negative definiteness of the differential form $\kappa$ leads to the relations

$$F' > 0, \quad F'' > 0, \quad 2\lambda_2(\lambda_2 - 2)FF'' - 4\lambda_2^2 F'^2 + \frac{\gamma_2(\lambda_2 + 2)F'F''}{\gamma_3} > 0.$$

**5.** $h(a) = \lambda_1 e^{\lambda_2}$

The pure thermodynamic part of the system symmetry algebra coincides with the symmetry Lie algebra of the Navier-Stokes system of differential equations on a two dimensional unit sphere. So, the classification of thermodynamic states can be found in [3].

**6.** $h(a) = \ln a$

Let a basis vector of a one-dimensional symmetry algebra be

$$\gamma_1 \partial_p + \gamma_2 \partial_s + \gamma_3 (p\,\partial_p + \rho\,\partial_\rho),$$

then in the general case expressions for the pressure and temperature have the form

$$p = \frac{-(\gamma_2 F' + C)\rho}{\gamma_3} - \frac{\gamma_1}{\gamma_3}, \quad T = F', \quad F = F\left(s - \frac{\gamma_2}{\gamma_3}\ln\rho\right).$$

The negative definiteness of the differential form $\kappa$ leads to the relations

$$F' > 0, \quad F'' > 0, \quad \frac{\gamma_2 F' + C}{\gamma_3} < 0$$

when $s \in (-\infty, s_0]$.

## 4. Differential invariants

As before in [3], we consider two group actions on the Navier-Stokes system $\mathcal{E}$. Specifically, the prolonged actions of the Lie algebras $\mathfrak{g}_{\mathfrak{m}}$ and $\mathfrak{g}_{\mathfrak{s}\mathfrak{y}\mathfrak{m}}$.

Recall that a function $J$ on the manifold $\mathcal{E}_k$ is a *kinematic differential invariant of order* $\leq k$ if

(1) $J$ is a rational function along fibers of the projection $\pi_{k,0} \colon \mathcal{E}_k \to \mathcal{E}_0$,

(2) $J$ is invariant with respect to the prolonged action of the Lie algebra $\mathfrak{g}_{\mathfrak{m}}$, i.e., for all $X \in \mathfrak{g}_{\mathfrak{m}}$,

$$X^{(k)}(J) = 0, \tag{4.1}$$

where $\mathcal{E}_k$ is the prolongation of the system $\mathcal{E}$ to $k$-jets, and $X^{(k)}$ is the $k$-th prolongation of a vector field $X \in \mathfrak{g}_{\mathfrak{m}}$.

Note that fibers of the projection $\mathcal{E}_k \to \mathcal{E}_0$ are irreducible algebraic manifolds.

A kinematic invariant is *an Navier-Stokes invariant* if condition (4.1) holds for all $X \in \mathfrak{g}_{\mathfrak{sym}}$.

We say that a point $x_k \in \mathcal{E}_k$ and the corresponding orbit $\mathcal{O}(x_k)$ ($\mathfrak{g}_{\mathfrak{m}}$- or $\mathfrak{g}_{\mathfrak{sym}}$-orbit) are *regular*, if there are exactly $m = \operatorname{codim} \mathcal{O}(x_k)$ independent invariants (kinematic or Navier-Stokes) in a neighborhood of this orbit. Otherwise, the point and the corresponding orbit are *singular*.

The Navier-Stokes system together with the symmetry algebras $\mathfrak{g}_{\mathfrak{m}}$ or $\mathfrak{g}_{\mathfrak{sym}}$ satisfies the conditions of Lie-Tresse theorem (see [6]), and, therefore, the kinematic and Navier-Stokes differential invariants separate regular $\mathfrak{g}_{\mathfrak{m}}$ and $\mathfrak{g}_{\mathfrak{sym}}$ orbits on the Navier-Stokes system $\mathcal{E}$ correspondingly.

By a $\mathfrak{g}_{\mathfrak{m}}$ or $\mathfrak{g}_{\mathfrak{sym}}$-invariant differentiation we mean a total differentiation

$$A\frac{\mathrm{d}}{\mathrm{d}t} + B\frac{\mathrm{d}}{\mathrm{d}a}$$

that commutes with prolonged action of algebra $\mathfrak{g}_{\mathfrak{m}}$ or $\mathfrak{g}_{\mathfrak{sym}}$. Here $A$, $B$ are rational functions on the prolonged system $\mathcal{E}_k$ for some $k \geq 0$.

### 4.1. Kinematic invariants.

**Theorem 4.1.** (1) *The field of kinematic invariants is generated by first-order basis differential invariants and by basis invariant differentiations. This field separates regular orbits.*

(2) *For the general cases of $h(a)$, as well as for $h(a) = \lambda_1 a^{\lambda_2}$, $h(a) = \lambda_1 e^{\lambda_2 a}$ and $h(a) = \ln a$, the basis differential invariants are*

$$a, \quad u, \quad \rho, \quad s, \quad u_t, \quad u_a, \quad \rho_a, \quad s_t, \quad s_a,$$

*and the basis invariant differentiations are*

$$\frac{\mathrm{d}}{\mathrm{d}t}, \quad \frac{\mathrm{d}}{\mathrm{d}a}.$$

(3) *For the cases $h(a) = const$, $h(a) = \lambda a$ the basis differential invariants are*

$$\rho, \quad s, \quad u_a, \quad u_t + uu_a, \quad \rho_a, \quad s_a, \quad s_t + us_a,$$

*and basis invariant differentiations are*

$$\frac{\mathrm{d}}{\mathrm{d}t} + u\frac{\mathrm{d}}{\mathrm{d}a}, \quad \frac{\mathrm{d}}{\mathrm{d}a}.$$

(4) *For the case $h(a) = \lambda a^2$ the basis differential invariants are*

$$\rho, \quad s, \quad u_a, \quad u_t + uu_a - 2\lambda g a, \quad \rho_a, \quad s_a, \quad s_t + us_a,$$

*and basis invariant differentiations are*

$$\frac{\mathrm{d}}{\mathrm{d}t} + u\frac{\mathrm{d}}{\mathrm{d}a}, \quad \frac{\mathrm{d}}{\mathrm{d}a}.$$

(5) *The number of independent invariants of pure order $k$ equals* 5 *for* $k \geq 1$.

### 4.2. Navier-Stokes invariants.

In this subsection we study the thermodynamic states that admit a one-dimensional symmetry algebra generated by the vector field $A$.

Considering the action of the thermodynamic vector field $A$ on the field of kinematic invariants and finding first integrals of this action we get basis Navier-Stokes differential invariants of the first order.

Below we list basis Navier-Stokes invariants for the different form of function $h(a)$.

**1.** $h(a) = const$

When the thermodynamic state admits a one-dimensional symmetry algebra generated by the vector field

$$\xi_1 X_2 + \xi_2 X_3 + \xi_3 X_6 + \xi_4 X_7 = \xi_1 \partial_p + \xi_2 \partial_s + \xi_3(t\,\partial_t + a\,\partial_a - p\,\partial_p - \rho\,\partial_\rho) + \xi_4(a\,\partial_a + u\,\partial_u - 2\rho\,\partial_\rho + 2T\,\partial_T),$$

then the field of Navier-Stokes invariants is generated by the first order differential invariants

$$s + \frac{\xi_2}{\xi_3 + 2\xi_4}\ln\rho, \quad u_a \rho^{-\frac{\xi_3}{\xi_3+2\xi_4}}, \quad \rho_a \rho^{\frac{\xi_4}{\xi_3+2\xi_4}-2}, \quad \frac{\rho^2(u_t + uu_a)}{\rho_a u_a}, \quad \frac{\rho s_a}{\rho_a}, \quad \frac{s_t + us_a}{u_a}$$

and by the invariant differentiations

$$\rho^{-\frac{\xi_3}{\xi_3+2\xi_4}}\left(\frac{\mathrm{d}}{\mathrm{d}t} + u\frac{\mathrm{d}}{\mathrm{d}a}\right), \quad \rho^{-\frac{\xi_3+\xi_4}{\xi_3+2\xi_4}}\frac{\mathrm{d}}{\mathrm{d}a}.$$

**2.** $h(a) = \lambda a$, $\lambda \neq 0$

When the thermodynamic state admits a one-dimensional symmetry algebra generated by the vector field

$$\xi_1 X_2 + \xi_2 X_3 + \xi_3 X_6 + \xi_4 X_7 = \xi_1 \partial_p + \xi_2 \partial_s + \xi_3(t\,\partial_t + 2a\,\partial_a + u\,\partial_u - p\,\partial_p - 3\rho\,\partial_\rho + 2T\partial_T) + \xi_4\left(t\,\partial_t + \left(\frac{\lambda g t^2}{2} + a\right)\partial_a + \lambda g t\,\partial_u - p\,\partial_p - \rho\,\partial_\rho\right),$$

then the field of Navier-Stokes differential invariants is generated by the differential invariants

$$s + \frac{\xi_2 \ln\rho}{3\xi_3 + \xi_4}, \quad u_a \rho^{\frac{-\xi_3-\xi_4}{3\xi_3+\xi_4}}, \quad \rho_a \rho^{\frac{\xi_3}{3\xi_3+\xi_4}-2}, \quad \frac{\rho^2(u_t + uu_a - \lambda g)}{\rho_a u_a}, \quad \frac{\rho s_a}{\rho_a}, \quad \frac{s_t + us_a}{u_a}$$

of the first order and by the invariant differentiations

$$\rho^{-\frac{\xi_3+\xi_4}{3\xi_3+\xi_4}}\left(\frac{\mathrm{d}}{\mathrm{d}t} + u\frac{\mathrm{d}}{\mathrm{d}a}\right), \quad \rho^{-\frac{2\xi_3+\xi_4}{3\xi_3+\xi_4}}\frac{\mathrm{d}}{\mathrm{d}a}.$$

**3.** $h(a) = \lambda a^2$, $\lambda \neq 0$

If the thermodynamic state admits a one-dimensional symmetry algebra generated by the vector field

$$\xi_1 X_2 + \xi_2 X_3 + \xi_3 X_6 = \xi_1 \partial_p + \xi_2 \partial_s + \xi_3(a\,\partial_a + u\,\partial_u - 2\rho\,\partial_\rho + 2T\partial_T),$$

then the field of Navier-Stokes differential invariants is generated by the first order differential invariants

$$s+\frac{\xi_2}{2\xi_3}\ln\rho,\quad u_a,\quad \rho(u_t+uu_a-2\lambda ga)^2,\quad \frac{\rho_a^2}{\rho^3},\quad \frac{s_a^2}{\rho},\quad s_t+us_a$$

and by the invariant differentiations

$$\frac{\mathrm{d}}{\mathrm{d}t}+u\frac{\mathrm{d}}{\mathrm{d}a},\quad \rho^{-\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}a}.$$

**4.** $h(a)=\lambda_1 a^{\lambda_2}$, $\lambda\neq 0,1,2$

If the thermodynamic state admits a one-dimensional symmetry algebra generated by the vector field

$$\xi_1X_2+\xi_2X_3+\xi_3X_4=\xi_1\partial_p+\xi_2\partial_s+\\ \xi_3\left(t\,\partial_t-\frac{2a}{\lambda_2-2}\partial_a-\frac{\lambda_2 u}{\lambda_2-2}\partial_u-p\,\partial_p+\frac{\lambda_2+2}{\lambda_2-2}\rho\,\partial_\rho-\frac{2\lambda_2}{\lambda_2-2}T\,\partial_T\right),$$

then the field of Navier-Stokes differential invariants is generated by the first order differential invariants

$$s+\frac{\xi_2(\lambda_2-2)}{2\xi_3}\ln a,\quad a^{-\lambda_2}u^2,\quad au\rho,\quad \frac{au_t}{u^2},\quad \frac{au_a}{u},\quad a^2u\rho_a,\quad \frac{as_t}{u},\quad as_a$$

and by the invariant differentiations

$$\rho^{\frac{\lambda_2-2}{\lambda_2+2}}\frac{\mathrm{d}}{\mathrm{d}t},\quad \rho^{\frac{-2}{\lambda_2+2}}\frac{\mathrm{d}}{\mathrm{d}a}.$$

**5.** $h(a)=\lambda_1 e^{\lambda_2 a}$

If the thermodynamic state admits a one-dimensional symmetry algebra generated by the vector field

$$\xi_1X_2+\xi_2X_3+\xi_3X_4=\xi_1\partial_p+\xi_2\partial_s+\xi_3\left(t\,\partial_t-\frac{2}{\lambda_2}\,\partial_a-u\,\partial_u-p\,\partial_p+\rho\,\partial_\rho-2T\partial_T\right),$$

then the field of Navier-Stokes differential invariants is generated by the first order differential invariants

$$s+\frac{\lambda_2\xi_2}{2\xi_3}a,\quad e^{-\lambda_2 a}u^2,\quad u\rho,\quad \frac{u_t}{u^2},\quad \frac{u_a}{u},\quad u\rho_a,\quad \frac{s_t}{u},\quad s_a$$

and by the differentiations

$$\rho\frac{\mathrm{d}}{\mathrm{d}t},\quad \frac{\mathrm{d}}{\mathrm{d}a}.$$

**6.** $h(a)=\ln a$

If the thermodynamic state admits a one-dimensional symmetry algebra generated by the vector field

$$\xi_1X_2+\xi_2X_3+\xi_3X_4=\xi_1\partial_p+\xi_2\partial_s+\xi_3\left(t\,\partial_t+a\,\partial_a-p\,\partial_p-\rho\,\partial_\rho\right),$$

then the field of Navier-Stokes differential invariants is generated by the first order differential invariants

$$s-\frac{\xi_2}{\xi_3}\ln a,\quad u,\quad a\rho,\quad au_t,\quad au_a,\quad a^2\rho_a,\quad as_t,\quad as_a$$

and by the invariant differentiations

$$\rho^{-1}\frac{\mathrm{d}}{\mathrm{d}t}, \quad \rho^{-1}\frac{\mathrm{d}}{\mathrm{d}a}.$$

## Appendix

The following table summarizes relations between the function $h$ and the symmetry algebra of the system (1.1), see Section 2 for details.

| | |
|---|---|
| $h(a)$ is arbitrary | $X_1 = \partial_t$,<br>$X_2 = \partial_p$,<br>$X_3 = \partial_s$ |
| $h(a) = const$ | $X_4 = \partial_a$,<br>$X_5 = t\,\partial_a + \partial_u$,<br>$X_6 = t\,\partial_t + a\,\partial_a - p\,\partial_p - \rho\,\partial_\rho$,<br>$X_7 = a\,\partial_a + u\,\partial_u - 2\rho\,\partial_\rho + 2T\,\partial_T$ |
| $h(a) = \lambda a$, $\lambda \neq 0$ | $X_4 = \partial_a$,<br>$X_5 = t\,\partial_a + \partial_u$,<br>$X_6 = t\,\partial_t + 2a\,\partial_a + u\,\partial_u - p\,\partial_p - 3\rho\,\partial_\rho + 2T\,\partial_T$,<br>$X_7 = t\,\partial_t + (\frac{\lambda g t^2}{2} + a)\,\partial_a + \lambda g t\,\partial_u - p\,\partial_p - \rho\,\partial_\rho$ |
| $h(a) = \lambda a^2$, $\lambda \neq 0$ | $X_4 = \exp(\sqrt{2\lambda g}\,t)\,\partial_a + \sqrt{2\lambda g}\exp(\sqrt{2\lambda g}\,t)\,\partial_u$,<br>$X_5 = \exp(-\sqrt{2\lambda g}\,t)\,\partial_a - \sqrt{2\lambda g}\exp(-\sqrt{2\lambda g}\,t)\,\partial_u$<br>$X_6 = a\partial_a + u\,\partial_u - 2\rho\,\partial_\rho + 2T\,\partial_T$, |
| $h(a) = \lambda_1 a^{\lambda_2}$,<br>$\lambda_2 \neq 0, 1, 2$ | $X_4 = (\lambda_2 - 2)t\,\partial_t - 2a\partial_a - \lambda_2 u\partial_u -$<br>$p\,\partial_p + (\lambda_2 + 2)\rho\,\partial_\rho - 2\lambda_2 T\partial_T$ |
| $h(a) = \lambda_1 e^{\lambda_2 a}$,<br>$\lambda_2 \neq 0$ | $X_4 = t\,\partial_t - \frac{2}{\lambda_2}\,\partial_a - u\,\partial_u - p\,\partial_p + \rho\,\partial_\rho - 2T\,\partial_T$ |
| $h(a) = \ln a$ | $X_4 = t\,\partial_t + a\,\partial_a - p\,\partial_p - \rho\,\partial_\rho$ |

## References

[1] Anderson, Ian M. and Torre, Charles G.: *The Differential Geometry Package* (2016). Downloads. Paper 4. http://digitalcommons.usu.edu/dg_downloads/4

[2] Batchelor G. K. *An introduction to fluid dynamics.* Cambridge university press, Cambridge, 2000.

[3] Duyunova A., Lychagin V. and Tychkov S. *Differential invariants for flows of fluids and gases.* ArXiv:2004.01567 [math-ph].

[4] Duyunova A., Lychagin V. and Tychkov S. Symmetries and differential invariants for inviscid flows on a curve. *Lobachevskii Journal of Mathematics*, **41** (12) (2020)

[5] Duyunova A., Lychagin V. and Tychkov S.: *Continuum mechanics of media with inner structures.* ArXiv:2005.05840 [math-ph].

[6] Kruglikov B. and Lychagin V.: Global Lie-Tresse theorem. *Selecta Math.* **22** (2016) 1357-1411.

Anna Duyunova: V. A. Trapeznikov Institute of Control Sciences of Russian Academy of Sciences, Moscow, 117997, Russia.
*E-mail address*: duyunova_anna@mail.ru

Valentin Lychagin: V. A. Trapeznikov Institute of Control Sciences of Russian Academy of Sciences, Moscow, 117997, Russia, University of Tromsø, Norway.
*E-mail address*: valentin.lychagin@uit.no

Sergey Tychkov: V. A. Trapeznikov Institute of Control Sciences of Russian Academy of Sciences, Moscow, 117997, Russia.
*E-mail address*: sergey.lab06@gmail.com