Global and Stochastic Analysis
Vol. 11 No. 4 (September, 2024)

Received: 10th June 2024 Revised: 12th July 2024 Accepted: 15th August 2024

ON ABSOLUTELY MONOTONIC FUNCTIONS AND SOME INEQUALITIES FOR SPECIAL FUNCTIONS

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Abstract. First we prove that the main theorem for absolutely monotonic functions on $(0,\infty)$ from the book Mitrinović D.S., Pečarić J.E., Fink A.M. "Classical And New Inequalities In Analysis", Kluwer Academic Publishers, 1993, is not valid without additional restrictions. After that connected inequalities for special functions of hypergeometric type is studied.

1. On the main theorem for absolutely monotonic functions

In this section we correct the main theorem for absolutely monotonic functions on $(0,\infty)$ from the book Mitrinović D.S., Pečarić J.E., Fink A.M. "Classical And New Inequalities In Analysis", Kluwer Academic Publishers, 1993. Also note that different classes of functions and inequalities for special functions are important in many areas, also including probability theory and stochastic analysis. For some references cf. [1].

In the classical book [2], chapter XIII, page 365, there is a definition of absolutely monotonic on $(0,\infty)$ functions.

Definition. A function $f(x)$ is said to be *absolutely monotonic on* $(0,\infty)$ if it has derivatives of all orders and

$$f^{(k)}(x) \geq 0, x \in (0,\infty), k = 0,1,2,\dots \ . \tag{1.1}$$

For absolutely monotonic functions the next integral representation is essential:

$$f(x) = \int_0^\infty e^{xt}\ d\sigma(t), \tag{1.2}$$

where $\sigma(t)$ is bounded and nondecreasing and the integral converges for all $x \in (0,\infty)$.

Also the basic set of inequalities is considered.

Let $f(x)$ be an absolutely monotonic function on $(0,\infty)$. Then

$$f^{(k)}(x) f^{(k+2)}(x) \geq \left(f^{(k+1)}(x)\right)^2, k = 0,1,2,\dots \ . \tag{1.3}$$

2010 *Mathematics Subject Classification.* Primary 33E99.
Key words and phrases. absolutely monotonic functions, Turan–type inequalities, concave functions, convex functions, ratios of special functions, hypergeometric functions.

After this definition in the book [2] the result which we classify as the main theorem for absolutely monotonic functions on $(0, \infty)$ is formulated (theorem 1, page 366).

The main theorem for absolutely monotonic functions.

The above definition (1.1), integral representation (1.2) and basic set of inequalities (1.3) are equivalent.

It means:

$$(1.1) \Leftrightarrow (1.2) \Leftrightarrow (1.3). \tag{1.4}$$

In the book [2] for (1.1) $\Leftrightarrow$ (1.2) the reference is given to [3], and an equivalence (1.2) $\Leftrightarrow$ (1.3) is proved, it is in fact a consequence of Chebyschev inequality.

In this note we consider a counterexample to the equivalence (1) $\Leftrightarrow$ (2) of Widder. So unfortunately it seems that the main theorem for absolutely monotonic functions in the book [2] **is not valid !!!**.

This counterexample is very simple so it is strange enough it was not found before.

Really, to construct a very simple counterexample, consider a function $f(x) = x^2 + 1$. Obviously for all $x \in [0, \infty)$

$$f(x) \geq 0, f'(x) = 2x \geq 0, f''(x) = 2 \geq 0, f^{(k)}(x) = 0 \geq 0, k > 2. \tag{1.5}$$

So this function $f(x)$ is in the class of absolutely monotonic functions on $(0, \infty)$ due to the definition (1). If (1) $\Rightarrow$ (3) is valid then the next inequality must be true as a special case of (3) for all $x \in (0, \infty)$

$$f(x)f''(x) \geq \left(f'(x)\right)^2 \Leftrightarrow 2(x^2 + 1) \geq 4x^2 \Leftrightarrow 1 \geq x^2$$

but this is not valid for all $x \in (0, \infty)$.

As a conclusion we see that implication (1) $\Rightarrow$ (3) in [2] is not valid. It also means that implication (1) $\Rightarrow$ (2) is also not valid. The implication (2) $\Rightarrow$ (3) is obviously valid due to the Chebyschev inequality.

And consequently also the theorem 2 in [2], pages 366–367 on determinant inequalities is not valid too if based only on definition (1).

In some papers the above implications are used to derive new results for absolutely monotonic functions. It seems not to be a correct way of reasoning. One way is to change the main theorem on absolutely monotonic functions to a proper one, otherwise for all special cases an integral representation must be proved independently.

Comment 1. On the other hand everything is OK with theorems on completely monotonic functions. An integral representation for them in [4] include the additional condition

$$\lim_{x \to \infty} f(x) = 0.$$

This condition is omitted in [3] but mysteriously mentioned in [4] with the reference again to [3]. May be something like it is needed also for absolutely monotonic functions.

Different aspects of completely monotonic functions are considered in ([3, 4, 5].

Comment 2. There are many ways to generalize notions of absolutely and completely monotonic functions. It seems that a first step was done by Sergei Bernstein and very important generalizations were investigated by Bulgarian mathematicians Nikola Obreshkov (also known for two celebrated named formulas: Obreshkov generalized Taylor expansion formula and the Obreshkov integral transform — the first integral transform which kernel depends on Meyer G–function and not depends on any hypergeometric function of any kind) and Jaroslav Tagamlitskii.

Comment 3. With absolute and complete monotonicity different functional classes are deeply connected: Stieltjes, Pick, Bernstein, Schoenberg, Schur and others.

So the next problems seem to be rather interesting and important.

Problem 1. Give a correct proof for the theorem under consideration from [2] and so give justification for equivalences (1.4).

Problem 2. Generalize the theorem under consideration from [2] *for fractional derivatives* and give justification for equivalences (1.4) for this case.

2. Ratio monotonicity for some classes of special functions

In the preprint [6] one of the authors formulated some conjectures on monotonicity of ratios for exponential series sections. They lead to more general conjecture on monotonicity of ratios of Kummer hypergeometric functions and was not proved from 1993. In this paper we prove some conjectures from [6] for Kummer hypergeometric functions and its further generalizations for Gauss and generalized hypergeometric functions. The results are also closely connected with Turán–type inequalities.

2.1. Introduction and statement of problems. Let us consider the series for the exponential function

$$\exp(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},\ x \geq 0,$$

its section $S_n(x)$ and series remainder $R_n(x)$ in the form

$$S_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!},\ R_n(x) = \exp(x) - S_n(x) = \sum_{k=n+1}^{\infty} \frac{x^k}{k!},\ x \geq 0. \qquad (2.1)$$

Besides simplicity and elementary nature of these functions many mathematicians studied problems for them. G. Szegő proved a remarkable limit distribution for zeroes of sections, accumulated along so–called the Szegő curve. S. Ramanujan seems was the first who proved the non–trivial inequality for exponential sections in the form: if

$$\frac{e^n}{2} = R_{n-1}(n) + \frac{n^n}{n!}\theta(n)$$

then

$$\frac{1}{3} < \theta(n) = \frac{n!\left(\frac{e^n}{2} - R_{n-1}(n)\right)}{n^n} < \frac{1}{2}.$$

This result is important as it also evaluates e^n in rational bounds

$$\frac{2n^n}{3n!} + 2R_{n-1}(n) < e^n < \frac{n^n}{n!} + 2R_{n-1}(n)$$

as it was specially pointed out in ([?], pp. 323–324).

In the preprint [6] were thoroughly studied inequalities of the form

$$m(n) \leq f_n(x) = \frac{R_{n-1}(x)R_{n+1}(x)}{[R_n(x)]^2} \leq M(n),\ x \geq 0. \tag{2.2}$$

The search for the best constants $m(n) = m_{best}(n)$, $M(n) = M_{best}(n)$ has some history. The left–hand side of (2.2) was first proved by Kesava Menon with $m(n) = \frac{1}{2}$ (not best) and by Horst Alzer with

$$m_{best}(n) = \frac{n+1}{n+2} = f_n(0), \tag{2.3}$$

cf. [6] for the more detailed history. In [6] it was also shown that in fact the inequality (2.2) with the sharp lower constant (2.3) is a special case of the stronger inequality proved earlier in 1982 by Walter Gautschi.

It seems that the right–hand side of (2.2) was first proved by the author in [6] with $M_{best} = 1 = f_n(\infty)$. In [6] dozens of generalizations of inequality (2.2) and related results were proved. May be in fact it was the first example of so called Turan–type inequality for special case of the Kummer hypergeometric functions.

Obviously the above inequalities are consequences of the next conjecture originally formulated in [6] .

Conjecture 1. *The function $f_n(x)$ in (2.2) is monotone increasing for $x \in [0; \infty), n \in \mathbb{N}$.* So the next inequality is valid

$$\frac{n+1}{n+2} = f_n(0) \leq f_n(x) < 1 = f_n(\infty). \tag{2.4}$$

In 1990's we tried to prove this conjecture in the straightforward manner by expanding an inequality $(f_n(x))' \geq 0$ in series and multiplying triple products of hypergeometric functions but failed.

Consider a representation via Kummer hypergeometric functions

$$f_n(x) = \frac{n+1}{n+2}\ g_n(x),\ \ g_n(x) = \frac{{}_1F_1(1; n+1; x){}_1F_1(1; n+3; x)}{[{}_1F_1(1; n+2; x)]^2}. \tag{2.5}$$

So the conjecture 1 may be reformulated in terms of this function $g_n(x)$ as conjecture 2.

Conjecture 2. *The function $g_n(x)$ in (2.5) is monotone increasing for $x \in [0; \infty), n \in \mathbb{N}$.*

This leads us to the next more general

Problem 1. *Find monotonicity in x conditions for $x \in [0; \infty)$ for all parameters a,b,c for the function*

$$h(a, b, c, x) = \frac{{}_1F_1(a; b-c; x){}_1F_1(a; b+c; x)}{[{}_1F_1(a; b; x)]^2}. \tag{2.6}$$

We may also call (2.6) mockingly (in Ramanujan way, remember his mock theta–functions!) "*The abc–problem*" for Kummer hypergeometric functions, why not?

Another generalization is to change Kummer hypergeometric functions to higher ones.

Problem 2. *Find monotonicity in x conditions for $x \in [0;\infty)$ for all vector–valued parameters a,b,c for the function*

$$h_{p,q}(a,b,c,x) = \frac{{}_pF_q(a;b-c;x)\,{}_pF_q(a;b+c;x)}{\left[{}_pF_q(a;b;x)\right]^2}, \tag{2.7}$$

$$a = (a_1,\ldots,a_p), b = (b_1,\ldots,b_q), c = (c_1,\ldots,c_q).$$

This is "*The abc–problem*" for generalized hypergeometric functions. The more complicated problems are obvious and may be considered for pairs or triplets of parameters and also for multivariable hypergeometric functions.

In 1941 while studying the zeros of Legendre polynomials, the Hungarian mathematician Paul Turán discovered the following inequality

$$P_{n-1}(x)P_{n+1}(x) < \left[P_n(x)\right]^2,$$

where $|x| < 1,\ n \in \mathbb{N} = 1, 2, \ldots$ and P_n stands for the classical Legendre polynomial. This inequality was published by P. Turán only in 1950. However, since the publication in 1948 by G. Szegö of the above famous Turán inequality for Legendre polynomials, many authors have deduced analogous results for classical (orthogonal) polynomials and special functions. In the last 62 years it has been shown by several researchers that the most important (orthogonal) polynomials (e.g. Laguerre, Hermite, Appell, Bernoulli, Jacobi, Jensen, Pollaczek, Lommel, Askey-Wilson, ultraspherical polynomials) and special functions (e.g. Bessel, modified Bessel, gamma, polygamma, Riemann zeta functions) satisfy a Turán inequality. In 1981 one of the PhD students of P. Turán, L. Alpár in Turán's biography mentioned that the above Turán inequality had a wide-ranging effect, this inequality was dealt with in more than 60 papers. Since Turán's inequality was investigated for the orthogonal polynomials having hypergeometric representation, it is worth studying the validity of such inequality for various hypergeometric functions as well. Recently Turán type inequalities for the q-Kummer's and q-hypergeometric functions were prooved and discussed and using the monotonicity property of ratios of Kummer, Gauss and generalized hypergeometric functions the author presented some Turán type inequalities for this functions.

The aim of this paper is to prove conjectures 1 and 2, and to find conditions for validity of problems 1 and 2 and so completely solve them.

3. Two lemmas

We formulate two useful lemmas which will be used below. These lemmas were first proved in ([7]), cf. also ([8])–([9]) for the detailed proof and further applications. The lemmas are modern variants of a classical Bernoulli rule from calculus.

Lemma 3.1. *Let* (a_n) *and* (b_n) $(n = 0, 1, 2...)$ *be real numbers, such that* $b_n > 0$, $n = 0, 1, 2, ...$ *and* $\left(\frac{a_n}{b_n}\right)_{n\geq 0}$ *is increasing (decreasing), then* $\left(\frac{a_{0+}...+a_n}{b_0+...+b_n}\right)_n$ *is increasing (decreasing).*

Lemma 3.2. . *Let* (a_n) *and* (b_n) $(n = 0, 1, 2...)$ *be real numbers and let the power series* $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $B(x) = \sum_{n=0}^{\infty} b_n x^n$ *be convergent if* $|x| < r$. *If* $b_n > 0$, $n = 0, 1, 2, ...$ *and if the sequence* $\left(\frac{a_n}{b_n}\right)_{n\geq 0}$ *is (strictly) increasing (decreasing) , then the function* $\frac{A(x)}{B(x)}$ *is also (strictly) increasing on* $[0, r[$.

4. Monotonicity for the Kummer hypergeometric function and associated Turán type inequality

Theorem 4.1. *Let* a, b, c *be real numbers such that* $0 < a < b - c$ *and* $b > 1$ *and the function* $x \longmapsto h(a, b, c, x)$ *is defined by*

$$h(a, b, c, x) = \frac{{}_1F_1(a; b - c; x)\,{}_1F_1(a; b + c; x)}{\left[{}_1F_1(a; b; x)\right]^2}. \tag{4.1}$$

Then this function is increasing on $[0, \infty[$. *Consequently, for* $n \in \mathbb{N}$, *the functions* $x \longmapsto f_n(x)$ *in (2.2) and* $x \longmapsto g_n(x)$ *in (2.5) are also increasing on* $[0, \infty[$.

Proof For all a, b, c be real numbers such that $0 < a < b - c$ and $b > 1$ we evaluate

$$\begin{aligned} h(a, b, c, x) &= \frac{{}_1F_1(a; b - c; x)\,{}_1F_1(a; b + c; x)}{\left[{}_1F_1(a; b; x)\right]^2} = \\ &= \frac{\left(\sum_{n=0}^{\infty} \frac{(a)_n}{(b-c)_n n!} x^n\right)\left(\sum_{n=0}^{\infty} \frac{(a)_n}{(b+c)_n n!} x^n\right)}{\left[\sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n n!} x^n\right]^2} = \\ &= \frac{\sum_{n=0}^{\infty} A_n x^n}{\sum_{n=0}^{\infty} B_n x^n}, \end{aligned}$$

where

$$A_n = \sum_{k=0}^{n} \frac{(a)_k (a)_{n-k}}{(b - c)_k (b + c)_{n-k} k!(n - k)!} \quad \text{and } B_n = \sum_{k=0}^{n} \frac{(a)_k (a)_{n-k}}{(b)_k (b)_{n-k} k!(n - k)!}.$$

Let define sequences $(u_{n,k})_{k\geq 0}$, $(v_{n,k})_{k\geq 0}$ and $(w_{n,k})_{k\geq 0}$ by

$$u_{n,k} = \frac{(a)_k (a)_{n-k}}{(b - c)_k (b + c)_{n-k} k!(n - k)!}, \quad v_{n,k} = \frac{(a)_k (a)_{n-k}}{(b)_k (b)_{n-k} k!(n - k)!},$$

and

$$w_{n,k} = \frac{u_{n,k}}{v_{n,k}} = \frac{(b)_k (b)_{n-k}}{(b - c)_k (b + c)_{n-k}}, \; k \geq 0.$$

It follows that

$$\begin{aligned}\frac{w_{n,k+1}}{w_{n,k}} &= \frac{u_{n,k+1}v_{n,k}}{v_{n,k+1}u_{n,k}} = \\ &= \frac{(b)_{k+1}(b)_{n-k-1}(b-c)_k(b+c)_{n-k}}{(b-c)_{k+1}(b+c)_{n-k-1}(b)_k(b)_{n-k}} = \\ &= \frac{\Gamma(b+k+1)}{\Gamma(b+k)}.\frac{\Gamma(b+n-k-1)}{\Gamma(b+n-k)}.\frac{\Gamma(b-c+k)}{\Gamma(b-c+k+1)}.\frac{\Gamma(b+c+n-k)}{\Gamma(b+c+n-k-1)} = \\ &= \frac{(b+k)}{(b-c+k)}.\frac{(b+c+n-k-1)}{(b+n-k-1)} \geq 1.\end{aligned}$$

We conclude that the sequence $(w_{n,k})_{k\geq 0}$ is increasing and consequently the sequence $(C_n = \frac{A_n}{B_n})_{n\geq 0}$ is also increasing by lemma 3.1. Thus the function $h(a,b,c,x)$ is increasing on $[0,\infty[$ by lemma 3.2. Finally, replacing a and c by 1 and b by $n+1$ for all $n \in \mathbb{N}$, we obtain that the functions $x \longmapsto g_n(x)$ and $x \longmapsto f_n(x)$ are also increasing on $[0,\infty[$. So both conjectures 1 and 2 from introduction are proved. And also we found the solution to the Problem 1 from introduction if restrictions of the theorem 1 are valid.

Corollary 4.2. *For all a,b,c be real numbers such that $0 < a < b-c$ and $b > 1$, the following Turán type inequality*

$$[{}_1F_1(a,b,x)]^2 \leq {}_1F_1(a,b-c,x).{}_1F_1(a,b+c,x) \tag{4.2}$$

holds for all $x \in [0,\infty[$.

Proof Since the function $x \longmapsto h(a,b,c,x)$ is increasing on $[0,\infty[$, we have

$$h(a,b,c,x) \geq h(a,b,c,0) = 1.$$

This result is interesting as a corollary of monotonicity property we consider, this inequality itself is not new is known. And in general Turán type inequalities always can be generalized to stronger results on monotonicity of function ratios with unit upper or lower constants.

5. Monotonicity for the hypergeometric function and associated Turán type inequality

Now we also solve the Problem 2 for general hypergeometric–type functions under some natural conditions.

Theorem 5.1. *Let $p,q \in \mathbb{N}$ be such that $p \leq q+1$, $a = (a_1,...,a_p)$, $b = (b_1,...,b_q)$, $c = (c_1,...,c_q)$, $b_i > 0$, $b_i - c_i > 0$ for $i = 1,2,...,q$ and $a_i > b_i$ for $i = 2,...,p$. If $b_i > 1$ for $i = 1,2,...,q$, then the function $x \longmapsto h_{p,q}(a,b,c,x)$ in (2.7) is strictly increasing on $[0,1[$.*

Proof

By using the power–series representations of the function ${}_pF_q(a;b;x)$ we have

$$h_{p,q}(a;b;c;x) = \frac{{}_pF_q(a;b-c;x).\,{}_pF_q(a;b+c;x)}{\left({}_pF_q(a;b;x)\right)^2} =$$

$$= \frac{\left[\sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n...(a_p)_n x^n}{(b_1-c_1)_n(b_2-c_2)_n...(b_q-c_q)_n n!}\right]}{\left[\sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n...(a_p)_n x^n}{(b_1)_n(b_2)_n...(b_q)_n n!}\right]^2} \cdot$$

$$\cdot \left[\sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n...(a_p)_n x^n}{(b_1+c_1)_n(b_2+c_2)_n...(b_q+c_q)_n n!}\right] = \frac{\sum_{n=0}^{\infty} A_n(a,b,c)x^n}{\sum_{n=0}^{\infty} B_n(a,b)x^n}$$

where

$$A_n(a,b,c) = \sum_{k=0}^{n} U_k(a,b,c) =$$

$$= \sum_{k=0}^{n} \frac{[(a_1)_k(a_1)_{n-k}]\,[(a_1)_k(a_1)_{n-k}]\,...\,[(a_p)_k(a_p)_{n-k}]}{[(b_1-c_1)_k...(b_q-c_q)_k]\,[(b_1+c_1)_{n-k}...(b_q+c_q)_{n-k}]\,k!(n-k)!},$$

and

$$B_n(a,b) = \sum_{k=0}^{n} V_k(a,b) =$$

$$= \sum_{k=0}^{n} \frac{[(a_1)_k(a_1)_{n-k}]\,[(a_2)_k(a_2)_{n-k}]\,...\,[(a_p)_k(a_p)_{n-k}]}{[(b_1)_k(b_1)_{n-k}]\,[(b_2)_k(b_2)_{n-k}]\,...\,[(b_q)_k(b_q)_{n-k}]\,k!(n-k)!}.$$

Now, for fixed $n \in \mathbb{N}$ we define sequences $(W_{n,k}(a,b,c))_{k\geq 0}$ by

$$W_{n,k}(a,b,c) = \frac{U_k(a,b,c)}{V_k(a,b)} = \frac{[(b_1)_k(b_1)_{n-k}]\,[(b_2)_k(b_2)_{n-k}]\,...\,[(b_q)_k(b_q)_{n-k}]}{[(b_1-c_1)_k...(b_q-c_q)_k]\,[(b_1+c_1)_{n-k}...(b_q+c_q)_{n-k}]}.$$

For $n, k \in \mathbb{N}$ we evaluate

$$\frac{W_{n,k+1}(a,b,c)}{W_{n,k}(a,b,c)} = \prod_{j=1}^{q} \left[\frac{(b_j)_{k+1}(b_j)_{n-k-1}(b_j-c_j)_k(b_j+c_j)_{n-k}}{(b_j)_k(b_j)_{n-k}(b_j-c_j)_{k+1}(b_j+c_j)_{n-k-1}}\right] =$$

$$= \prod_{j=1}^{q} \left[\left(\frac{\Gamma(b_j+k+1)}{\Gamma(b_j+k)}\right)\left(\frac{\Gamma(b_j+n-k-1)}{\Gamma(b_j+n-k)}\right) \cdot\right.$$

$$\left.\cdot \left(\frac{\Gamma(b_j-c_j+k)}{\Gamma(b_j-c_j+k+1)}\right)\left(\frac{\Gamma(b_j+c_j+n-k-1)}{\Gamma(b_j+c_j+n-k)}\right)\right] =$$

$$= \prod_{j=1}^{q} \left[\frac{b_j+k}{b_j-c_j+k}\right]\left[\frac{b_j+c_j+n-k-1}{b_j+n-k-1}\right] > 1.$$

And now we conclude that $(W_{n,k})_{k\geq 0}$ is increasing and consequently $\left(C_n = \frac{A_n}{B_n}\right)_{n\geq 0}$ is increasing too by the Lemma 3.1. Thus the function $x \longmapsto h_{p,q}(a;b;c;x)$ is increasing on $[0,1[$ by the Lemma 3.2. It completes the proof of the theorem 2.

Corollary 5.2. *Let* $p, q \in \mathbb{N}$ *be such that* $p \leq q+1$, $a = (a_1, ..., a_p)$, $b = (b_1, ..., b_q)$, $c = (c_1, ..., c_q)$, $b_i > 0$, $b_i - c_i > 0$ *for* $i = 1, 2, ..., q$ *and* $a_i > b_i$ *for* $i = 2, ..., p$. *If* $b_i > 1$ *for* $i = 1, 2, ..., q$, *then the following Turán type inequality*

$$ {}_pF_q(a; b - c; x) {}_pF_q(a; b + c; x) > ({}_pF_q(a; b; x))^2 \tag{5.1} $$

holds for all $x \in [0, 1[$.

Proof Follows immediately from the monotonicity of the function $h_{p,q}(a; b; c; x)$.

6. Conclusion

In this paper first we prove that the main theorem for absolutely monotonic functions on $(0, \infty)$ from the book Mitrinović D.S., Pečarić J.E., Fink A.M. "Classical And New Inequalities In Analysis", Kluwer Academic Publishers, 1993, is not valid without additional restrictions. After that connected inequalities for special functions of hypergeometric type is studied. We prove monotonicity of ratios for some special functions of hypergeometric type. Our technique use a kind of a modern Bernoulli-type rule proved by M. Biernacki and J. Krzyz in 1995 [7].

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