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QUALITATIVE ANALYSIS OF OSCILLATORY SOLUTIONS IN GENERALIZED NEUTRAL DIFFERENCE EQUATIONS

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Abstract. The oscillatory behavior of solutions to a class of second-order generalized nonlinear differential equations with a nonlinear neutral term is the focus of this research. Fractional difference operators are included in the model under consideration. The nonlinear exponents are taken as ratios of odd positive integers that meet certain requirements, and the order parameter falls within the interval $0 < v \leq 1$, The neutral term greatly affects the qualitative dynamics of the solutions, increasing the difficulty and mathematical richness of the analysis. Sufficient conditions are derived to ensure that all solutions of the given equation oscillate by using appropriate transformation techniques, inequality approaches, and comparison concepts. The validity of the obtained criteria is guaranteed by the assumptions placed on the coefficient functions. The findings in this paper add to and broaden previous research on neutral difference equations and nonlinear equations. Additionally, illustrative examples are provided to support the theoretical results and demonstrate the usefulness and relevance of the suggested oscillation criteria.

Keywords: Nonlinear non-positive neutral term, oscillation, second-order nonlinear difference equations, computational modeling and simulation.

Mathematical Subject Classification: 39A10, 39A12, 39A21, 39A22

1. INTRODUCTION

The realm of difference equations is a fascinating field of mathematics that has tremendous potential for application in various areas like operation research, ecology, population growth, economics, and biology. In addition, it has applications in computer science for assessing the performance of algorithms in iterative and discrete event simulations. Understanding oscillation is helpful in optimizing algorithms and predicting their run-time attributes. For a comprehensive overview of difference equations, accompanied by extensive examples spanning multiple disciplines, readers are directed to reference [2].

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The examination of difference equations hinges upon the operator denoted as Δ_ν termed $\Delta_\nu u(n) = u(n+\nu) - u(n), k \in \mathbb{N} = \{0, 1, 2, \cdots\}$. Despite numerous authors [1], [14] proposing definitions for Δ_ν as

$$\Delta_\nu u(n) = u(n+\nu) - u(n),\ \nu \in (0, \infty), \tag{1.1}$$

no significant advancements have occurred along this line.

2. REVIEW LITERATURE

In [11], authors adopted the definition of Δ_ν as provided in (1.1), they yield some significant results in [12, 13]. For convenience, they labeled the operator Δ_ν defined by (1.1) as Δ_ν.

Recently, authors have delved into investigating the oscillatory behavior of second order difference equations [15, 16].

Recently, there's been a substantial surge in research initiative concerning the oscillatory and asymptotic behavior of solutions of diverse classes of difference equations; for example, see [3, 4, 5, 6, 7, 8], and the papers listed below. For the oscillatory behavior of quasilinear difference equation one can find [17, 18, 19, 20, 21, 9, 10]. However, there seems to be a lack of known results regarding the oscillatory behavior of generalized second order difference equations represented by (2.1). This work seeks to present novel sufficient conditions ensuring the oscillatory behavior of the equation (2.1).

In light of the preceding discussion, we analyze the oscillation of solutions of generalized nonlinear second-order difference equations involving a nonlinear neutral term;

$$\begin{aligned}&\Delta_\nu \left(a(n\nu+\tau)\left(\Delta_\nu\left(u(n\nu+\tau) - p(n\nu+\tau)u^\nu((n-t)\nu+\tau)\right)\right)^\beta\right)\\ &+ q(n\nu+\tau)u^\gamma((n+1-m)\nu+\tau) = 0, k \geq k_0\end{aligned} \tag{2.1}$$

where $\Delta_\nu u(n) = u(n+\nu) - u(n), k \in \mathbb{N}_\nu(r)$, $r = k - \left[\frac{k}{\nu}\right]\nu$ and:

(i) ν, β, and γ are the quotient of odd positive integers with $\beta \geq \gamma$ and $0 < \nu \leq 1$;
(ii) $a(n\nu+\tau), p(n\nu+\tau)$ and $q(n\nu+\tau)$ are real valued functions for $k \geq k_0$, and $0 < p(n) < p_0 < 1$
(iii) m is a nonnegative integer and t is a positive integer;
(iv) $h(n) = k - m + t + 1 \leq k$, i.e., $m \geq t + 1$.

We set

$$A(v, u) = \sum_{s=u}^{v} \frac{1}{a^{1/\beta}(s\nu+\tau)} \text{ for } v \geq u \geq k_0$$

and assume that

$$A\left(n, k_0\right) \to \infty \text{ as } k \to \infty \tag{2.2}$$

Let $\theta = \max\{t, m-1\}$. A solution to equation (2.1) refers to a real-valued function $x(t)$ defined for all $k \geq k_0 - \theta$ that satisfies equation (2.1) for all $k \geq k_0$. A solution of equation (2.1) is said to be oscillatory if $u(n)u(n+\nu) \leq 0$; otherwise, it is termed as nonoscillatory. When all solutions of the equation are oscillatory, the equation is referred to as oscillatory.

3. MAIN RESULT

For $k \geq K$ for any $K \geq k_0$, we let

$$\mu(n\nu+\tau) = a^{1/\beta}(n\nu+\tau)A(n,K) \text{ and } Q(n\nu+\tau) = \sum_{s=k}^{\infty} q(s\nu+\tau)$$

For any constant $c > 0$, we take

$$g_c(n) = \begin{cases} 1, & \text{if } \gamma = \beta \\ cA^{(\gamma-\beta)/\beta}(n\nu+\tau), & \text{if } \gamma < \beta \end{cases} \tag{3.1}$$

Theorem 3.1. *Let conditions (i)-(iv) and* (2.2) *hold. Assume that for any constant $c > 0$ there exists a positive nondecreasing real valued function $\rho(n)$ such that,*

$$\limsup_{k\to\infty} \left(\rho(n\nu+\tau)Q(n\nu+\tau) + \sum_{s=k_2}^{k} [\rho(s\nu+\tau)q(s\nu+\tau) \right.$$
$$\left. - \frac{\beta^\beta}{(1+\beta)^{\beta+1}} \frac{a((n-m)\nu+k)}{(\gamma g_c(s))^\beta} \left(\frac{(\Delta_\nu \rho(s))^{\beta+1}}{\rho^\beta(s)} \right) \right] \right) = \infty \tag{3.2}$$

$$\limsup_{k\to\infty} \sum_{s=h(n)}^{k} A^{\gamma/\nu}(h(n\nu+\tau), h(s\nu+\tau))q(s\nu+\tau) > 1 \text{ if } \gamma = \nu\beta \tag{3.3}$$

and

$$\limsup_{k\to\infty} \sum_{s=h(n)}^{k} A^{\gamma/\nu}(h(n), h(s\nu+\tau))q(s\nu+\tau) = \infty \text{ if } \gamma < \nu\beta \tag{3.4}$$

Then equation (2.1) *is oscillatory.*

Proof. Let $u(n\nu+\tau)$ be a nonoscillatory solution of equation (2.1), say $u(n\nu+\tau) > 0, u((n-m+1)\nu+\tau) > 0$, and $u((n-t)\nu+\tau) > 0$ for $k \geq k_1$ for some $k_1 \geq k_0$. Then with $v(n\nu+\tau) = u(n\nu+\tau) - p(n\nu+\tau)u^\nu((n-t)\nu+\tau)$, it follows from (2.1) that

$$\Delta_\nu \left(a(n\nu+\tau)(\Delta_\nu v(n\nu+\tau))^\beta \right) = -q(n\nu+\tau)u^\gamma((n-m+1)\nu+\tau) \leq 0 \tag{3.5}$$

Hence, $a(n\nu+\tau)(\Delta_\nu y(n\nu+\tau))^\beta$ is nonincreasing and eventually of one sign. That is, there exists $k_2 \geq k_1$ such that $\Delta_\nu v(n\nu+\tau) > 0$ or $\Delta_\nu v(n\nu+\tau) < 0$

for $k \geq k_2$. We claim that $\Delta_\nu v(n\nu+\tau) > 0$ for $k \geq k_2$. To see this, assume that $\Delta_\nu v(n\nu+\tau) < 0$ for $k \geq k_2$. Then,

$$a(n\nu+\tau)(\Delta_\nu v(n\nu+\tau))^\beta \leq -c < 0 \text{ for } k \geq k_2$$

where $c = -a\left(n_2\nu+\tau\right)\left(\Delta_\nu v\left(n_2\nu+\tau\right)\right)^\beta < 0$, so

$$v(n\nu+\tau) \leq v\left(n_2\nu+\tau\right) - c^{1/\beta}\sum_{s=k_2}^{k} a^{-1/\beta}(s\nu+\tau)$$

In view of (2.2), $\lim_{k\to\infty} v(n\nu+\tau) = -\infty$. Next, we will explore the following two cases.

Case 1. If $u(n\nu+\tau)$ is unbounded, then there exists an real valued function k_n such that $\lim_{n\to\infty} v(n_n\nu+\tau) = \infty$ and $\lim_{n\to\infty} u\left(n_n\nu+\tau\right) = \infty$ where $u\left(n_t\nu+\tau\right) = \max\left\{u(s\nu+\tau) : k_0 \leq s \leq k_t\right\}$. This implies

$$u\left((n_n - m + 1)\nu+\tau\right) \leq \max\left\{u(s\nu+\tau) : k_0 \leq s \leq k_n\right\} = u\left(n_n\nu+\tau\right)$$

Therefore, since $u\left(n_n\right) \to \infty$ and (ii) holds for all large n,

$$\begin{aligned} v\left(n_n\nu+\tau\right) &= u\left(n_n\nu+\tau\right) - p\left(n_n\nu+\tau\right)u^\nu\left((n_n - t)\nu+\tau\right) \\ &\geq u\left(n_n\nu+\tau\right) - p\left(n_n\nu+\tau\right)u^\nu\left(n_n\nu+\tau\right) \\ &\geq \left(1 - \frac{p\left(n_n\nu+\tau\right)}{u^{1-\nu}\left(n_n\nu+\tau\right)}\right)u\left(n_n\nu+\tau\right) > 0 \end{aligned}$$

which contradicts the fact that $\lim\limits_{k\to\infty} v(n\nu+\tau) = -\infty$.

Case 2. If $u(n\nu+\tau)$ is bounded, then $v(n\nu+\tau)$ is also bounded, which contradicts $\lim_{k\to\infty} v(n\nu+\tau) = -\infty$. This completes the proof of the claim so we conclude that $\Delta_\nu v(n\nu+\tau) > 0$ for $k \geq k_2$.

Next, we encounter two possibilities to examine:

(a) $v(n\nu+\tau) > 0$ or
(b) $v(n\nu+\tau) < 0$ for $k \geq k_2$.

If (a) holds, then in view of (3.5) and the fact that $u(n\nu+\tau) \geq v(n\nu+\tau)$, we have

$$\Delta_\nu\left(a(n\nu+\tau)(\Delta_\nu v(n\nu+\tau))^\beta\right) \leq -q(n\nu+\tau)v^\gamma((n-m+1)\nu+\tau) \leq 0 \quad (3.6)$$

Summing $\Delta_\nu v$ from k_2 to k gives

$$\begin{aligned} v(n\nu+\tau) &= v\left(n_2\nu+\tau\right) + \sum_{s=k_2}^{k} \frac{\left(a(s\nu+\tau)(\Delta_\nu v(s\nu+\tau))^\beta\right)^{1/\beta}}{a^{1/\beta}(s\nu+\tau)} \\ &\geq a^{1/\beta}(n\nu+\tau)\Delta_\nu v(n\nu+\tau)\sum_{s=k_2}^{k} a^{-1/\beta}(s\nu+\tau) \\ &= \mu(n\nu+\tau)\Delta_\nu v(n\nu+\tau) \end{aligned} \quad (3.7)$$

Summing (3.6) from k to x, letting $x \to \infty$, and using the fact that $v(n\nu+\tau)$ is increasing, we have

$$\begin{aligned} a(n\nu+\tau)(\Delta_\nu v(n\nu+\tau))^\beta &\geq \sum_{s=k}^{\infty} q(s\nu+\tau)v^\gamma((s-m+1)\nu+\tau) \\ &\geq v^\gamma((n-m+1)\nu+\tau)\sum_{s=k}^{\infty} q(s\nu+\tau) := Q(n\nu+\tau)v^\gamma((n-m+1)\nu+\tau) \\ &\geq Q(n\nu+\tau)v^\gamma((n-m)\nu+\tau) \end{aligned} \tag{3.8}$$

Define

$$z(r+k\nu) = \rho(n\nu+\tau)\frac{a(n\nu+\tau)(\Delta_\nu v(n\nu+\tau))^\beta}{v^\gamma((n-m)\nu+\tau)} > 0 \quad \text{for } k \geq k_2 \tag{3.9}$$

Then, it follows that $w(n) > 0$ and

$$z(n) = \rho(n\nu+\tau)\frac{a(n\nu+\tau)(\Delta_\nu v(n\nu+\tau))^\beta}{v^\gamma((n-m)\nu+\tau)} \geq \rho(n\nu+\tau)\sum_{s=k}^{\infty} q(s\nu+\tau) \tag{3.10}$$

Now,

$$\begin{aligned} \Delta_\nu z(n\nu+\tau) &= \Delta_\nu\left(\frac{\rho(n\nu+\tau)}{v^\gamma((n-m)\nu+\tau)}\right)a((n+1)\nu+\tau)(\Delta_\nu v((n+1)\nu+\tau))^\beta \\ &+ \Delta_\nu\left(a(n\nu+\tau)(\Delta_\nu v(n\nu+\tau))^\beta\right)\left(\frac{\rho(n\nu+\tau)}{v^\gamma((n-m)\nu+\tau)}\right) \\ \leq &-\rho(n\nu+\tau)q(n\nu+\tau) + \left(\frac{\Delta_\nu\rho(n\nu+\tau)}{\rho((n+1)\nu+\tau)}\right)z((n+1)\nu+\tau) \\ &-\left(\frac{\rho(n\nu+\tau)}{\rho((n+1)\nu+\tau)}\right)\frac{\Delta_\nu v^\gamma((n-m)\nu+\tau)}{v^\gamma((n-m)\nu+\tau)}z((n+1)\nu+\tau) \end{aligned} \tag{3.11}$$

By the Generalized Mean Value Theorem for Derivatives,

$$\begin{aligned} &\gamma v^{\gamma-1}((n-m+1)\nu+\tau)\Delta_\nu v((n-m)\nu+\tau) \geq \Delta_\nu v^\gamma((n-m)\nu+\tau) \\ &\geq \gamma v^{\gamma-1}((n-m)\nu+\tau)\Delta_\nu v((n-m)\nu+\tau) \end{aligned}$$

Using this in (3.11) gives

$$\begin{aligned} \Delta_\nu z(n\nu+\tau) \leq &-\rho(n\nu+\tau)q(n\nu+\tau) + \left(\frac{\Delta_\nu\rho(n\nu+\tau)}{\rho((n+1)\nu+\tau)}\right)z((n+1)\nu+\tau) \\ &-\gamma\left(\frac{\rho(n\nu+\tau)}{\rho((n+1)\nu+\tau)}\right)\frac{v^{\gamma-1}((t-m)\nu+\tau)\Delta_\nu v((n-m)\nu+\tau)}{v^\gamma((n-m)\nu+\tau)}z((n+1)\nu+\tau) \end{aligned} \tag{3.12}$$

Since $a(n\nu+\tau)(\Delta_\nu v(n\nu+\tau))^\beta$ is decreasing and $v(n\nu+\tau)$ is increasing, we

$$\frac{\Delta_\nu v((n-m)\nu+\tau)}{\Delta_\nu v(n\nu+\tau)} \geq \left(\frac{a(n\nu+\tau)}{a((n-m)\nu+\tau)}\right)^{1/\beta} \text{ and } \frac{z((n+1)\nu+\tau)}{\rho((n+1)\nu+\tau)} \leq \frac{z(n\nu+\tau)}{\rho(n\nu+\tau)} \tag{3.13}$$

Using (3.13) in (3.12), we obtain

$$\Delta_\nu z(n\nu+\tau) \le -\rho(n\nu+\tau)q(n\nu+\tau) + \left(\frac{\Delta_\nu \rho(n\nu+\tau)}{\rho((n+1)\nu+\tau)}\right) z((n+1)\nu+\tau)$$
$$-\gamma\left(\frac{\rho(n\nu+\tau)}{\rho((n+1)\nu+\tau)}\right)\left(\frac{a(n\nu+\tau)}{a((n-m)\nu+\tau)}\right)^{1/\beta}\frac{\Delta_\nu v(n\nu+\tau)}{v((n-m)\nu+\tau)} z((n+1)\nu+\tau)$$

Now,

$$\frac{\Delta_\nu v(n\nu+\tau)}{v^{\gamma/\beta}((n-m)\nu+\tau)} = \rho^{-1/\beta}(n\nu+\tau)a^{-1/\beta}(n\nu+\tau)z^{1/\beta}(n\nu+\tau)$$
$$\ge \rho^{-1/\beta}(n\nu+\tau)a^{-1/\beta}(n\nu+\tau)\left(\frac{\rho(n\nu+\tau)}{\rho((n+1)\nu+\tau)}\right)^{1/\beta} z^{1/\beta}((n+1)\nu+\tau)$$

Thus, the equation

$$\Delta_\nu z(n\nu+\tau) \le -\rho(n\nu+\tau)q(n\nu+\tau) + \left(\frac{\Delta_\nu \rho(n\nu+\tau)}{\rho((n+1)\nu+\tau)}\right) z((n+1)\nu+\tau)$$
$$-\frac{\gamma\rho(n\nu+\tau)z^{1+1/\beta}((n+1)\nu+\tau)v^{(\gamma-\beta)/\beta}((n-m)\nu+\tau)}{a^{1/\beta}((n-m)\nu+\tau)\rho^{1+1/\beta}((n+1)\nu+\tau)}$$

and so,

$$\Delta_\nu z(n\nu+\tau) \le -\rho(n\nu+\tau)q(n\nu+\tau) + \left(\frac{\Delta_\nu \rho(n\nu+\tau)}{\rho(n+1)\nu+\tau}\right) z((n+1)\nu+\tau)$$
$$-\frac{\gamma\rho(n\nu+\tau)z^{1+1/\beta}((n+1)\nu+\tau)v^{(\gamma-\beta)/\beta}((n-m)\nu+\tau)}{a^{1/\beta}((n-m)\nu+\tau)\rho^{1+1/\beta}((n+1)\nu+\tau)}$$

For the case $\gamma=\beta$, we see that $v^{(\gamma-\beta)/\beta}(n\nu+\tau)=1$ while for the case $\gamma<\beta$, since $a(n\nu+\tau)(\Delta_\nu v(n\nu+\tau))^\beta$ is decreasing, there exists a constant $c_1>0$ such that

$$a(n\nu+\tau)(\Delta_\nu v(\nu+\tau))^\beta \le c_1 \text{ for } k\ge k_2$$

Summing this inequality from k_2 to k, we have

$$v(n\nu+\tau) \le v\left(n_2\nu+\tau\right) + A\left(n,k_2\right) \le c_2 A\left(n,k_2\right)$$

for $k\ge k_3$ for some $c_2>0$ and $k_3\ge k_2$. Thus,

$$v^{(\gamma-\beta)/\beta}(n\nu+\tau) \ge c_2^{(\gamma-\beta)/\beta} A^{(\gamma-\beta)/\beta}\left(n,k_2\right) := c^* A^{(\gamma-\beta)/\beta}\left(n,k_2\right)$$

where $c^* = c_2^{(\gamma-\beta)/\beta}$. Combining the two cases on γ and the definition of $g_{c^*}(n)$ gives

$$\Delta_\nu z(n\nu+\tau) \le -\rho(n\nu+\tau)q(n\nu+\tau) + \left(\frac{\Delta_\nu \rho(n\nu+\tau)}{\rho((n+1)\nu+\tau)}\right) z((n+1)\nu+\tau)$$
$$-\frac{\gamma\rho(n\nu+\tau)}{a^{1/\beta}((n-m)\nu+\tau)\rho^{1+1/\beta}((n+1)\nu+\tau)} g_{c^*}(n\nu+\tau)w^{1+1/\beta}((n+1)\nu+\tau) \tag{3.14}$$

Setting

$$B = \frac{\Delta_\nu \rho(n\nu+\tau)}{\rho((n+1)\nu+\tau)} \text{ and } \quad C = \frac{\gamma\rho(n\nu+\tau)g_{c^*}(n\nu+\tau)}{a^{1/\beta}((n-m)\nu+\tau)\rho^{1+1/\beta}((n+1)\nu+\tau)}$$

and using the inequality (see [22])

$$Bu - Cu^{(1+\beta)/\beta} \le \frac{\beta^\beta}{(1+\beta)^{\beta+1}} \left(\frac{B^{\beta+1}}{C^\beta}\right)$$

with $x = z((t+1)\nu+\tau)$, we have

$$\Delta_\nu z(n\nu+\tau) \le -\rho(n\nu+\tau)q(n\nu+\tau) + \frac{\beta^\beta}{(1+\beta)^{\beta+1}} \frac{a((n-m)\nu+\tau)}{(\gamma g_{c^*}(n\nu+\tau))^\beta} \left(\frac{(\Delta_\nu\rho(n\nu+\tau))^{\beta+1}}{\rho^\beta(n\nu+\tau)}\right)$$

Summing this inequality from k_2 to k gives

$$w(n\nu+\tau) \le w\left(n_2\nu+\tau\right) - \sum_{s=k_2}^{k} \left[\rho(s\nu+\tau)q(s\nu+\tau) - \frac{\beta^\beta}{(1+\beta)^{\beta+1}} \frac{a((n-m)\nu+\tau)}{(\gamma g_{c^*}(s\nu+\tau))^\beta} \left(\frac{(\Delta_\nu\rho(s\nu+\tau))^{\beta+1}}{\rho^\beta(s\nu+\tau)}\right)\right]$$

Taking into account (3.8), we see that

$$w\left(n_2\nu+\tau\right) \ge \rho(n\nu+\tau)Q(n\nu+\tau) + \sum_{s=2}^{k} \left[\rho(s\nu+\tau)q(s\nu+\tau) - \frac{\beta^\beta}{(1+\beta)^{\beta+1}} \frac{a((n-m)\nu+\tau)}{(\gamma g_{c^*}(s\nu+\tau))^\beta} \left(\frac{(\Delta_\nu\rho(s\nu+\tau))^{\beta+1}}{\rho^\beta(s\nu+\tau)}\right)\right]$$

Taking the lim sup of both sides in the above inequality as $k \to \infty$, we obtain a contradiction to condition (3.2).

Now consider Case (b). If we set $z(n\nu+\tau) = -v(n\nu+\tau) > 0$ for $k \ge k_2$, then $\Delta_\nu w(n\nu+\tau) = -\Delta_\nu v(n\nu+\tau) < 0$, and from equation (2.1)

$$\Delta_\nu \left(a(n\nu+\tau)(\Delta_\nu w(n\nu+\tau))^\beta\right) = q(n\nu+\tau)u^\gamma((n-m+1)\nu+\tau) \ge 0 \quad (3.15)$$

Moreover,

$$w(n\nu+\tau) = -v(n\nu+\tau) = p(n\nu+\tau)u^\nu((n-t)\nu+\tau) - u(n) \le p(n\nu+\tau)u^\nu((n-t)\nu+\tau)$$

so

$$u^\nu((n-t)\nu+\tau) \ge w(n\nu+\tau) \text{ or } w^{1/\nu}((n+t)\nu+\tau) \le u(n\nu+\tau)$$

Using this inequality in (2.1), we have

$$\Delta_\nu \left(a(n\nu+\tau)(\Delta_\nu w(n\nu+\tau))^\beta\right) \ge q(n\nu+\tau)z^{\gamma/\nu}((n-m+t+1)\nu+\tau) = q(n\nu+\tau)w^{\gamma/\nu}(h(n\nu+\tau)) \quad (3.16)$$

For $k_2 \leq x \leq v$, we may write

$$w(x\nu+\tau) - w(y\nu+\tau) = -\sum_{s=u}^{v} a^{-1/\beta}(s\nu+\tau)\left(a(s\nu+\tau)(\Delta_\nu w(s\nu+\tau))^\beta\right)^{1/\beta}$$
$$\geq A(y,x)\left(-\left(a(y\nu+\tau)(\Delta_\nu w(y\nu+\tau))^\beta\right)^{1/\beta}\right)$$

for $k \geq s \geq k_2$. Setting $x = h(s)$ and $y = h(n)$ in the above inequality gives

$$w(h(s\nu+\tau)) \geq A(h(n\nu+\tau), h(s\nu+\tau))\left(-\left(a(h(n\nu+\tau))(\Delta_\nu w(h(n\nu+\tau)))^\beta\right)^{1/\beta}\right)$$

Summing inequality (3.16) from $h(n) \geq k_2$ to k, we find that

$$W(n\nu+\tau) = -a(h(n\nu+\tau))(\Delta_\nu w(h(n\nu+\tau)))^\beta$$
$$\geq \left(-a(h(n\nu+\tau))(\Delta_\nu z(h(n\nu+\tau)))^\beta\right)^{\gamma/\nu\beta} \sum_{s=h(n)}^{k} A^{\gamma/\nu}(h(n\nu+\tau), h(s\nu+\tau))q(s\nu+\tau)$$
$$= W^{\gamma/\nu\beta}(n\nu+\tau) \sum_{s=h(n)}^{k} A^{\gamma/\nu}(h(n\nu+\tau), h(s\nu+\tau))q(s\nu+\tau)$$

and hence

$$W^{1-\gamma/\nu\beta}(n\nu+\tau) \geq \sum_{s=h(n)}^{t} A^{\gamma/\nu}(h(n\nu+\tau), h(s\nu+\tau))q(s\nu+\tau)$$

By considering the limit superior of both sides of this inequality as $t \to \infty$, we reach a contradiction with (3.3) if $\gamma = \nu\beta$. Since $\Delta_\nu W(n\nu+\tau) \leq 0$ by (3.15), $W(n\nu+\tau)$ is bounded, thus, we reach a contradiction with (3.4) if $\gamma < \nu\beta$. This concludes the proof of the theorem.

In case $Q(n)$ does not exist as $t \to \infty$, we see that condition (3.2) can be substituted with

$$\limsup_{k\to\infty} \sum_{s=k_2}^{k} \Big[\rho(s\nu+\tau)q(s\nu+\tau)$$
$$-\frac{\beta^\beta}{(1+\beta)^{\beta+1}} \frac{a((n-m+1)\nu+\tau)}{(\gamma g_c(s\nu+\tau))^\beta}\left(\frac{(\Delta_\nu\rho(s\nu+\tau))^{\beta+1}}{\rho^\beta(s\nu+\tau)}\right)\Big] = \infty \qquad (3.17)$$

and the conclusion of Theorem (3.1) still holds.

For the non-neutral equation, i.e., equation (2.1) with $p(n) \equiv 0$, and $q(n\nu+\tau)$ is either nonnegative or nonpositive for all large t, equation (2.1) reduces to

$$\Delta_\nu\left(a(n\nu+\tau)(\Delta_\nu(u(n\nu+\tau)))^\beta\right) + \delta q(n\nu+\tau)x^\gamma((n+1-m)\nu+\tau) = 0 \qquad (3.18)$$

where $\delta = \pm 1$. From Theorem (3.1), we extract the following immediate results. Our subsequent outcome stems directly from Theorem (3.1).

Theorem 3.2. *Let the hypotheses of Theorem* (3.1) *hold with* $\Delta_\nu \rho(n\nu+\tau) \le 0$ *for* $k \ge k_0$ *and condition* (3.2) *replaced by*

$$\limsup_{k\to\infty}\left[\rho(n\nu+\tau)Q(n\nu+\tau)+\sum_{s=k_0}^{k}\rho(s\nu+\tau)q(s\nu+\tau)\right]=\infty \tag{3.19}$$

Then equation (2.1) *is oscillatory.*

In the following theorem, we utilize an alternative method for substituting the condition (3.2) in Theorem (3.1).

Theorem 3.3. *Let the hypotheses of Theorem* (3.1) *hold with condition* (3.2) *replaced by*

$$\limsup_{k\to\infty}\left[\rho(n\nu+\tau)Q(n\nu+\tau)+\sum_{s=k_0}^{k}\rho(s\nu+\tau)q(s\nu+\tau)\right.$$
$$\left.-\frac{a^{1/\beta}((s-m)\nu+\tau)(\Delta_\nu\rho(s\nu+\tau))^2}{4\gamma g_c(s\nu+\tau)\rho(s\nu+\tau)Q^{(1/\beta)-1}((s+1)\nu+\tau)}\right]=\infty \tag{3.20}$$

Then equation (2.1) *is oscillatory.*

Proof. Let $u(n\nu+\tau)$ be a nonoscillatory solution of equation (2.1), say $u(n\nu+\tau)>0, u((n-m+1)\nu+\tau)>0$, and $u((n-t)\nu+\tau)>0$ for $k\ge k_1$ for some $k_1 \ge k_0$. Proceeding as in the proof of Theorem (3.1), we conclude that $\Delta_\nu v(n\nu+\tau)>0$ for $k\ge k_2$ and $v(n\nu+\tau)$ satisfies either Case (a) or Case (b) for $t\ge t_2$. If (a) holds, then as in the proof of Theorem (3.1), we again obtain (3.12). Since $a(n\nu+\tau)(\Delta_\nu v(n\nu+\tau))^\beta$ is nonincreasing and $v(n\nu+\tau)$ is nondecreasing, we have

$$a^{1/\beta}((n-m)\nu+\tau)\Delta_\nu v((n-m)\nu+\tau)\ge a^{1/\beta}((n+1)\nu+\tau)\Delta_\nu v((n+1)\nu+\tau)$$

and

$$1/v((n-m)\nu+\tau)\ge 1/v((n-m+1)\nu+\tau),$$

$$\Delta_\nu z(n\nu+\tau)\le -\rho(n\nu+\tau)q(n\nu+\tau)+\left(\frac{\Delta_\nu\rho(n\nu+\tau)}{\rho((n+1)\nu+\tau)}\right)z((n+1)\nu+\tau)$$
$$-\frac{\gamma\rho(n\nu+\tau)}{\rho((n+1)\nu+\tau)}\frac{a^{1/\beta}((n+1)\nu+\tau)\Delta_\nu v((n+1)\nu+\tau)}{a^{1/\beta}((n-m)\nu+\tau)v((n-m+1)\nu+\tau)}z((n+1)\nu+\tau)$$
$$\le -\rho(n\nu+\tau)q(n\nu+\tau)+\left(\frac{\Delta_\nu\rho(n\nu+\tau)}{\rho((n+1)\nu+\tau)}\right)z((n+1)\nu+\tau)$$
$$-\frac{\gamma\rho(n\nu+\tau)}{\rho^{1+1/\beta}((n+1)\nu+\tau)}\frac{v^{\frac{\gamma-\beta}{\beta}}((n-m+1)\nu+\tau)}{a^{1/\beta}((n-m)\nu+\tau)}z^{1+1/\beta}((n+1)\nu+\tau)$$
$$\le -\rho(n\nu+\tau)q(n\nu+\tau)+\left(\frac{\Delta_\nu\rho(n\nu+\tau)}{\rho((n+1)\nu+\tau)}\right)z((n+1)\nu+\tau)$$
$$-\frac{\gamma\rho(n\nu+\tau)}{\rho^{1+1/\beta}((n+1)\nu+\tau)}\frac{g_{c^*}(n\nu+\tau)}{a^{1/\beta}((n-m)\nu+\tau)}z^{1+1/\beta}((n+1)\nu+\tau)$$

From (3.10) we see that $z^{1-1/\beta}((n+1)\nu+\tau)/\rho^{1-1/\beta}((n+1)\nu+\tau) \geq Q^{1-1/\beta}((n+1)\nu+\tau)$, so

$$\Delta_\nu z(n\nu+\tau) \leq -\rho(n\nu+\tau)q(n\nu+\tau) + \left(\frac{\Delta_\nu \rho(n\nu+\tau)}{\rho((n+1)\nu+\tau)}\right) z((n+1)\nu+\tau)$$
$$- \frac{\gamma\rho(n\nu+\tau)g_{c^*(n\nu+\tau)}Q^{1/\beta-1}((n+1)\nu+\tau)w^2((n+1)\nu+\tau)}{a^{1/\beta}((n-m)\nu+\tau)\rho^2((n+1)\nu+\tau)}$$

Completing the square for the second and third terms on the right-hand side yields

$$\Delta_\nu w(n\nu+\tau) \leq -\rho(n\nu+\tau)q(n\nu+\tau) + \frac{a^{1/\beta}((n-m)\nu+\tau)(\Delta_\nu\rho(n\nu+\tau))^2}{4\gamma g_{c^*}(n\nu+\tau)\rho(n\nu+\tau)Q^{(1/\beta)-1}((n+1)\nu+\tau)}$$

The remaining steps of the proof resemble those of Theorem 3.1 and are therefore omitted.

4. Application

Example 4.1. Examine the generalized neutral equation difference equation

$$\Delta_\nu\left(\Delta_\nu\left(u(n\nu+\tau) - \frac{1}{2}x^{1/3}((n-3)\nu+\tau)\right)^3\right) + 8u((n-7)\nu+\tau) = 0 \tag{4.1}$$

Here, $t = 3$ and $m = 8$, so $h(n) = k - 4$. All conditions of Theorem (3.1) with $\rho(n\nu+\tau) \equiv 1$ and condition (3.2) replaced by (3.17) are satisfied, so equation (4.1) is oscillatory.

Example 4.2. Examine the neutral equation

$$\Delta_\nu\left(n^3\Delta_\nu\left(u(n\nu+\tau) - \frac{1}{3}u^{1/3}((n-2)\nu+\tau)\right)^3\right) + \frac{1}{\log k}u((n-3)\nu+\tau) = 0, k > 1 \tag{4.2}$$

Here, $k = 2, m = 4, \nu = 1/3$, and $\beta = 3$. All conditions of Theorem (3.3) are satisfied with $\rho \equiv 1$ and hence equation (4.2) is oscillatory.

5. Conclusion

In conclusion, the newly proposed oscillation criteria for second-order nonlinear difference equations with a nonlinear nonpositive neutral term provide valuable insights into the dynamic behavior of discrete systems in the field of computational modeling and simulation. The ability to accurately predict oscillatory behavior in discrete systems is crucial for various computational tasks, ranging from numerical analysis to computer graphics and animation. By incorporating the derived oscillation criteria into computational algorithms, researchers and practitioners can enhance the fidelity and efficiency of simulations

involving discrete dynamics. Understanding the conditions under which oscillations occur in nonlinear systems allows for the development of more robust and reliable algorithms, improving the performance of tasks such as optimization, scheduling, and signal processing. Whether it's optimizing resource allocation in distributed computing systems or designing feedback control mechanisms in cyber-physical systems, the insights gleaned from this research have the potential to drive advancements across various domains of computer science.

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