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LIGHTNING BIRTH MODEL: EXACT SOLUTION

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Abstract. This note is devoted to the study of differential equations describing the process preceding the formation of lightning in a charged gas. It is based on the ideas of V. I. Pustovojt, according to which weakly ionized gas moves in the Earth's magnetic field under the influence of pressure and temperature. The result is a system of differential equations, which in the case of a one-dimensional space is reduced to a nonlinear partial differential equation with respect to the normalized electric field strength. To construct its exact solutions, we use the symmetry method. The Lie algebra of point symmetries allowed by this equation turned out to be infinite-dimensional. This made it possible to construct classes of exact solutions of the equation, depending on arbitrary functions and arbitrary constants. An analysis of these solutions leads us to the conclusion that the singularity points of the electric field are discrete. It is in the vicinity of these points that electrical discharges can occur.

1. Introduction

The physical aspects of the occurrence of lightning and other electrical atmospheric phenomena are described by Ya. I. Frenkel [1]. However, until recently, it was unclear what caused the increase in the electromagnetic field in the cloud, leading to lightning. A description of the mechanism of the occurrence of the electric potential difference, based on the hydrodynamic model of a charged gas, was proposed by V. I. Pustovojt [5].

A weakly ionized cloud (gaseous medium) moves under the influence of a pressure difference and a temperature gradient at a certain speed. In this case, charged particles inside the cloud collide with neutral scattering centers. In addition to hydrodynamic forces, electric forces act on particles from other particles and from the external field of the Earth.

In this article, we examine the model proposed by Pustovojt. We have found classes of exact solutions of model equations that admit singularities of the pole type. Such solutions correspond to the features of the electric field strength. At these special points, a concentration of electric charges occurs and lightning can occur.

To construct exact solutions, we use symmetry methods (see [2, 3, 4]).

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2. Model

The following model of the formation of the difference of electric potentials in the cloud was proposed in [5].

Suppose that a cloud moves at a speed of v_0 under the influence of a pressure difference and a temperature gradient. Assume that its general electroneutrality is violated and the cloud is ionized. Moreover, charged particles inside the cloud collide with neutral scattering centers with a frequency of ν. In addition to hydrodynamic forces, particles are affected by electromagnetic forces from other particles and the external electric field of the Earth. We assume that the problem is one-dimensional and that predominantly negative particles prevail in the region under consideration.

Under these assumptions, Pustovoit obtained the following system of differential equations:

$$\begin{cases} enE - k_B T \dfrac{\partial n}{\partial x} - mn\nu\,(v - v_0) = 0, \\ \dfrac{\partial n}{\partial t} + \dfrac{\partial (nv)}{\partial x} = 0, \\ \varepsilon \dfrac{\partial E}{\partial x} = -4\pi e n. \end{cases} \tag{2.1}$$

Here e, m, v are particle charge, mass and velocity, respectively, ν is the effective collision frequency of a particle with all scattering centers, n is the concentration of negatively charged particles, T is a temperature, k_B is the Boltzmann constant, ε is the dielectric permeability of the medium. The direction x corresponds to the direction along the electric field of the Earth.

Introducing new variables

$$l \equiv \frac{e^2}{k_B T}, \qquad \xi = \frac{x}{l}, \qquad \tau_0 \equiv \frac{ml^2\nu}{k_B T}, \qquad \tau = \frac{t}{\tau_0}, \qquad \alpha = \frac{mv_0 l\nu}{k_B T},$$
$$\rho(\xi,\tau) = n(\xi,\tau) l^3, \qquad u(\xi,\tau) = \frac{elE(\xi,\tau)}{k_B T},$$

we obtain the following system of differential equations

$$\begin{cases} \dfrac{\partial^2 u(\xi,\tau)}{\partial \xi \partial \tau} = \dfrac{\partial}{\partial \xi}\left(\dfrac{\partial^2 u(\xi,\tau)}{\partial \xi^2} - (u(\xi,\tau) - \alpha)\dfrac{\partial u(\xi,\tau)}{\partial \xi} \right), \\ \dfrac{\partial u(\xi,\tau)}{\partial \xi} = -\dfrac{4\pi}{\varepsilon}\rho(\xi,\tau). \end{cases} \tag{2.2}$$

Note that the first equation of this system can be considered independently of the second.

Using the methods of group analysis of differential equations we will construct the exact solutions of the first equation of system (2.2).

3. Symmetries

We find the infinitesimal point symmetries of the first equation of system (2.2) in the form of vector fields on the space of 0-jets. Using the standard procedure [2, 4], we find that the Lie algebra of symmetries is infinite-dimensional and generated by the vector fields

$$\begin{aligned} X_1 =& F_1(\tau)\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \tau} + \dot{F}_1(\tau)\frac{\partial}{\partial u}, \\ X_2 =& \left(F_2(\tau) + \frac{\xi}{2}\right)\frac{\partial}{\partial \xi} + \tau\frac{\partial}{\partial \tau} + \left(\dot{F}_2(\tau) + \frac{\alpha - u}{2}\right)\frac{\partial}{\partial u}, \\ X_3 =& \left(F_3(\tau) + \frac{\tau\xi}{2}\right)\frac{\partial}{\partial \xi} + \frac{\tau^2}{2}\frac{\partial}{\partial \tau} + \left(\dot{F}_3(\tau) + \frac{\tau(\alpha - u) + \xi}{2}\right)\frac{\partial}{\partial u}, \end{aligned}$$

where $F_1(\tau), F_2(\tau), F_3(\tau)$ are arbitrary functions. The dot above means the derivative of the variable τ.

4. Exact solutions

Here we construct solutions of equation (2.2_1) that are invariant under shifts along the vector fields X_1, X_2, X_3.

4.1. X_1-invariant solutions. Solutions that are invariant with respect to the vector field X_1 have the form

$$u(\tau, \xi) = \dot{A}(\tau) + V(\xi - A(\tau)),$$

where $= \dot{A}(\tau) = F_1(\tau)$. The corresponding reduced ordinary differential equation has the form

$$V''' - (V - \alpha)\, V'' - (V')^2 = 0, \tag{4.1}$$

where $V = V(z)$, $z = \xi - A(\tau)$.

In order to integrate this equation, we use the substitution $y(z) = V(z) - \alpha$. Then equation (4.1) takes the form

$$y''' - yy'' - (y')^2 = y''' - (yy')' = 0,$$

then

$$y'' - yy' - a = \left(y' - \frac{y^2}{2}\right)' - a.$$

We get the Riccati equation

$$y' - \frac{y^2}{2} - az - b = 0.$$

It can be solved in terms of the Airy functions. Then the general solution of equation (4.1) is

$$V(z) = \frac{2^{2/3}a^{1/3}\left(c\mathrm{Ai}\left(1, -\dfrac{2^{2/3}(za+b)}{2a^{2/3}}\right) + \mathrm{Bi}\left(1, -\dfrac{2^{2/3}(za+b)}{2a^{2/3}}\right)\right)}{c\mathrm{Ai}\left(-\dfrac{2^{2/3}(za+b)}{2a^{2/3}}\right) + \mathrm{Bi}\left(-\dfrac{2^{2/3}(za+b)}{2a^{2/3}}\right)} + \alpha. \tag{4.2}$$

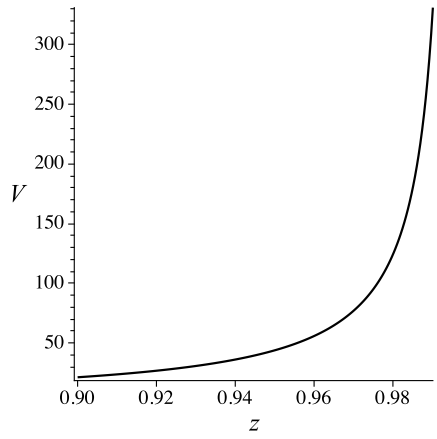

FIGURE 1. The graph of solution (4.5) with $C_1 = -1, C_3 = 0, a = 2$.

Here a, b, c are arbitrary constants, Ai and Bi are the Airy functions of the first and second kind respectively. Therefore X_1-invariant solution of equation (2.2_1) is

$$u(\tau,\xi) = \dot{A}(\tau) + \alpha + \frac{(4a)^{1/3}\left(c\mathrm{Ai}\left(1, -\frac{2^{2/3}((\xi - A(\tau))a + b)}{2a^{2/3}}\right) + \mathrm{Bi}\left(1, -\frac{2^{2/3}((\xi - A(\tau))a + b)}{2a^{2/3}}\right)\right)}{c\mathrm{Ai}\left(-\frac{2^{2/3}((\xi - A(\tau))a + b)}{2a^{2/3}}\right) + \mathrm{Bi}\left(-\frac{2^{2/3}((\xi - A(\tau))a + b)}{2a^{2/3}}\right)}. \tag{4.3}$$

4.2. X_2-invariant solutions. These solutions have the form

$$u(\tau,\xi) = \dot{B}(\tau)\sqrt{\tau} + \frac{B(\tau)}{2\sqrt{\tau}} + \frac{1}{\tau}V\left(\frac{\xi}{\sqrt{\tau}} - B(\tau)\right) + \alpha, \tag{4.4}$$

where $F_2(\tau) = \dot{B}(\tau)\tau^{3/2}$. The reduced ordinary differential equation has the form

$$V''' + \left(\frac{z}{2} - V\right)V'' - (V')^2 + V' = 0,$$

where $V = V(z)$, $z = \dfrac{\xi}{\sqrt{\tau}} - B(\tau)$. Its general solution is

$$V(z) = \frac{4C_3 a(1-2a)\mathrm{U}\left(a+1, \frac{3}{2}, q\right) - 4a\mathrm{M}\left(a+1, \frac{3}{2}, q\right)}{(4C_1+z)\left(2C_3\mathrm{U}\left(a, \frac{3}{2}, q\right) + \mathrm{M}\left(a, \frac{3}{2}, q\right)\right)}$$
$$+ \frac{4(2a-4C_1^2-1-C_1 z)\left(C_3\mathrm{U}\left(a, \frac{3}{2}, q\right) + \frac{1}{2}\mathrm{M}\left(a, \frac{3}{2}, q\right)\right)}{(4C_1+z)\left(2C_3\mathrm{U}\left(a, \frac{3}{2}, q\right) + \mathrm{M}\left(a, \frac{3}{2}, q\right)\right)} \tag{4.5}$$

where U and M are the Kummer functions [6] and

$$a = C_1^2 - \frac{1}{2}C_2 + \frac{1}{2}, \qquad q = -\frac{(4C_1+z)^2}{4}.$$

The graph of solution (4.5) with $C_1 = -1, C_3 = 0, a = 2$ is shown in Fig. 1.

4.3. X_3-invariant solutions. These solutions have the form

$$u(\tau, \xi) = \frac{\xi}{\tau} - \frac{1}{\tau}V\left(\frac{\xi}{\tau} - C(\tau)\right) + \tau\dot{C}(\tau) + \alpha. \tag{4.6}$$

Therefore the reduced equation is

$$V''' + VV'' + (V')^2 = 0. \tag{4.7}$$

Here $V = V(z)$, $z = \dfrac{\xi}{\tau} - C(\tau)$. Write this equation in the form

$$V''' + (VV')' = 0.$$

and integrate it:

$$V'' + VV' + a = (VV')' + a = 0.$$

Integrating this equation, we get the following Riccati equation:

$$V' + \frac{1}{2}V^2 + az + b = 0,$$

where a, b are arbitrary constant. Its general solution is

$$V(z) = \frac{-2^{2/3}a^{1/3}\left(c\mathrm{Ai}\left(1, -\frac{az+b}{2^{1/3}a^{2/3}}\right) + \mathrm{Bi}\left(1, -\frac{az+b}{2^{1/3}a^{2/3}}\right)\right)}{c\mathrm{Ai}\left(-\frac{az+b}{2^{1/3}a^{2/3}}\right) + \mathrm{Bi}\left(-\frac{az+b}{2^{1/3}a^{2/3}}\right)}. \tag{4.8}$$

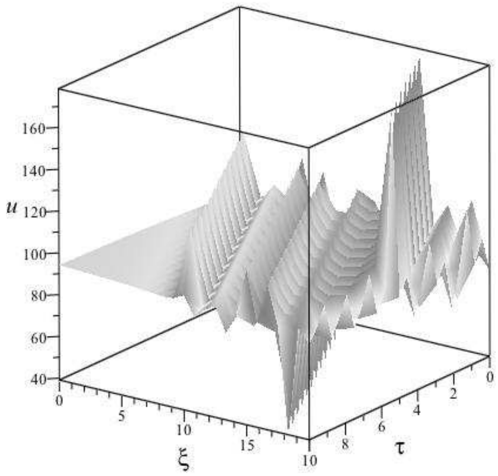

FIGURE 2. The graph of solution (5.1).

Therefore X_3-invariant solutions of equation (2.2_1) have the following form:

$$u(\tau,\xi)=\frac{\xi}{\tau}+\tau\dot{C}(\tau)+\alpha$$
$$-\frac{1}{\tau}\frac{-2^{2/3}a^{1/3}\left(c\mathrm{Ai}\left(1,-\dfrac{a\left(\dfrac{\xi}{\tau}-C(\tau)\right)+b}{2^{1/3}a^{2/3}}\right)+\mathrm{Bi}\left(1,-\dfrac{a\left(\dfrac{\xi}{\tau}-C(\tau)\right)+b}{2^{1/3}a^{2/3}}\right)\right)}{c\mathrm{Ai}\left(-\dfrac{a\left(\dfrac{\xi}{\tau}-C(\tau)\right)+b}{2^{1/3}a^{2/3}}\right)+\mathrm{Bi}\left(-\dfrac{a\left(\dfrac{\xi}{\tau}-C(\tau)\right)+b}{2^{1/3}a^{2/3}}\right)}.$$

5. The physical meaning of solutions

Consider solution (4.3) with parameters $a=b=1, c=0, \alpha=10, A(\tau)=\tau$:

$$u(\tau,\xi)=\frac{4^{1/3}\mathrm{Bi}\left(1,-2^{-1/3}(x-t+1)\right)}{\mathrm{Bi}\left(-2^{-1/3}(x-t+1)\right)}+10. \tag{5.1}$$

The graph of the solution is shown in Fig. 2.

Analyzing the location of the zeros of the numerator and denominator of the fraction in formula (5.1), we see that the points of discontinuity of function (5.1) are discrete on the straight line $\tau=\xi+1$ lying on the coordinate plane (τ,ξ) (see Fig. 3). This means that singularities of the electromagnetic field strength arise at these points and an electric discharge can occur in their vicinity.

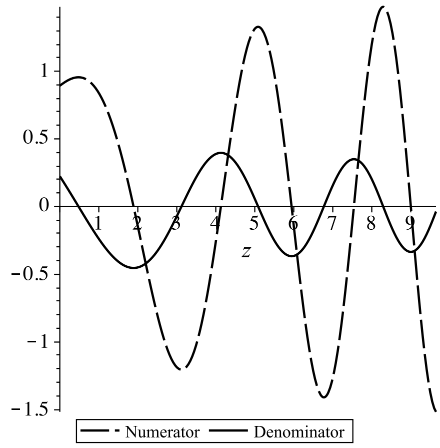

FIGURE 3. The graphs of the numerator and the denomirator of the fraction in solution (5.1).

Remark 5.1. The distribution of the relative density of negatively charged particles in time and space can be calculated explicitly by formula (2.2_2).

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