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D-DISTANCE MAGIC LABELING ON REGULAR GRAPHS

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Abstract. Distance-based labeling is an important area in graph theory due to its applications in coding theory, network design, communication systems, and related areas. A D-distance magic labeling (D-DML) of a graph G with n vertices is a bijection $f : V(G) \to \{1, 2, \ldots, n\}$ such that, for every vertex $v \in V(G)$,

$$w(v) = \sum_{u \in N_D(v)} f(u)$$

is a constant, say k, which is called the magic constant, where

$$N_D(v) = \{u \in V(G) : d(u, v) \in D\}.$$

In this paper, I study D-distance magic labeling (D-DML) of simple undirected regular graphs for a distance set $D \subseteq \{1, 2, \ldots, d\}$, where d is the diameter of the graph G. I investigate D-DML of disjoint unions of cycles, complete graphs, and r-regular graphs. In addition, I investigate D-DML of non-isomorphic regular graphs.

1. Introduction

The idea of magic labeling got inspired by construction of magic squares. Using this concept, in 1963 Sedlacek presented magic labeling of $G = (V, E)$. Later O'Neal and Slater defined D-distance magic labeling (D-DML) where D is subset of nonnegative integers.

Definition 1.1 (1)**.** A bijection

$$g : V \longrightarrow \{1, 2, \ldots, v\}$$

is called a D-DML if there is a constant k such that, for every $x \in V$,

$$w(x) = \sum_{y \in N_D(x)} g(y) = k,$$

where

$$N_D(x) = \{y \in V : d(x, y) \in D\}.$$

A graph that admits a D-DML is called a D-DMG.

Definition 1.2. [2] The *join* of graphs G_1 and G_2 contains all the vertices and edges of G_1 and G_2, together with edges joining each vertex of G_1 to every vertex of G_2. It is denoted by

$$G_1 + G_2.$$

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2. Main Results

Theorem 2.1. *A 2-regular graph G is a $(0,d)$-DMG if G is a disjoint union of m identical even cycles.*

Proof. Let $C_{n_1}, C_{n_2}, \ldots, C_{n_m}$ be the m cycles of length n, where n is even. Let

$$G = C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_m}.$$

Note that, for each C_{n_i}, $i = 1, 2, \ldots, m$, the diameter is

$$d = \frac{n}{2}.$$

Let

$$v_{i1}, v_{i2}, \ldots, v_{in}$$

be the vertices of C_{n_i} in clockwise direction.

Note that

$$(v_{i1}, v_{i,\frac{n}{2}+1}), (v_{i2}, v_{i,\frac{n}{2}+2}), (v_{i3}, v_{i,\frac{n}{2}+3}), \ldots$$

are the pairs of vertices at distance d. There are

$$\frac{n}{2}$$

such pairs.

Define

$$g : V \longrightarrow \{1, 2, \ldots, mn\}$$

as follows.

For $i = 1$,

$$g(v_{ij}) = j, \qquad j = 1, 2, \ldots, d,$$

and

$$g(v_{ij}) = mn - j + d + 1, \qquad j = d+1, d+2, \ldots, n.$$

For $i = 2$,

$$g(v_{ij}) = j + d, \qquad j = 1, 2, \ldots, d,$$

and

$$g(v_{ij}) = mn - (j + d) + d + 1, \qquad j = d+1, d+2, \ldots, n.$$

For $i = 3$,

$$g(v_{ij}) = j + 2d, \qquad j = 1, 2, \ldots, d,$$

and

$$g(v_{ij}) = mn - (j + 2d) + d + 1, \qquad j = d+1, d+2, \ldots, n.$$

Continuing the process, in general, for $i = k$,

$$g(v_{ij}) = j + (k-1)d, \qquad j = 1, 2, \ldots, d,$$

and

$$g(v_{ij}) = mn - \big(j + (k-1)d\big) + d + 1, \qquad j = d+1, d+2, \ldots, n.$$

Thus, for every pair of vertices at distance d, the sum of their labels is constant. Hence, G admits a $(0,d)$-distance magic labeling.

Therefore, G is a $(0,d)$-DMG.

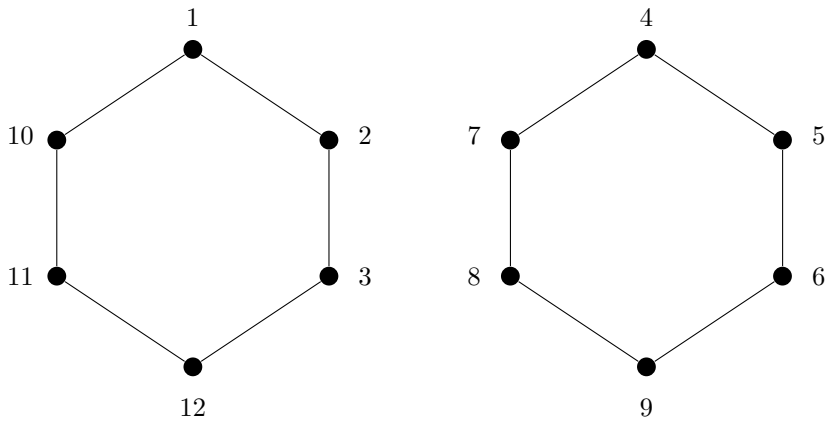


FIGURE 1. $(0,3)$-DML of the graph $C_6 \cup C_6$.

Theorem 2.2. *The disjoint union of m copies of complete graphs*

$$G = mK_n, \qquad n > 1,$$

is a $(0,1)$-DMG if and only if either

(1) *n is even; or*
(2) *both n and m are odd.*

Proof. Let

$$G = mK_n.$$

For $i = 1, 2, \ldots, m$, let

$$V_i = \{v_i^1, v_i^2, \ldots, v_i^n\}$$

be the vertex set of the ith copy of K_n. Then

$$V(G) = V_1 \cup V_2 \cup \cdots \cup V_m.$$

Note that

$$|V(G)| = nm.$$

For each v_i^j of the ith copy of K_n,

$$N_{(0,1)}(v_i^j) = V_i, \qquad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n.$$

Since there are nm vertices, the sum of all labels is

$$\frac{nm(nm+1)}{2}.$$

Hence, if G admits a $(0,1)$-DML, the magic constant is

$$k = \frac{n(nm+1)}{2}.$$

Suppose that n is odd and m is even. Then nm is even, and hence $nm+1$ is odd. Therefore,

$$n(nm+1)$$

is odd. Consequently,

$$2 \nmid n(nm+1),$$

which contradicts the fact that

$$k = \frac{n(nm+1)}{2}$$

must be an integer. Therefore, when n is odd, m must be odd.

Conversely, we consider the following two cases.

Case 1: n is even.

Define

$$g : V(G) \longrightarrow \{1, 2, \ldots, nm\}$$

as follows. For $i = 1, 2, \ldots, m$,

$$\begin{aligned} g(v_i^1) &= i, \\ g(v_i^2) &= 2m + 1 - i, \\ g(v_i^3) &= 2m + i, \\ g(v_i^4) &= 4m + 1 - i, \\ g(v_i^5) &= 4m + i, \\ g(v_i^6) &= 6m + 1 - i, \end{aligned}$$

and so on. In general,

$$g(v_i^j) = \begin{cases} (j-1)m + i, & \text{if } j \text{ is odd,} \\ jm + 1 - i, & \text{if } j \text{ is even.} \end{cases}$$

Thus, g is a bijection from $V(G)$ onto $\{1, 2, \ldots, nm\}$.

For every $i = 1, 2, \ldots, m$, the sum of the labels assigned to the vertices of K_n is

$$\sum_{j=1}^{n} g(v_i^j) = \frac{n(nm+1)}{2} = k.$$

Since

$$N_{(0,1)}(v_i^j) = V_i,$$

we have

$$w(v_i^j) = \sum_{v_i^\ell \in V_i} g(v_i^\ell) = k.$$

Hence, G admits a $(0, 1)$-DML.

Case 2: m and n are both odd.

When $n = 3$, we have

$$K_3 = C_3,$$

and the result follows from the preceding theorem.

Let $n \geq 5$.

The sum of the nm labels is

$$1 + 2 + \cdots + nm = \frac{nm(nm+1)}{2}.$$

Therefore, if G is a $(0, 1)$-DMG, then the magic constant is

$$k = \frac{n(nm+1)}{2}.$$

Since m and n are both odd, k is an integer. We can therefore partition the set

$$\{1, 2, \ldots, nm\}$$

into m subsets, each containing n elements, such that the sum of the elements in each subset is k.

Assign the elements of the first subset to the vertices of the first copy of K_n. Next, choose n elements from the remaining labels whose sum is k and assign them to the vertices of the second copy of K_n. Continue this process until all m copies of K_n are labeled.

Thus, each copy of K_n has label sum k. Hence, for every vertex v_i^j,

$$w(v_i^j) = \sum_{y \in N_{(0,1)}(v_i^j)} g(y) = k.$$

Therefore, G admits a $(0,1)$-DML.

Hence the disjoint union mK_n is a $(0,1)$-DMG if and only if n is even or both n and m are odd. □

For example, refer Fig 2.

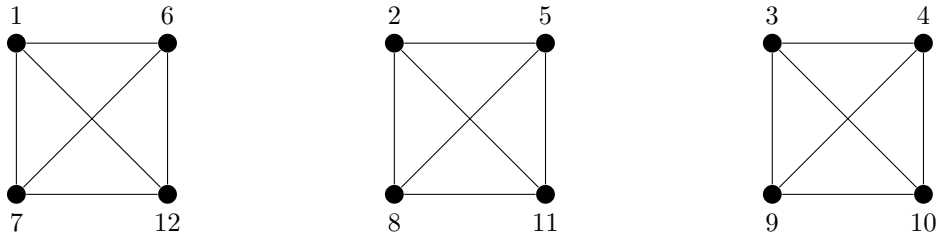


FIGURE 2. $(0,1)$-DMG of $3K_4$.

Theorem 2.3. *For both n and r even, an r-regular graph is not a $(0,1)$-DMG.*

Proof. If G is an r-regular $(0,1)$-DMG, then

$$K' = \frac{(r+1)(n+1)}{2}$$

is a magic constant from[6].

When n and r are both even, $(r+1)(n+1)$ is odd. Hence,

$$K' = \frac{(r+1)(n+1)}{2} \notin \mathbb{Z}^+,$$

which is a contradiction. Hence the proof.

Theorem 2.4. *For even r, an r-regular graph is a $(0,2)$-DMG if the number of vertices is $2n = r + 2$.*

Proof. Construct an r-regular graph G by assigning all the vertices around a circle and joining each vertex to its s nearest neighbors on either side, where

$$r = 2s.$$

Now, starting from any arbitrary vertex, label the first n vertices clockwise as

$$v_1, v_2, \ldots, v_n.$$

The remaining vertices in the same direction are labeled as

$$u_1, u_2, \ldots, u_n.$$

Now, for any vertex v_i, $i = 1, 2, \ldots, n$,

$$N_{(0,2)}(v_i) = \{v_i, u_i\}.$$

Similarly,

$$N_{(0,2)}(u_i) = \{v_i, u_i\}.$$

To have a $(0, 2)$-DMG,

$$w(v_i) = w(u_i) = g(v_i) + g(u_i)$$

for any i.

Define the function

$$g : V \longrightarrow \{1, 2, \ldots, 2n\}$$

as follows:

$$g(v_i) = i, \qquad i = 1, 2, \ldots, n,$$

and

$$g(u_i) = 2n - (i - 1), \qquad i = 1, 2, \ldots, n.$$

Therefore,

$$g(v_i) + g(u_i) = i + 2n - (i - 1) = 2n + 1,$$

which is constant for every i.

Hence, G is a $(0, 2)$-DMG.

For example, see Fig 3.

Corollary 2.5. *The magic constant of a* $(0, 2)$*-distance magic* r*-regular graph is* $r + 3$.

Proof. From Theorem 2.4,

$$w(v_i) = i + 2n - (i - 1).$$

Therefore,

$$w(v_i) = 2n + 1.$$

Since

$$2n = r + 2,$$

we obtain

$$w(v_i) = r + 2 + 1.$$

Hence,

$$w(v_i) = r + 3.$$

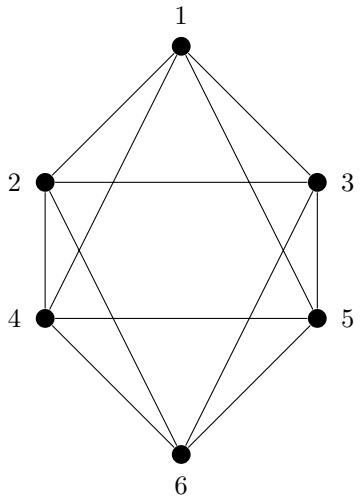


FIGURE 3. $(0,2)$-DML of a 4-regular graph

Therefore, the magic constant is

$$\boxed{r+3}.$$

Corollary 2.6. *The join of an r-regular graph containing $r+2$ vertices and the cycle C_4 is a $(0,2)$-DMG.*

Proof. Let G_1 be an r-regular graph containing $r+2$ vertices.

Join each vertex of G_1 with every vertex of C_4 to obtain

$$G = G_1 + C_4.$$

The resulting graph G is an $(r+4)$-regular graph having $r+6$ vertices.

Therefore, by Theorem 2.4, G is a $(0,2)$-DMG.

Observation 1. *Two non-isomorphic r-regular graphs with the same number of vertices need not be $(0,2)$-distance magic graphs.*

Example 2.7. Consider two non-isomorphic 3-regular graphs with 8 vertices.

Note that in Fig 4, the graph is a $(0,2)$-DMG.

However, in Fig 5, we observe that

$$N_{(0,2)}(c) = \{c, a, h, f, e\},$$

and

$$N_{(0,2)}(h) = \{h, b, c, e, f\}.$$

We know that a given graph is a $(0,2)$-DMG if

$$w(c) = w(h).$$

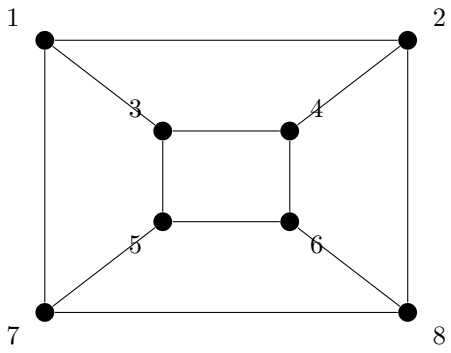


FIGURE 4. $(0, 2)$-distance magic graph.

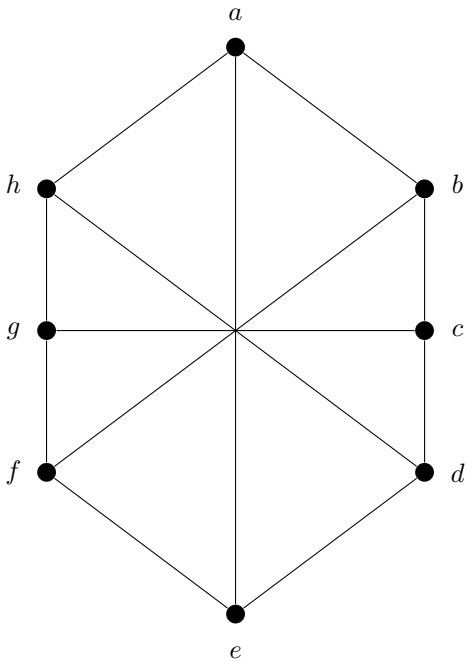


FIGURE 5. A graph which is not $(0, 2)$-DMG.

Therefore,

$$g(c) + g(a) + g(h) + g(f) + g(e) = g(h) + g(b) + g(c) + g(e) + g(f).$$

This implies

$$g(a) = g(b),$$

which is a contradiction since g is a bijection.

Therefore, the given graph cannot be a $(0, 2)$-DMG.

Theorem 2.8. *A complete graph G with diameter d is not a $\{d\}$-DMG.*

Proof. For a complete graph G, we have

$$d = 1.$$

Let x, y be two distinct vertices of G.

Then

$$w(x) = \sum_{v \in V(G)} g(v) - g(x).$$

Similarly,

$$w(y) = \sum_{v \in V(G)} g(v) - g(y).$$

If G is a $\{d\}$-DMG, then

$$w(x) = w(y).$$

Therefore,

$$\sum_{v \in V(G)} g(v) - g(x) = \sum_{v \in V(G)} g(v) - g(y),$$

which implies

$$g(x) = g(y).$$

This contradicts the fact that g is a bijection.

Hence, G is not a $\{d\}$-DMG.

Conclusion

D-DML in regular graphs has a wide range of applications, particularly in fields involving network analysis, error correction, optimization, and cryptography. Further investigation is necessary to develop and apply more D-DML techniques in these fields.

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