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ON THE RELATION BETWEEN SEQUENTIAL AND PRIME LABELINGS OF DIGRAPHS AND THEIR COMPUTATIONAL APPLICATIONS

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Abstract. In this paper, we bridge the gap in directed graph analysis by combining sequential and prime labeling techniques, establishing a relationship between these two labeling methods for certain classes of directed graphs. This study offers valuable insights into the structural properties of digraphs, revealing novel connections that enhance the theoretical foundation of graph labeling. Our findings contribute to a deeper understanding of directed graph labeling and open new avenues for research in graph theory and its applications in both mathematical and computational domains.

Building upon this, we propose a novel framework that integrates coined quantum walks, entropy-guided encryption, and secure robotic path planning over sequentially labeled digraphs. Two key security mechanisms are introduced: *Path Entropy Masking*, which obfuscates the robot's route by embedding dummy transitions, and *Message Entropy Masking*, which ensures message confidentiality through the use of entropy-derived noise vectors. These mechanisms are implemented within a quantum-inspired navigation model utilizing coin and shift operators defined on a Hilbert space formed by graph positions and transition directions.

The robot evaluates multiple acyclic paths from source to destination using quantum entropy metrics, trust scores, and machine learning-based safety predictions. A multi-criteria decision-making strategy is then employed to select the most secure and efficient path. Encrypted communication is conducted along this trusted path using entropy-masked encoding, thereby ensuring both safe traversal and secure message delivery.

1. Introduction

This study is confined to finite directed graphs, represented by $G = (N, A)$, where $N(G)$ denotes the collection of vertices and $A(G)$ signifies the set of directed edges. Each edge $\langle x, y\rangle \in A(G)$ indicates a directed edge from vertex x to vertex y. The incoming degree of a vertex y, written as $\delta^-(y)$, counts the number of edges directed into y, while the outgoing degree, denoted $\delta^+(x)$, represents the number of edges originating from x.

Graph labeling is a process of assigning integers to the vertices and/or edges of a graph following certain conditions. One notable labeling method is the prime labeling, originally introduced by Entringer and later extended by researchers such

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as Hameeda Begum and N. Tout [8]. Under this scheme, each vertex is assigned a unique integer such that any two adjacent vertices have relatively prime labels.

An advanced version of this technique is the edge-vertex prime labeling, where a function g maps distinct positive integers to both the vertices and edges. For every arc $\langle x, y\rangle \in A(G)$, the values $g(x)$, $g(y)$, and $g(\langle x, y\rangle)$ are required to be pairwise coprime.

Definition 1.1 (Edge-Vertex Prime Labeling)**.** [3] Let G be a directed graph. An Edge-vertex prime labeling is a bijective function $h : N(G) \cup A(G) \to \{1, 2, \ldots, |N(G) \cup A(G)|\}$ such that for every directed edge $a = \langle p, q\rangle \in A(G)$, the values $h(p)$, $h(q)$, and $h(\langle p, q\rangle)$ are pairwise relatively prime.

For directed graphs, additional prime labeling variants include:

- **Incoming degree of prime labeling**:
  A function $h : N(G) \to \{1, 2, \ldots, |N(G)|\}$ assigns integers to vertices such that for every edge $\langle m, n\rangle \in A(G)$, the integers $h(m)$ and $h(n)$ are coprime.
- **Outgoing degree of prime labeling**: For a directed graph G with vertices m and n, a function $h : N(G) \to \{1, 2, \ldots, m + n\}$ defines an outgoing prime assignment if, for each arc $\langle n, m\rangle \in A(G)$, the assigned labels are mutually prime.

Another significant graph labeling method is the k-sequential labeling, initially proposed by Peter J. Slater [5] and later extended to directed graphs by Shivaraj Kumar [7]. Bloom and Golomb [1] also discussed its structural implications in graph theory.

Definition 1.2 (r-sequential digraph)**.** [7] A digraph G with a total of $\tau = |N| + |A|$ vertices and edges is termed *r-sequential* if there exists a bijective mapping $\sigma : N(G) \cup A(G) \to \{r, r + 1, \ldots, r + \tau - 1\}$ such that, for every arc $\langle i, j\rangle \in A(G)$, the labeling satisfies:

$$\sigma(\langle i, j\rangle) \equiv |\sigma(j) - \sigma(i)| \pmod{(\tau + r)}.$$

When $r = 1$, the labeling is simply called *sequential*. Such sequential schemes are useful in encoding topological information and have been explored in specific graph classes like bi-star graphs, coconut trees, and their variants [2, 3, 4].

Quantum walks have emerged as a powerful computational paradigm, inspired by quantum mechanics. Introduced as a quantum analogue of classical random walks [9], they have demonstrated advantages in quantum search algorithms [10], graph traversal [11], and robotic path planning [12]. In discrete-time *coined quantum walks*, the state of a quantum system evolves over a graph through the application of coin and shift operators defined on a composite Hilbert space. This framework captures quantum interference and superposition, enabling more efficient exploration of large and structured networks.
Graph based models are also critical in intelligent robotic navigation and secure communication systems. In particular, sequentially labeled digraphs where edges are assigned unique integer weights model structured environments such as industrial automation systems, sensor networks, and multi agent robotic platforms. These arc weights may represent traversal costs, entropy values, or communication

delays relevant for path evaluation. Shannon's foundational work on information theory [13] introduced the concept of entropy as a measure of uncertainty or information content. In cryptographic contexts, entropy based masking is employed to obscure data patterns and resist statistical attacks [14].

Our framework employs quantum walk dynamics to evaluate multiple acyclic paths from a source to a target vertex, computing quantum entropy, trust scores, and machine learning-based security predictions. A multi-criteria decision strategy identifies the most secure and efficient route. The final communication is encrypted using entropy-masked encoding, ensuring both safe traversal and secure message delivery in adversarial environments.

2. Relation between Sequential and Prime labeling

In this work, the biconditional "if and only if" is represented using the symbol $\iff$. Additionally, we adopt the notation $f(x) \perp f(y)$ to indicate that $\gcd(f(x), f(y)) = 1$, meaning the assigned labels are relatively prime [6].

Utilizing the earlier concepts of prime labeling based on in-degree and out-degree criteria for directed graphs, we now define a generalized prime labeling for digraphs as follows:

Theorem 2.1. *An In-Star digraph $\langle S_{1,n}\rangle$ for $(n \geq 2)$ is r-sequential $\iff$ it admits a prime labeling.*

Proof. Let us assume that $\mathcal{D} = \langle S_{1,n}\rangle$ is r-sequential digraph.
Define a bijection $\phi : V(\mathcal{D}) \cup E(\mathcal{D}) \to \{r, r+1, \ldots, r+\tau-1\}$ such that $\phi(x_0) = r$.
Let x_0 be the central vertex and x_i for $1 \leq i \leq n$ be the outer (pendent) vertices.
Since the central vertex is adjacent to all outer vertices,
we consider two cases:

(a) If r is even, i.e., $r = 2^m$ for $m > 0$: Assign labels to the vertices $\{x_1, x_2, \ldots, x_n\}$ as $\{r+1, r+3, r+5, \ldots, r+\tau-2\}$, and to the edges as $\{r+\tau-1, r+\tau-3, \ldots, r+4, r+2\}$ with $mod(\tau + r)$.
Since $\delta^-(x_0) = n$ and $\delta^-(x_i) = 0$ for all $1 \leq i \leq n$, and $\phi(x_0) \perp \phi(x_i)$ for all $\langle x_i, x_0\rangle \in E(\mathcal{D})$. Thus, $\mathcal{D}$ satisfies the prime labeling condition. This labeling is valid when $r = 2^m$ for some $m > 0$; otherwise, the prime labeling condition fails.

(b) If $r = 1$, i.e., $r = 2^m$ for $m = 0$: Assign labels to the vertices $\{x_1, x_2, \ldots, x_n\}$ as $\{r+2, r+4, \ldots, r+\tau-1\}$. By the definition of r-sequential labeling, the edge labels are $\{r+\tau-2, r+\tau-4, \ldots, r+3, r+1\}$. Since $\delta^-(x_0) = n$ and $\delta^-(x_i) = 0$ for all $1 \leq i \leq n$, and $\phi(x_0) \perp \phi(x_i)$ for all $\langle x_i, x_0\rangle \in E(\mathcal{D})$, we conclude that $\mathcal{D}$ admits a prime labeling. Hence, the prime labeling condition is satisfied when $r = 2^m$ for $m = 0$.

Conversely,
Assume that the In-Star digraph $\mathcal{D} = \langle S_{1,n}\rangle$ admits a prime labeling.
Let the central vertex be x_0, and the outer (pendent) vertices be $x_1, x_2, \ldots, x_n$, such that the edge set is $E = \{\langle x_i, x_0\rangle \mid 1 \leq i \leq n\}$. Since $\mathcal{D}$ admits a prime

labeling, there exists a bijection $\phi : V(\mathcal{D}) \cup E(\mathcal{D}) \to \{r, r+1, \ldots, r+\tau-1\}$ such that:

- $\phi(x_0) = r$ (central vertex),
- All other labels (for x_1 to x_n and edges $\langle x_i, x_0\rangle$) are distinct elements of the labeling set,
- For all i, $\gcd(\phi(x_i), \phi(x_0)) = 1$.

We now construct a labeling σ based on ϕ, and prove that it satisfies the condition of an r-sequential labeling under the modulus condition. For every edge $\langle x_i, x_0\rangle \in E(\mathcal{D})$, define:

$$\sigma(x_j) := \phi(x_j), \quad \text{for all } x_j \in V(\mathcal{D}),$$

$$\sigma(\langle x_i, x_0\rangle) := |\phi(x_0) - \phi(x_i)| \pmod{\tau + r}.$$

Since ϕ is injective and defined over $\{r, r+1, \ldots, r+\tau-1\}$, and exactly $\tau+1$ labels are assigned (vertices and edges), the assignment is bijective over the union of $V(\mathcal{D})$ and $E(\mathcal{D})$. Thus, for each edge $\langle x_i, x_0\rangle \in E(\mathcal{D})$, we have:

$$\sigma(\langle x_i, x_0\rangle) \equiv |\sigma(x_0) - \sigma(x_i)| \pmod{\tau + r},$$

which satisfies the condition for an r-sequential labeling under the given modular definition. Therefore, $\mathcal{D}$ is r-sequential. □

Example 2.2. Figures 1 and 2 illustrate the in-star and out-star digraphs,respectively.

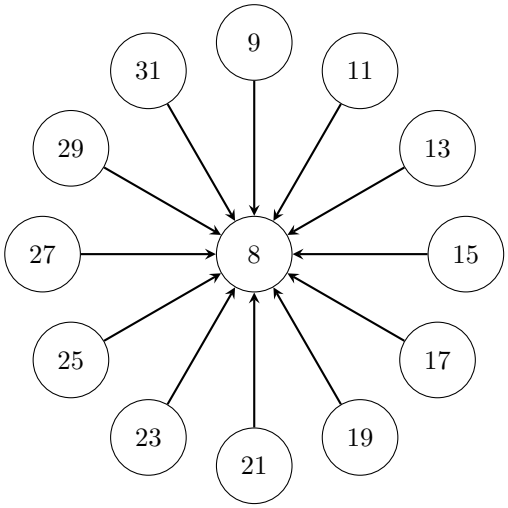


FIGURE 1. A 8-sequential and prime labeling of $\overrightarrow{S_{1,12}}$

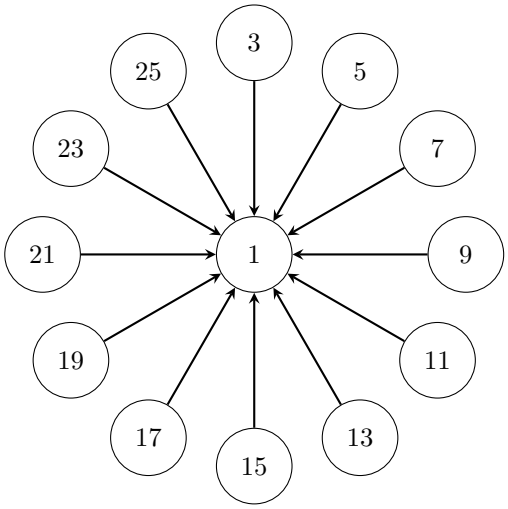


FIGURE 2. A 1-sequential and prime labeling of $\overrightarrow{S_{1,12}}$

Note 1. For remaining r values(other than $r = 2^m$ for m greater than or equal to 0) star digraph is only sequential.

In a similar manner, we observe

Theorem 2.3. *The Out-Star digraph* $\langle S'_{1,n} \rangle$ $(n \geq 2)$ *is* 1*-sequential* $\iff$ *it satisfies prime labeling.*

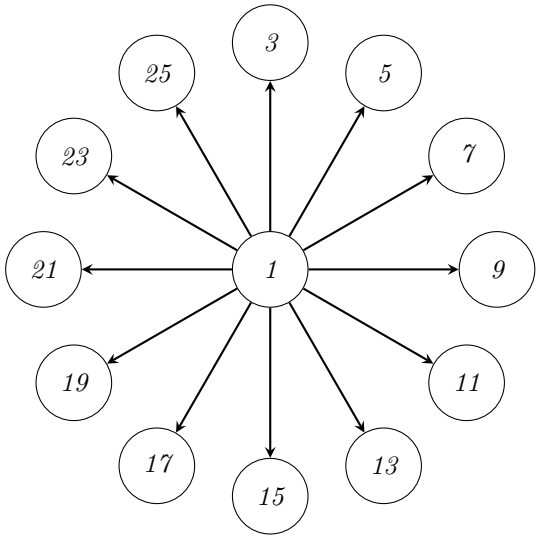


FIGURE 3. A 1-sequential $\langle S'_{1,12} \rangle$

This follows trivially from the fact that reversing the edge orientation of an In-Star yields an Out-Star. Since both vertex and edge labelings set remain unchanged, labeling remains sequential as well as prime.

Note 2. The Out-Star $\langle S'_{1,n}\rangle$ is not r-sequential for any $r > 1$ since repeated vertex or edge labels violate the r-sequential labeling condition.

Theorem 2.4. *The Bi-Star digraph $\overrightarrow{B}(m,n)$ is sequential $\iff$ it admits edge-vertex prime labeling.*

Proof. Assume $\mathcal{D} = \overrightarrow{B}(m,n)$ is sequentially labeled.
Let vertex set be $V(\mathcal{D}) = \{x_i, y_j \mid 0 \le i \le m, 0 \le j \le n\}$, where x_0 and y_0 are the apex vertices and others are pendent.
Let the edge set be $E(\mathcal{D}) = \{\langle x_0, x_i\rangle \mid 1 \le i \le m\} \cup \{\langle y_0, y_j\rangle \mid 1 \le j \le n\} \cup \{\langle y_0, x_0\rangle\}$.
Define a bijection $\phi : V(\mathcal{D}) \cup E(\mathcal{D}) \to \{1, 2, \ldots, \tau\}$ such that $\phi(x_0) = 1$, $\phi(y_0) = \tau$.

Two labeling cases arise:
a) If all x_i's are odd labeling and y_j's are even labeling:
Then $\phi(\langle x_0, y_0\rangle) = \tau - 1$, $\phi(x_i) = 2i + 1$, $\phi(y_j) = 2m + 2j$, $\phi(\langle x_0, x_i\rangle) = 2i$, and $\phi(\langle y_0, y_j\rangle) = 2m + 2j + 1$ with $mod(\tau + 1)$.

b) If x_i's are even labelings and y_j's are odd labelings:
Then $\phi(\langle y_0, x_0\rangle) = 2$, $\phi(x_i) = 2i + 2$, $\phi(y_j) = 2m + 2j + 1$, $\phi(\langle x_0, x_i\rangle) = 2i + 1$, and $\phi(\langle y_0, y_j\rangle) = 2m + 2j + 2$.
In both cases, we check that $\phi(u)$, $\phi(v)$, and $\phi(\langle u, v\rangle)$ are pairwise coprime for all connected vertex pairs and bridge edge directions are different.
Hence, Bi-Star is edge-vertex prime labeled when τ is a prime number.
Conversely,
Assume that the Bi-Star digraph $\mathcal{D} = \overrightarrow{B}(m,n)$ admits an edge-vertex prime labeling.
Let the vertex set be: $V(\mathcal{D}) = \{x_0, x_1, \ldots, x_m, y_0, y_1, \ldots, y_n\}$, where x_0 and y_0 are the central vertices, and others are pendent vertices .
Let the edge set be: $E(\mathcal{D}) = \{\langle x_0, x_i\rangle \mid 1 \le i \le m\} \cup \{\langle y_0, y_j\rangle \mid 1 \le j \le n\} \cup \{\langle y_0, x_0\rangle\}$.
Suppose $\phi : V(\mathcal{D}) \cup E(\mathcal{D}) \to \{1, 2, \ldots, \tau\}$ is a bijection such that:

- $\phi(u)$, $\phi(v)$, and $\phi(\langle u, v\rangle)$ are pairwise relatively prime for every edge $\langle u, v\rangle \in V(\mathcal{D})$.
- All labels are distinct and from the set $\{1, 2, \ldots, \tau\}$, where $\tau = |V| + |E| = (m + n + 2) + (m + n + 1) = 2m + 2n + 3$.

We now define a new labeling σ based on ϕ, and show that it satisfies the definition of a sequential labeling:
$\sigma : V(\mathcal{D}) \cup E(\mathcal{D}) \to \{1, 2, \ldots, \tau\}$ such that for every edge $\langle u, v\rangle \in E(\mathcal{D})$, the following congruence holds:
$\sigma(\langle u, v\rangle) \equiv (\sigma(v) - \sigma(u)) \pmod{\tau + 1}$.

Since ϕ is already a bijection over τ elements and assigns pairwise coprime values to all adjacent structures, we set $\sigma(x) := \phi(x) \quad$ for all $x \in V(\mathcal{D}) \cup E(\mathcal{D})$.

Now for each edge $\langle u, v\rangle$, consider $\sigma(\langle u, v\rangle) \equiv |\sigma(v) - \sigma(u)| \pmod{\tau + 1}$.

Since σ maps to a consecutive range starting from 1 and all values are distinct, the modulus $\tau + 1$ ensures that the absolute difference condition wraps correctly, and no collision occurs due to modulus overflow. Additionally, due to pairwise coprimality:$\gcd(\sigma(u), \sigma(v)) = 1 \quad \Rightarrow \quad |\sigma(v) - \sigma(u)|$ is coprime with both $\sigma(u), \sigma(v)$, which supports the modular difference congruence with uniqueness.

Hence, such a mapping σ satisfies the modulus-based definition of a sequential labeling.

Therefore, if the Bi-Star digraph admits an edge-vertex prime labeling, then it is sequential. □

Note 3. If the labels assigned to the pendent vertices $\{x_i\}$ and $\{y_j\}$ are all of the same parity (i.e., every x_i and every y_j are all even, or every x_i and every y_j are all odd), then neither an edge–vertex prime labeling nor a sequential labeling exists for $\overrightarrow{B}(m, n)$.

Note 4. A coconut tree digraph is a directed graph obtained by attaching a directed star to one end of a directed path.

Theorem 2.5. *The Coconut Tree digraph $\mathcal{C}\overrightarrow{T}_{m,n}$ is sequential $\iff$ it admits edge-vertex prime labeling.*

Proof. Let $\mathcal{D} = \mathcal{C}\overrightarrow{T}_{m,n}$ be a coconut tree. Assume $\mathcal{C}\overrightarrow{T}_{m,n}$ sequentially labeled digraph.

Let vertex set be $V(\mathcal{D}) = \{x_i, z_j \mid 1 \le i \le n, 1 \le j \le m\}$, where x_i are vertices on a directed path and z_j are pendant vertices.

Edge set:

$$E(\mathcal{D}) = \{\langle x_i, x_{i+1}\rangle \mid 1 \le i \le n-1\} \cup \{\langle x_1, z_j\rangle \mid 1 \le j \le m\}$$

Define a bijection $\phi : V(\mathcal{D}) \cup E(\mathcal{D}) \to \{1, 2, \ldots, \tau\}$, where $\tau = 2m + 2n - 1$.

Assign: $\phi(x_i) = \{1, \tau - 2(n-2), \tau, \tau - 2(n-3), \tau - 2, \tau - 2(n-4), \tau - 4, \tau - 2(n-5), \ldots\}$, $\phi(z_j) = \{3, 5, 7, \ldots, \tau - 2(n+1), \tau - 2(n), \tau - 2(n-1)\}$ for$1 \le i \le n,\ 1 \le j \le m$, $\phi(E(\mathcal{D})) = \{2, 4, 6, \ldots, \tau - 3, \tau - 1\}$.

We claim that each $\phi(u)$, $\phi(v)$, and $\phi(\langle u, v\rangle)$ for adjacent vertices and edges are pairwise coprime.

That is: $\phi(x_i) \perp \phi(x_{i+1}), \quad \phi(x_1) \perp \phi(z_j), \quad \phi(x_i) \perp \phi(\langle x_i, x_{i+1}\rangle), \quad \phi(x_1) \perp \phi(\langle x_1, z_j\rangle)$.

Hence, $\mathcal{C}\overrightarrow{T}_{m,n}$ is edge-vertex prime labeled if τ is a prime number.

Conversely,

Suppose $\mathcal{C}\overrightarrow{T}_{m,n}$ admits an edge-vertex prime labeling $\phi : V(\mathcal{D}) \cup E(\mathcal{D}) \to \{1, 2, \ldots, \tau\}$, where $\tau = |V(\mathcal{D})| + |E(\mathcal{D})| = 2m + 2n - 1$, and all labels are distinct. Assume ϕ satisfies the condition that for every edge $\langle u, v\rangle$, the values $\phi(u)$, $\phi(v)$, and $\phi(\langle u, v\rangle)$ are pairwise coprime.

Define a new labeling $\sigma : V(\mathcal{D}) \cup E(\mathcal{D}) \to \{1, 2, \ldots, \tau\}$ as $\sigma = \phi$. For every edge $\langle u, v\rangle \in E(\mathcal{D})$, we compute $|\sigma(v) - \sigma(u)|$. Since $\phi(\langle u, v\rangle)$ is coprime with both

$\phi(u)$ and $\phi(v)$, and all values lie in $\{1, \ldots, \tau\}$, it follows that
$\sigma(\langle u, v\rangle) \equiv (\sigma(v) - \sigma(u)) \pmod{\tau + 1}$.
Therefore, σ satisfies the condition for $\mathcal{C}\overrightarrow{T}_{m,n}$ to be sequential with $r = 1$.
Hence, $\mathcal{C}\overrightarrow{T}_{m,n}$ is sequential. □

Example 2.6. A Sequential and Edge vertex prime labeling of Coconut tree $\mathcal{C}\overrightarrow{T}_{13,8}$.

Note 5. We observe that reversing the orientation of edges while retaining the same labeling still yields both sequential and prime labeling in directed Coconut trees.

Note 6. A Double Coconut Tree digraph is a directed graph formed by attaching two directed stars at the two terminal ends of a directed path.

Theorem 2.7. *The Double Coconut Tree digraph $\overrightarrow{D}(n, p, m)$ for $p = 3$ is sequential $\iff$ it admits edge-vertex prime labeling.*

Proof. Let $\overrightarrow{D}(n, 3, m)$ be sequentially labeled with vertex set: $V(\overrightarrow{D}) = \{x_i, y_j, z_k \mid 1 \le i \le n, 1 \le j \le m, 1 \le k \le 3\}$ where x_i and y_j are pendant vertices and z_k are path vertices.
Edge set:$E(\overrightarrow{D}) = \{\langle z_1, x_i\rangle \mid 1 \le i \le n\} \cup \{\langle z_3, y_j\rangle \mid 1 \le j \le m\} \cup \{\langle z_1, z_2\rangle, \langle z_3, z_2\rangle\}$
Define a bijection $\phi : V(\overrightarrow{D}) \cup E(\overrightarrow{D}) \to \{1, 2, \ldots, \tau\}$ as:

$$\phi(z_1) = 1, \quad \phi(z_2) = 2n + 3, \quad \phi(z_3) = \tau \quad where \quad \tau = 2n + 2m + p + 2$$

$$\phi(x_i) = 2i + 1, \quad \phi(\langle z_1, x_i\rangle) = 2i$$

$$\phi(y_j) = 2n + 2j + 3, \quad \phi(\langle z_1, z_2\rangle) = 2n + 2, \quad \phi(\langle z_3, z_2\rangle) = 2n + 4$$

Now verify pairwise coprimality:

$$\phi(z_1) \perp \phi(x_i), \quad \phi(z_2) \perp \phi(z_1), \quad \phi(z_2) \perp \phi(z_3), \quad \phi(z_3) \perp \phi(y_j)$$

Hence, $\overrightarrow{D}$ is edge-vertex prime labeled when τ is prime.
Conversely,
Suppose $\overrightarrow{D}(n, 3, m)$ admits prime labeling. Then:

$$|V(\overrightarrow{D})| = n + m + 3, \quad |E(\overrightarrow{D})| = n + m + 2, \quad \Rightarrow \tau = 2n + 2m + 5.$$

Since labels are distinct and fulfill the edge-vertex coprimality, $\mathcal{D}$ is sequential. □

Note 7. We observe that changing the edge directions in $\overrightarrow{D}(n, 3, m)$ with the same labels still results in sequential and edge-vertex prime labeling.

Theorem 2.8. *If a directed tree $\mathcal{T}$ is prime labeled and every vertex $x \in V(\mathcal{T})$ has an odd label $\psi(x)$, then the disjoint union of two copies of $\mathcal{T}$ is sequential.*

Proof. Let $\mathcal{T}$ be a directed tree with each vertex x labeled by an odd number $\psi(x)$ with m vertices.
Assign:

$\psi(x_i) \in \{2i+1 \mid 0 \le i \le m-1\}$, and edges: $\psi(\langle x_i, x_j\rangle) \in \{2i+2 \mid 0 \le i \le m-2\}$

For the disjoint union $\mathcal{T} \cup \mathcal{T}$, total vertices $= 2m$, edges $= 2m - 2$.
Label the second copy's vertices $\{x'_1, x'_2, \ldots, x'_m\}$ corresponding to $\psi(x_i)$ as:

$$\psi(x'_i) = 4m - 1 - \psi(x_i) \quad \text{(ensures all values are even)}$$

These are coprime with all odd-labeled original vertices and form a valid sequential labeling over the combined structure. □

Example 2.9. Below Figure 4 shows the disjoint union of two directed trees. Each directed tree is an out-star in which the central vertex has outgoing edges to all its pendant vertices. The first tree is rooted at vertex 1 with outgoing edges to the vertices $\{3, 5, 7, 9, 11, 13\}$. The second tree is rooted at vertex 26 with outgoing edges to the vertices $\{14, 16, 18, 20, 22, 24\}$. Since the two trees share no vertices and no edges, their union forms a disjoint union of two directed out-trees.

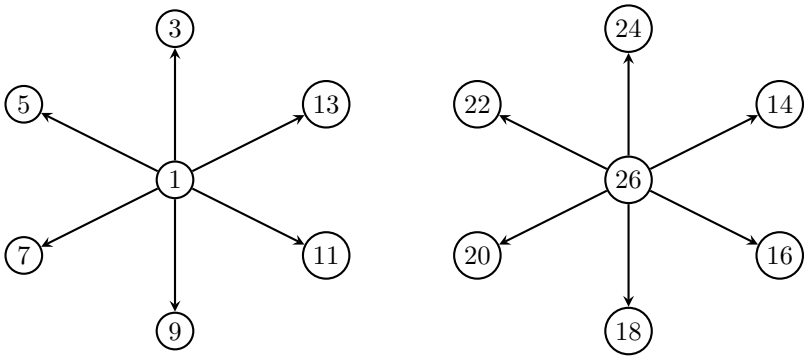


FIGURE 4. A Disjoint union of two copies of directed tree

Theorem 2.10. *If directed cycles $\overrightarrow{\mathcal{C}}_n$ and $\overrightarrow{\mathcal{C}}_m$ are prime labeled and $n \neq m$, then their disjoint union is sequential.*

Proof. Let $\overrightarrow{D} = \overrightarrow{\mathcal{C}}_n$ and $\overrightarrow{\mathcal{C}}_m$ be prime labeled digraphs with vertex set:

$$V(\overrightarrow{D}) = \{1, 2, 3, \ldots, t\}, \quad \text{where } t = 2m + 2n, \quad \phi(x) \perp \phi(y) \text{ for } \langle x, y\rangle \in E(\overrightarrow{D})$$

In the disjoint union $\overrightarrow{\mathcal{C}}_n \cup \overrightarrow{\mathcal{C}}_m$, the t unique labels produce t uniquely labeled edges. Therefore, the digraph is sequential. □

Example 2.11. Sequential and Prime labeling of disjoint union of $\overrightarrow{\mathcal{C}}_5 \cup \overrightarrow{\mathcal{C}}_6$.

Note 8. If $n = m$, then some vertex labels may share common divisors, violating the prime condition.

3. Prime and Sequentially Labeled Digraph Description

3.1. Quantum Walk Framework for Secure Navigation. Quantum walks, introduced in foundational work by Aharonov, Davidovich, and Zaguary [15] and later reviewed by Kempe [11], serve as the backbone of quantum-inspired algorithms in search, navigation, and cryptographic applications. We model the robot's behavior using a discrete-time coined quantum walk over a specially labeled

digraph, where the robot's state evolves through a sequence of coin and shift operations. This framework leverages quantum-inspired randomness while enabling deterministic, cost-sensitive traversal decisions.

3.2. Hilbert Space Representation. The system's quantum state resides in the composite Hilbert space[9, 11]:

$$\mathcal{H} = \mathcal{H}_{\text{position}} \otimes \mathcal{H}_{\text{direction}},$$

where $\mathcal{H}_{\text{position}}$ represents the current vertex, and $\mathcal{H}_{\text{direction}}$ stores the previous vertex (simulating direction of travel).

3.3. Coin Operator Definition. At each vertex v with outgoing edges to neighbors $\{u_1, u_2, \ldots, u_d\}$ having respective weights $w_1, w_2, \ldots, w_d$, the normalized transition probabilities are given by:

$$p(u_i|v) = \frac{w_i}{\sum_{j=1}^{d} w_j}.$$

The coin operator creates a superposition over possible directions with amplitudes proportional to the square roots of these probabilities:

$$C_v|v, u_{\text{prev}}\rangle = \sum_{i=1}^{d} \sqrt{p(u_i|v)}|v, u_i\rangle.$$

In our modified model for secure navigation, the robot does not evolve into a full superposition but deterministically chooses the path corresponding to the **minimum amplitude**, i.e., the least probable direction.

3.4. Shift Operator Definition. The shift operator simulates movement across an edge by swapping the current and previous vertices:

$$S|v, u\rangle = |u, v\rangle.$$

This captures both the robot's current position and its movement history.

3.5. Quantum Evolution. The quantum walk evolves over discrete time steps by alternating the coin and shift operations [11, 10, 12]:

$$|\psi(t+1)\rangle = S \cdot C|\psi(t)\rangle,$$

where $C = \bigoplus_v C_v$ is the global coin operator acting vertex-wise, and S is the shift operator defined over the digraph topology .

4. Objective and Initialization

The digraph in Fig. 5 forms the structural basis for our quantum walk simulation, supporting entropy analysis, cryptographic encoding, and secure path evaluation. Our objective is to navigate from vertex 1 to vertex 13 using a minimum amplitude traversal strategy to maintain stealth and security.

Before initiating the quantum walk, the robot first analyzes all feasible paths in the digraph and evaluates their transition amplitudes. Using the coin and shift

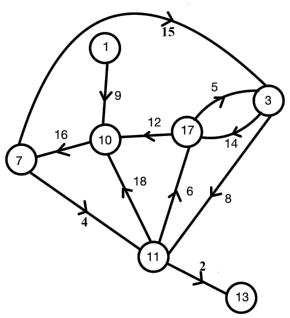


FIGURE 5. A prime sequentially labeled digraph.

operators, it then predicts the optimal traversal sequence. The initial quantum state of the robot is defined as

$$|\psi_0\rangle = |1, 10\rangle.$$

5. Illustrative Example of the Minimum-Amplitude Traversal Strategy

To illustrate the proposed navigation strategy, we simulate a quantum-inspired traversal from the source vertex 1 to the destination vertex 13 over a prime and sequentially labeled digraph using the **minimum amplitude strategy**. At each step, the robot selects the outgoing edge with the lowest normalized transition probability (i.e., least amplitude), which corresponds to the most unpredictable or least detectable path.

Step 1: Shift to Vertex 10

The initial state of the robot is:

$$|\psi_0\rangle = |1, 10\rangle,$$

indicating it begins at vertex 1 and is about to move to vertex 10. The shift operator swaps the current and previous vertices:

$$S|1, 10\rangle = |10, 1\rangle.$$

Now, the robot is at vertex 10, having come from vertex 1.

Step 2: Coin at Vertex 10

At vertex 10, there is only one outgoing edge:

$$10 \to 7 \quad \text{(weight 16)},$$

so the coin operator applies trivially:

$$C_{10}|10, 1\rangle = |10, 7\rangle.$$

Step 3: Shift to Vertex 7

The robot transitions from vertex 10 to vertex 7:

$$S|10, 7\rangle = |7, 10\rangle.$$

Step 4: Coin at Vertex 7
Vertex 7 has two outgoing edges:

- $7 \to 3$ with weight 15,
- $7 \to 11$ with weight 4.

We compute the normalized probabilities:

$$p(3) = \frac{15}{15+4} = \frac{15}{19}, \quad p(11) = \frac{4}{19}.$$

According to the minimum amplitude policy, we select the path with the **lowest probability**:

$$C_7|7,10\rangle = |7,11\rangle.$$

Step 5: Shift to Vertex 11
The robot now transitions to vertex 11:

$$S|7,11\rangle = |11,7\rangle.$$

Step 6: Coin at Vertex 11
Vertex 11 has three outgoing edges:

- $11 \to 10$ with weight 18,
- $11 \to 17$ with weight 6,
- $11 \to 13$ with weight 2.

The transition probabilities are:

$$p(10) = \frac{18}{26}, \quad p(17) = \frac{6}{26}, \quad p(13) = \frac{2}{26}.$$

The minimum amplitude corresponds to the least probable transition:

$$C_{11}|11,7\rangle = |11,13\rangle.$$

Step 7: Shift to Vertex 13
Final shift operation takes the robot to vertex 13:

$$S|11,13\rangle = |13,11\rangle.$$

At this point, the robot has successfully reached its destination while strictly adhering to the minimum amplitude policy at each decision point.

Final Path (Secured). The full transition sequence is:

$$|1,10\rangle \to |10,1\rangle \to |10,7\rangle \to |7,10\rangle \to |7,11\rangle \to |11,7\rangle \to |11,13\rangle \to |13,11\rangle$$

Final Quantum State: $|13,11\rangle$

Final Path$1 \to 10 \to 7 \to 11 \to 13$ (Entropy = 31)
This path avoids commonly used or high-probability transitions, favoring edges with lower transition likelihoods. From a security standpoint, such paths:

- Reduce the risk of interception or prediction by adversaries.
- Increase entropy across the path, maximizing unpredictability.
- Enable entropy-masked message transmission by embedding the encryption route within a minimally observable trajectory.

This demonstrates how entropy-aware quantum-inspired traversal can achieve both secure navigation and cryptographic robustness in robotic systems.

Although this model does not preserve full quantum coherence, it reflects a hybrid behavior where quantum inspired probabilistic logic is fused with classical decision-making principles.

This framework may be extended into hybrid quantum-classical algorithms, where initial exploration uses unitary quantum walks, followed by amplitude-based pruning for path optimization. Analyzing the theoretical limits, classical simulability, and real-time deployment feasibility remains an open direction.

5.1. Path Entropy Masking. Dummy edges increase apparent entropy. The true path is:

$$1 \to 10 \to 7 \to 11 \to 13 \quad (\text{Entropy} = 31)$$

Masked path:

$$1 \to 10 \to 7 \to 3 \to 11 \to 13 \quad (\text{Entropy} = 50)$$

TABLE 1. Cipher table for character-to-integer and binary mapping

Sl. No.	Character	Sl. No.	Character	Sl. No.	Character	Sl. No.	Character	Sl. No.	Character
1	A	21	U	41	o	61	&	81	ˆ
2	B	22	V	42	p	62	’	82	-
3	C	23	W	43	q	63	(	83	‘
4	D	24	X	44	r	64	)	84	{
5	E	25	Y	45	s	65	*	85	—
6	F	26	Z	46	t	66	+	86	\
7	G	27	a	47	u	67	,	87	∼
8	H	28	b	48	v	68	-	88	≠
9	I	29	c	49	w	69	.	89	∇·
10	J	30	d	50	x	70	/	90	×
11	K	31	e	51	y	71	:	91	¥
12	L	32	f	52	z	72	;	92	@
13	M	33	g	53	α	73	¡	93	/
14	N	34	h	54	β	74	=	94	≤
15	O	35	i	55	γ	75	¿	95	≥
16	P	36	j	56	!	76	?	96	...
17	Q	37	k	57	”	77	±	97	space
18	R	38	l	58	#	78	[	98	©
19	S	39	m	59	$	79	]	99	≈
20	T	40	n	60	%	80	\		

5.2. Message Entropy Masking.

Step 1: Character Encoding. Using TABLE 1, we extract the corresponding integer values for each character in the message. These integers are then encoded into binary strings to facilitate entropy-based masking.

- `O` $\Rightarrow$ 15
- `k` $\Rightarrow$ 37

These are then converted to 8-bit binary:

$$\texttt{msg_bin} = 00001111 \quad 00100101$$

Step 2: Key Generation from edge Weights. The edge weights used in the path are: [9, 16].

Their binary representations are:

$$\texttt{key_bin} = 00001001 \quad 00010000$$

Step 3: Entropy Mask from Dummy edge. Using dummy edge weight 15:

$$\texttt{entropy_mask} = 00001111 \quad 00001111$$

Step 4: Encryption Process.

(1) Intermediate XOR of message and key:

$$\texttt{intermediate} = 00001111 \oplus 00001001 = 00000110, \quad 00100101 \oplus 00010000 = 00110101$$

(2) Final XOR with entropy mask:

$$\texttt{cipher} = 00000110 \oplus 00001111 = 00001001, \quad 00110101 \oplus 00001111 = 00111010$$

Step 5: Final Ciphertext. Binary to decimal conversion:

- $00001001 = 9 \Rightarrow$ `I`
- $00111010 = 58 \Rightarrow$ `#`

$$\texttt{Ciphertext} = \texttt{I\#}$$

5.3. Message Decryption Process. To validate the correctness of our encryption technique, we now decrypt the ciphertext `"I#"` back to the original message `"Ok"` using the same key and entropy mask.

Step 1: Ciphertext to Binary. From the cipher table:

- `I` $\Rightarrow 9 \Rightarrow$ `00001001`
- `#` $\Rightarrow 58 \Rightarrow$ `00111010`

$$\texttt{cipher} = 00001001 \quad 00111010$$

Step 2: Remove Entropy Mask. Entropy mask from dummy edge (weight 15) is:

$$\texttt{entropy_mask} = 00001111 \quad 00001111$$

Now, perform XOR with the cipher to retrieve the intermediate:

$$\begin{aligned} \texttt{intermediate} &= \texttt{cipher} \oplus \texttt{entropy_mask} \\ &= 00001001 \oplus 00001111 = 00000110 \\ &\quad\ 00111010 \oplus 00001111 = 00110101 \end{aligned}$$

Step 3: Remove Key to Get Message. The key derived from edge weights [9, 16] is:

$$\texttt{key_bin} = 00001001 \quad 00010000$$

Now perform XOR:

$$\begin{aligned} \texttt{msg_bin} &= \texttt{intermediate} \oplus \texttt{key_bin} \\ &= 00000110 \oplus 00001001 = 00001111 \\ &\quad\ 00110101 \oplus 00010000 = 00100101 \end{aligned}$$

Step 4: Convert Binary to Characters. Use the cipher table to decode:

- $00001111 = 15 \Rightarrow$ `O`
- $00100101 = 37 \Rightarrow$ `k`

`Original Message` = `Ok`

This confirms the correctness of the encryption and decryption process using entropy masking.

6. conclusion

This study has explored the relationship between the sequential and Prime labeling of directed graphs, with a specific focus on the Star graph, Bi-Star graph, Coconut Tree, Double Coconut tree, $\overrightarrow{C}_n$ and $\overrightarrow{C}_m$, and $T \cup T$. We discovered that for some graphs, achieving both sequential and prime labeling is not possible due to the condition for prime labeling not being met. Specifically, if a digraph is sequential, it may not fulfill the requirements for prime labeling, as the longest common divisor of adjacent vertices is not equal to 1. It would be interesting to conduct future research work on this topic and find more graphs that satisfy the sequential and prime labeling. The proposed system leverages quantum-inspired navigation and entropy-based encryption to enhance security in robotic communication. By embedding entropy at both path and message levels, and using coined quantum walk evolution, the robot achieves robust, trust-aware, and stealthy movement through structured environments.

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