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BURGERS FISHER EQUATION AND ITS GENERALIZATIONS VIA GENERALIZED COLE-HOPF TRANSFORMATION

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Abstract. Using the simplified homogeneous balance method, the generalized Cole–Hopf transformation to the generalized forced Burgers Fisher equation, with space and time-variable coefficients, has been obtained. Using this transformation, exact solutions of the equation are obtained. A more general version of the Cole-Hopf transformation is studied to convert a class of non-linear parabolic equations into linear parabolic equations.

1. Introduction

The viscous Burgers equation [1]

$$u_t + uu_x - \nu u_{xx} = 0, \tag{1.1}$$

is a fundamental nonlinear convection–diffusion equation in which nonlinear steepening is balanced by viscous diffusion. Owing to this simple but representative structure, it has been used in the study of nonlinear waves, shock propagation, gas dynamics, turbulence and several related transport processes [2, 3, 4, 5, 6, 7, 8]. A distinctive property of equation (1.1) is that it can be reduced to the linear heat equation

$$\psi_t - \nu\psi_{xx} = 0, \tag{1.2}$$

by the Cole–Hopf transformation

$$u(x,t) = -2\nu \ (\ln\psi)_x = -2\nu \ \frac{\psi_x}{\psi}. \tag{1.3}$$

This linearization has motivated the construction of Cole–Hopf-type and Bäcklund transformations for wider classes of nonlinear evolution equations.

Several generalizations of this idea have been developed for higher-dimensional, higher-order, non-homogeneous and variable-coefficient Burgers equations. Wang [9] obtained a Bäcklund transformation for a compound higher-order Burgers equation, while Chen et al. [10] extended the Cole–Hopf transformation to an n-dimensional Burgers system. Xu et al. [11] considered a forced Burgers model with

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space- and time-dependent coefficients and derived a generalized Hopf–Cole transformation together with the corresponding constraints on the coefficient and forcing functions. Generalized transformations for nonlinear parabolic and Burgers-type equations were also studied in [12, 13, 14, 15, 16]. More recently, Wang et al. [17] used the simplified homogeneous balance method to derive a generalized Cole–Hopf transformation for a variable-coefficient Burgers equation with a linear damping term.

Another basic nonlinear diffusion model is the Fisher equation

$$u_t - u_{xx} = u(1-u), \tag{1.4}$$

which was introduced in population genetics [18]. By incorporating nonlinear convection, one obtains the Burgers–Fisher equation

$$u_t + \alpha u u_x - u_{xx} = \beta u(1-u), \tag{1.5}$$

which combines reaction, convection and diffusion in a single model. Exact and travelling-wave solutions of Burgers equations with reaction terms and of their generalized or fractional forms have been investigated by several authors [19, 20, 21]. Unlike the classical Burgers equation, however, a general Burgers–Fisher equation is not directly linearized by the standard Cole–Hopf transformation.

Variable coefficients are useful when the strengths of convection, diffusion and reaction change with position or time. Shang and Chen [22] studied a generalized homogeneous Burgers–Fisher equation with spatiotemporal variable coefficients. By the simplified homogeneous balance method, they established a Bäcklund transformation to a variable-coefficient linear parabolic equation and, for a suitable subclass, a generalized Cole–Hopf transformation to the heat equation. Their work raises the related question of how an external source may be incorporated into such a linearizable variable-coefficient Burgers–Fisher model and whether the same logarithmic-derivative idea can be placed in a broader framework for nonlinear parabolic equations.

Motivated by this question, we consider the non-homogeneous generalized Burgers–Fisher equation

$$u_t + \alpha(x,t)uu_x - \beta(x,t)u_{xx} + \gamma(x,t)u + \delta(x,t)u^2 = G(x,t), \tag{1.6}$$

which is converted into a linear parabolic partial differential equation by obtaining a suitable transformation using the simplified homogeneous balance method. Exact solutions of the Burgers–Fisher equation (1.6) are obtained in terms of solutions to related linear heat conduction equations. We characterize a class of coefficient-source combinations for which the forced equation remains exactly reducible to a linear heat conduction equation. Thus, the source term forms part of the compatibility structure of the linearizable equation. This distinguishes the present problem both from the homogeneous variable-coefficient Burgers–Fisher equation in [22] and from forced Burgers equations without the quadratic reaction term [11, 12].

Furthermore, a generalized version of the Cole-Hopf transformation is also derived to convert a class of non-linear parabolic partial differential equations into a linear parabolic heat conduction equation.

The structure of the paper is as follows. In Section 2, the simplified homogeneous balance method is used to obtain a generalized Cole–Hopf transformation and the compatibility conditions under which equation (1.6) is reduced to a linear heat equation; exact solutions and illustrative special cases are presented. In Section 3, a more general transformation is introduced to identify a class of nonlinear parabolic equations that can be mapped to linear variable-coefficient parabolic equations. Section 4 contains the concluding remarks.

2. Reduction of Forced Burgers Fisher Equation to a Linear PDE

Consider the Burgers–Fisher equation;

$$u_t + \alpha(x,t)uu_x - \beta(x,t)u_{xx} + \gamma(x,t)u + \delta(x,t)u^2 = G(x,t). \tag{2.1}$$

By means of the simplified homogeneous balance method [23], we propose the following transformation,

$$u(x,t) = f(x,t)\frac{\phi_x(x,t)}{\phi(x,t)} + g(x,t), \tag{2.2}$$

where $f(x,t)$, $g(x,t)$ and $\phi(x,t)$ are functions to be determined so that equation (2.1) is transformed to the linear partial differential equation. From equation (2.2), we obtain;

$$u_t = f_t\frac{\phi_x}{\phi} + f\frac{\phi_{xt}}{\phi} - f\frac{\phi_x\phi_t}{\phi^2} + g_t, \tag{2.3}$$

$$\begin{aligned}\alpha uu_x = {} & \alpha f f_x\frac{\phi_x^2}{\phi^2} + \alpha f^2\frac{\phi_x\phi_{xx}}{\phi^2} - \alpha f^2\frac{\phi_x^3}{\phi^3} + \alpha f g_x\frac{\phi_x}{\phi} + \alpha f_x g\frac{\phi_x}{\phi} \\ & + \alpha fg\frac{\phi_{xx}}{\phi} - \alpha fg\frac{\phi_x^2}{\phi^2} + \alpha gg_x,\end{aligned} \tag{2.4}$$

$$\begin{aligned}\beta u_{xx} = {} & \beta f_{xx}\frac{\phi_x}{\phi} + 2\beta f_x\frac{\phi_{xx}}{\phi} - 2\beta f_x\frac{\phi_x^2}{\phi^2} + \beta f\frac{\phi_{xxx}}{\phi} - 3f\beta\frac{\phi_{xx}\phi_x}{\phi^2} \\ & + \beta g_{xx} + 2f\beta\frac{\phi_x^3}{\phi^3}.\end{aligned} \tag{2.5}$$

Substituting equations (2.2)-(2.5) into non-homogeneous Burgers Fisher equation (2.1) and then multiplying both sides of equation by ϕ^3, we find a polynomial equation of variables ϕ_x and ϕ as follows;

$$\begin{aligned}&\phi_x^3(-f(2\beta+\alpha f)) + \phi_x^2\phi(\alpha f f_x - \alpha fg + 2f_x\beta + \delta f^2) + \phi_x\phi^2(f_t + \alpha fg_x + \alpha f_x g \\ &- \beta f_{xx} + \gamma f + 2\delta fg) + \phi_x\phi(-f\phi_t + \alpha f^2\phi_{xx} + 3f\beta\phi_{xx}) \\ &+ \phi^2(f\phi_{xt} + \alpha fg\phi_{xx} - 2\beta f_x\phi_{xx} - \beta f\phi_{xxx}) \\ &+ \phi^3(g_t + \alpha gg_x - \beta g_{xx} + \gamma g + \delta g^2 - G(x,t)) = 0.\end{aligned} \tag{2.6}$$

Equating the coefficient of ϕ_x^3 to zero, we get

$$f(x,t) = \frac{-2\beta(x,t)}{\alpha(x,t)}. \tag{2.7}$$

By plugging equation (2.7) into equation (2.2), $u(x,t)$ may be represented as,

$$u(x,t) = \frac{-2\beta(x,t)}{\alpha(x,t)}\frac{\phi_x(x,t)}{\phi(x,t)} + g(x,t). \tag{2.8}$$

This transforms equation (2.6) into following equation

$$\begin{aligned}
&\phi_x^2\phi\left(2\beta g + \frac{4\delta\beta^2}{\alpha^2}\right) + \phi_x\phi^2\left(-\frac{2\beta_t}{\alpha} + \frac{2\beta\alpha_t}{\alpha^2} - 2\beta g_x - 2\beta_x g + \frac{2\beta\alpha_x g}{\alpha}\right.\\
&\left. + \frac{2\beta\beta_{xx}}{\alpha} - \frac{4\beta\beta_x\alpha_x}{\alpha^2} - \frac{2\beta^2\alpha_{xx}}{\alpha^2} + \frac{4\beta^2\alpha_x^2}{\alpha^3} - \frac{2\beta\gamma}{\alpha} - \frac{4\beta\delta g}{\alpha}\right) + \frac{2\beta}{\alpha}\phi_x\phi(\phi_t - \beta\phi_{xx})\\
&+ \phi^2\left(\frac{-2\beta}{\alpha}\phi_{xt} - 2\beta g\phi_{xx} + \frac{4\beta\beta_x}{\alpha}\phi_{xx} - \frac{4\beta^2\alpha_x}{\alpha^2}\phi_{xx} + \frac{2\beta^2}{\alpha}\phi_{xxx}\right)\\
&+ \phi^3\left(g_t + \alpha g g_x - \beta g_{xx} + \gamma g + \delta g^2 - G(x,t)\right) = 0.
\end{aligned} \tag{2.9}$$

One can determine $g(x,t)$ by equating coefficient of $\phi_x^2\ \phi$ to zero.

$$g(x,t) = \frac{-2\beta(x,t)\delta(x,t)}{\alpha^2(x,t)}. \tag{2.10}$$

Therefore, transformation, given in equation (2.8), will be turned into,

$$u(x,t) = -\frac{2\beta(x,t)}{\alpha(x,t)}\frac{\phi_x(x,t)}{\phi(x,t)} - \frac{2\beta(x,t)\delta(x,t)}{\alpha^2(x,t)}. \tag{2.11}$$

Applying which, equation (2.9) can be rewritten as,

$$\begin{aligned}
&\beta_x\left[\phi_x\phi^2\left(\frac{4\beta\delta}{\alpha^2} + \frac{4\beta}{\alpha^2}(\delta - \alpha_x)\right) + \frac{4\beta}{\alpha}\phi_{xx}\phi^2 + \phi^3\left(\frac{4\beta\delta^2}{\alpha^3} + \frac{4\beta\delta_x}{\alpha^2} - \frac{8\beta\delta\alpha_x}{\alpha^3}\right)\right]\\
&+ \beta_{xx}\left(\frac{2\beta}{\alpha}\phi_x\phi^2 + \frac{2\beta\delta}{\alpha^2}\phi^3\right) + \phi_x\phi^2\left(\frac{-2\beta_t}{\alpha} + \frac{2\beta\alpha_t}{\alpha^2} - \frac{2\beta\gamma}{\alpha} + \frac{\beta^2}{\alpha^2}(4\delta_x - 2\alpha_{xx})\right.\\
&\left. + \frac{\beta^2}{\alpha^3}(-8\delta\alpha_x - 4\alpha_x\delta + 4\alpha_x^2 + 8\delta^2)\right) + \frac{2\beta}{\alpha}\phi\phi_x(\phi_t - \beta\phi_{xx}) + \phi^2\left(-\frac{2\beta}{\alpha}\phi_{xt}\right.\\
&\left. + \frac{4\beta^2}{\alpha^2}\phi_{xx}(\delta - \alpha_x) + \frac{2\beta^2}{\alpha}\phi_{xxx}\right) + \phi^3\left(\frac{-2\beta_t\delta}{\alpha^2} - \frac{2\beta\delta_t}{\alpha^2} + \frac{4\beta\delta\alpha_t}{\alpha^3} + \frac{4\beta^2}{\alpha^3}(\delta\delta_x - \delta\alpha_{xx}\right.\\
&\left. - 2\delta_x\alpha_x) + \frac{4\beta^2}{\alpha^4}(\delta^3 - 2\delta^2\alpha_x + 3\delta\alpha_x^2) + \frac{2\beta^2}{\alpha^2}\delta_{xx} - \frac{2\beta\delta\gamma}{\alpha^2} - G(x,t)\right) = 0.
\end{aligned} \tag{2.12}$$

Let us consider the following two constraint conditions,

$$\beta_x(x,t) = 0, \quad \delta(x,t) = \alpha_x(x,t). \tag{2.13}$$

Rewriting equation (2.12) using equation (2.13) and multiplying by $\frac{-1}{\phi^4}$ gives,

$$\left(\frac{2\beta\gamma}{\alpha}+\frac{2\beta_t}{\alpha}-\frac{2\beta\alpha_t}{\alpha^2}-\frac{2\beta^2\alpha_{xx}}{\alpha^2}\right)\left(\frac{1}{\phi}\frac{\alpha_x}{\alpha}+\frac{\phi_x}{\phi^2}\right)+$$
$$\frac{1}{\phi}\left(\frac{2\beta\alpha_{xt}}{\alpha^2}-\frac{2\beta\alpha_x\alpha_t}{\alpha^3}+\frac{10\beta^2\alpha_x\alpha_{xx}}{\alpha^3}-\frac{8\beta^2\alpha_x^3}{\alpha^4}-\frac{2\beta^2\alpha_{xxx}}{\alpha^2}+G(x,t)\right)+$$
$$\frac{1}{\phi}\left(\frac{2\beta(\phi_t-\beta\phi_{xx})}{\alpha\phi}\right)_x+\frac{2\beta\alpha_x}{\alpha^2\phi^2}(\phi_t-\beta\phi_{xx})=0. \tag{2.14}$$

Outcomes from equation (2.14) are written in the following theorem.

Theorem 2.1. *If α, β, γ, δ satisfy constraint conditions*

$$\beta_x(x,t)=\ 0,\quad \delta(x,t)\ =\ \alpha_x(x,t)$$

and

$$\gamma+\frac{\beta_t}{\beta}-\frac{\alpha_t}{\alpha}-\beta\frac{\alpha_{xx}}{\alpha}=0 \tag{2.15}$$

and $G(x,t)$ is of the form,

$$-2\beta\left(\frac{\alpha_{xt}}{\alpha^2}-\frac{\alpha_x\alpha_t}{\alpha^3}\right)-\beta^2\left(\frac{10\alpha_x\alpha_{xx}}{\alpha^3}-\frac{8\alpha_x^3}{\alpha^4}-\frac{2\alpha_{xxx}}{\alpha^2}\right), \tag{2.16}$$

then, Burgers–Fisher equation;

$$u_t+\alpha(x,t)uu_x-\beta(x,t)u_{xx}+\gamma(x,t)u+\delta(x,t)u^2=G(x,t) \tag{2.17}$$

has a solution of the form

$$u(x,t)=-\frac{2\beta(t)}{\alpha(x,t)}\ \frac{\phi_x(x,t)}{\phi(x,t)}-\frac{2\beta(t)\alpha_x}{\alpha^2}, \tag{2.18}$$

where $\phi(x,t)$ is a solution of the linear heat conduction equation

$$\phi_t-\beta(t)\phi_{xx}=0. \tag{2.19}$$

Remark 2.2. If $\alpha_x(x,t)=0$, then from Theorem 2.1, $\gamma(t)=-\frac{\beta_t}{\beta}+\frac{\alpha_t}{\alpha}$, $\delta(x,t)=0$ and $G(x,t)=0$ and we can conclude that a solution of the equation

$$u_t+\alpha(t)uu_x-\beta(t)u_{xx}+\gamma(t)u=0, \tag{2.20}$$

can be written as,

$$u(x,t)=-\frac{2\beta(t)}{\alpha(t)}\frac{\phi_x(x,t)}{\phi(x,t)}, \tag{2.21}$$

where $\phi(x,t)$ satisfies equation (2.19).

Results of [17] can be treated as a special case of our paper.

Remark 2.3. If $\alpha=\alpha(x)$, then from Theorem 2.1, we can find that equation

$$u_t+\alpha(x)uu_x-\beta(t)u_{xx}+\left(\frac{\beta\alpha_{xx}}{\alpha}-\frac{\beta_t}{\beta}\right)u+\alpha_xu^2=G(x,t), \tag{2.22}$$

where $G(x,t)=-\beta^2(t)\left(\frac{10\alpha_x\alpha_{xx}}{\alpha^3}-\frac{8\alpha_x^3}{\alpha^4}-\frac{2\alpha_{xxx}}{\alpha^2}\right)$, has a solution of the form,

$$u(x,t) = -\frac{2\beta(t)}{\alpha(x)}\,\frac{\phi_x(x,t)}{\phi(x,t)} - \frac{2\beta(t)\alpha_x(x)}{\alpha^2(x)}, \tag{2.23}$$

where $\phi(x,\ t)$ is a solution of equation (2.19).

Remark 2.4. If $\alpha(x,t) = h(t)e^{ax}$, where $h(t)$ is a function of t and a is a constant, then equation

$$u_t + h(t)e^{ax}uu_x - \beta(t)u_{xx} + \left(-\frac{\beta_t}{\beta} + \frac{h_t}{h} + a^2\beta\right)u + ah(t)e^{ax}u^2 = 0, \tag{2.24}$$

has a solution of the form

$$u(x,t) = -\frac{2\beta(t)e^{-ax}}{h(t)}\left(\frac{\phi_x(x,t)}{\phi(x,t)} + a\right). \tag{2.25}$$

Remark 2.5. An exact solution [17] of equation (2.19) is given by

$$\phi(x,t) = 1 + \sum_{i=1}^{N} \exp\xi_i, \tag{2.26}$$

where $\xi_i = p_i x + p_i^2 \int^t \beta(\tau)d\tau + c_i$, p_i and c_i are constants. Using equation (2.26), an exact solution of the non-homogeneous Burgers-Fisher equation (2.17) can be written as

$$u(x,t) = -\frac{2\beta(t)}{\alpha(x,t)}\,\frac{\sum_{i=1}^{N} p_i \exp\xi_i}{1 + \sum_{i=1}^{N}\exp\xi_i} - \frac{2\beta(t)\alpha_x}{\alpha^2}. \tag{2.27}$$

3. Generalized Cole-Hopf Transformation

In this section, we propose a generalized form of the Cole-Hopf transformation (2.2) which can be written as;

$$F(u) = f(x,t)\frac{\phi_x(x,t)}{\phi(x,t)} + g(x,t). \tag{3.1}$$

We seek non-linear parabolic equations that are transformed by equation (3.1) into the following linear parabolic equation;

$$\phi_t + b(x,t)\phi_x + c(x,t)\phi = \epsilon a(x,t)\phi_{xx}. \tag{3.2}$$

From (3.1) and (3.2), we find;

$$
\begin{aligned}
f\frac{\phi_x}{\phi} &= F-g,\\
f\frac{\phi_{xx}}{\phi} &= F'u_x - \frac{f_x(F-g)}{f} + \frac{(F-g)^2}{f} - g_x,\\
f\frac{\phi_t}{\phi} &= -b(F-g) - cf + \epsilon a\bigg(F'u_x - g_x - \frac{f_x(F-g)}{f} + \frac{(F-g)^2}{f}\bigg),\\
f\frac{\phi_{xt}}{\phi} &= (F-g)\bigg[\frac{-b(F-g)}{f} - c + \frac{a\epsilon}{f}\bigg(F'u_x - g_x - \frac{f_x(F-g)}{f}\\
&\quad + \frac{(F-g)^2}{f}\bigg)\bigg] + F'u_t - \frac{f_t(F-g)}{f} - g_t,\\
f\frac{\phi_{xxx}}{\phi} &= \bigg(F''u_x^2 + F'u_{xx} - \bigg(\frac{f_x}{f}\bigg)_x(F-g) - \frac{F'u_xf_x}{f} + \frac{2F'u_x(F-g)}{f}\\
&\quad - \frac{f_x(F-g)^2}{f^2} - \frac{f_xg_x}{f} - g_{xx}\bigg) + (F-g-f_x)\bigg(\frac{F'u_x}{f} - \frac{f_x(F-g)}{f^2}\\
&\quad - \frac{3g_x}{f} + \frac{(F-g)^2}{f^2}\bigg).
\end{aligned}
\tag{3.3}
$$

Differentiating equation (3.2) with respect to x and then multiplying both sides by $\frac{f}{\phi}$, we get

$$
f\frac{\phi_{tx}}{\phi} + (b-\epsilon a_x)f\frac{\phi_{xx}}{\phi} + (b_x+c)f\frac{\phi_x}{\phi} + fc_x = a\epsilon f\frac{\phi_{xxx}}{\phi}. \tag{3.4}
$$

Substituting expressions of (3.3) into equation (3.4), we get

$$
\begin{aligned}
&u_t + u_x\bigg(-2\epsilon a\frac{(F-g)}{f} + b - \epsilon a_x + 2\epsilon a\frac{f_x}{f}\bigg) + u_x^2\bigg(-\epsilon a\frac{F''}{F'}\bigg) - \epsilon a u_{xx}\\
&+ \frac{(F-g)}{F'}\bigg(-\frac{f_t}{f} + 2g_x\frac{\epsilon a}{f} - \frac{bf_x}{f} + \epsilon a_x\frac{f_x}{f} + b_x - 2\epsilon a\frac{f_x^2}{f^2} + \epsilon a\frac{f_{xx}}{f}\bigg)\\
&+ \frac{(F-g)^2}{F'}\bigg(2\epsilon a\frac{f_x}{f^2} - \frac{\epsilon a_x}{f}\bigg) - \frac{g_t}{F'} - \frac{g_x}{F'}(b-\epsilon a_x) - 2\epsilon a\frac{f_xg_x}{fF'} + \epsilon a\frac{g_{xx}}{F'}\\
&+ \frac{fc_x}{F'} = 0.
\end{aligned}
\tag{3.5}
$$

If we choose $F(u)$ as a linear function then without the quadratic component u_x^2, equation (3.5) can be studied for different types of non-linear diffusion equations.

Further, if

$$
f = a,\ b = \epsilon a_x,\ g = -a_x, \tag{3.6}
$$

then the following theorem holds.

Theorem 3.1. *With the transformation*

$$
F(u) = a\frac{\phi_x}{\phi} - a_x, \tag{3.7}
$$

we can transform the following non-linear partial differential equation;

$$u_t - 2\epsilon F u_x - \epsilon a \frac{F''}{F'} u_x^2 - \epsilon a u_{xx} + \frac{(F + a_x)}{F'} \left(-\frac{a_t}{a} - \frac{2\epsilon a_x^2}{a} \right)$$
$$\frac{(F + a_x)^2 \epsilon a_x}{F' a} + \frac{a_{xt}}{F'} + \frac{2\epsilon a_x a_{xx}}{F'} - \frac{\epsilon a a_{xxx}}{F'} + \frac{a c_x}{F'} = 0, \tag{3.8}$$

into a linear parabolic partial differential equation

$$\phi_t + \epsilon a_x \phi_x + c\phi = \epsilon a \phi_{xx}. \tag{3.9}$$

Remark 3.2. If $F(u) = u$ in the Theorem 3.1, then equation (3.8) converts into following Burgers-Fisher equation;

$$u_t - 2\epsilon u u_x - \epsilon a u_{xx} - \frac{a_t}{a} u + \frac{\epsilon a_x}{a} u^2$$
$$+ \left(a_{xt} - \frac{a_x a_t}{a} - \frac{\epsilon a_x^3}{a} + 2\epsilon a_x a_{xx} - \epsilon a a_{xxx} + a c_x \right) = 0, \tag{3.10}$$

which can be transformed to the linear parabolic PDE (3.9), using transformation (3.7).

Remark 3.3. Equation (3.10) in remark (3.2) is the same as equation (2.1) with

$$\alpha = -2\epsilon,\ \beta = \epsilon a,\ \gamma = -\frac{a_t}{a},\ \delta = \frac{\epsilon a_x}{a}$$

and

$$G(x,t) = -\left(a_{xt} - \frac{a_x a_t}{a} - \frac{\epsilon a_x^3}{a} + 2\epsilon a_x a_{xx} - \epsilon a a_{xxx} + a c_x \right).$$

This can be considered as a case of equation (2.1), when α is a constant but other coefficients are in variable form.

4. Conclusion

We investigated a non-homogeneous Burgers-Fisher equation (1.6) with variable coefficients in this study. We applied the extended Cole-Hopf transformation (2.2) to equation (1.6) and discovered an exact solution of (1.6) when variable coefficients satisfy the constraint conditions (2.13) and (2.15) and the force term follows specific constraints. We also found a more general version of the Cole-Hopf transformation to find linear parabolic partial differential equations from a class of non-linear parabolic partial differential equations.

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