Global and Stochastic Analysis
Vol. 13 No. 4 (2026)
DOI: 10.64837/GSA.13.4.9

FILIPPOV LEMMA FOR LEONTIEF-TYPE INCLUSION

M. RYAZANTSEV

Abstract. In this paper we study a Leontief-type inclusion with symmetric first-order mean derivative whose matrix pencil is regular and satisfies the rank-degree condition. For this type of inclusion, we prove an analogue of the well-known Filippov lemma.

Introduction

The notion of mean derivatives was introduced by E. Nelson (see [1, 2, 3]) for the needs of his Stochastic Mechanics, a version of quantum mechanics. The equation of motion in the Stochastic Mechanics was the so called Newton-Nelson equation, a second order stochastic equation with mean derivatives where a special second order mean derivative was in use.

Later, an additional mean derivative, called quadratic, was introduced, and a lot of new applications of the theory of mean derivatives appeared that described physical processes.

Among those the algebraic-differential equation, called the Leontief-type one, should be mentioned. It is based on the idea of measurement of dynamically distorted signals in an electronic devise [4, 5] but also it takes into account the noise. Mainly the symmetric mean derivative (current velocity) is in use in the investigation of such equations. The Leontief-type equations with mean derivatives have been investigated by G.A. Sviridyuk's school, by Yu.E. Gliklikh, E.Yu. Mashkov and others.

In this paper, we continue the research started in the works [6, 7] on the study of Leontief-type inclusions. A well-known analogue of Filippov lemma for the type of inclusions under consideration is formulated and proved.

1. Some facts from matrix theory

Everywhere below we deal with the n dimensional linear space $\mathbb{R}^n$, vectors from $\mathbb{R}^n$ and $n \times n$ matrices.

Consider two $n \times n$ constant matrices A and B where A is degenerate while B is non-degenerate. The expression $\lambda A + B$, where λ is real parameter, is called the matrix pencil. The polynomial $\theta(\lambda) = \det(\lambda A + B)$ is called the characteristic

Date: Date of Submission May 15, 2025; Date of Acceptance June 25, 2025, Communicated by Yuri E. Gliklikh .

2010 *Mathematics Subject Classification.* Primary 60H10; Secondary 60H30,60H99.

Key words and phrases. Mean derivative, stochastic algebraic-differential inclusions, Filippov lemma.

polynomial of the pencil $\lambda A + B$. The pencil is called regular, if its characteristic polynomial is not identically equal to zero.

If the matrix pencil $\tilde{M} + \lambda \tilde{L}$ is regular, one can apply the Kronecker-Weierstrass transformation and reduce the matrices $\tilde{L}$ and $\tilde{M}$ to the canonical quasi-diagonal form (see [8]) that we call L and M, respectively. In the canonical quasi-diagonal form, under appropriate numeration of basis vectors, in the matrix L first along diagonal there are Jordan boxes with zeros on diagonal, and the last matrix along diagonal is the unit one. In M in the lines corresponding to Jordan boxes, there is the unit matrix and the last block along diagonal is a certain non-degenerate matrix. For the sake of convenience, we present matrices L and M in explicit form:

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \tag{1.1}$$

and

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{n-q}^{n-q} & a_{n-q+1}^{n-q} & \dots & a_n^{n-q} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{n-q}^{n-q+1} & a_{n-q+1}^{n-q+1} & \dots & a_n^{n-q+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{n-q}^{n} & a_{n-q+1}^{n} & \dots & a_n^{n} \end{pmatrix}. \tag{1.2}$$

2. Preliminaries on the multivalued mappings

A multivalued mapping F from a set X to a set Y is a mapping that associates a non-empty set $F(x) \subset Y$ with each point $x \in X$.

Definition 2.1. A multivalued mapping is called upper semicontinuous at a point $x \in X$ if, for any $\varepsilon > 0$, there exists a neighborhood $U(x)$ of x such that $x' \in U(x)$ implies that $F(x')$ lies in the ε-neighborhood of the set $F(x)$. The mapping F is called upper semicontinuous on X if it is upper semicontinuous at every point.

Definition 2.2. For a given $\varepsilon > 0$, a continuous single-valued mapping $f_\varepsilon : X \to Y$ is called an ε-approximation of a multivalued mapping $F : X \to Y$ if the graph of f_ε as a set in $X \times Y$ lies in an ε-neighborhood of the graph of F.

Theorem 2.3. [10, Theorem 43.3]. *If a multivalued mapping $F : X \to C(Y)$ is upper semicontinuous and the space Y is regular, then F is closed.*

Theorem 2.4. [10, Theorem 43.4]. *If a multivalued mapping $F_0 : X \to C(Y)$ is closed, a multivalued mapping $F_1 : X \to K(Y)$ is upper semicontinuous, and $F_0(x) \cap F_1(x) \neq \emptyset$ for all $x \in X$, then the intersection $F = F_0 \cap F_1 : X \to K(Y)$ is upper semicontinuous.*

3. Preliminary information on mean derivatives

Everywhere below we deal with processes, equations, etc., given on a certain finite time interval $[0, T]$.

Consider a stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0, T]$, given on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and such that $\xi(t)$ is L_1-random variable for all t.

Every stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0, l]$, determines three families of σ-subalgebras of σ-algebra $\mathcal{F}$:
(i) the "past" $\mathcal{P}_t^\xi$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $0 \le s \le t$;
(ii) the "future" $\mathcal{F}_t^\xi$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $t \le s \le l$;
(iii) the "present" ("now") $\mathcal{N}_t^\xi$ generated by pre-images of Borel sets in $\mathbb{R}^n$ by the mapping $\xi(t)$.
All families are supposed to be complete, i.e., containing all sets of probability 0.

For convenience we denote the conditional expectation of $\xi(t)$ with respect to $\mathcal{N}_t^\xi$ by $E_t^\xi(\cdot)$.

Ordinary ("unconditional") expectation is denoted by E.

Strictly speaking, almost surely (a.s.) the sample paths of $\xi(t)$ are not differentiable for almost all t. Thus its "classical" derivatives exist only in the sense of generalized functions. To avoid using the generalized functions, following Nelson (see, e.g., [1, 2, 3]) we give

Definition 3.1. (i) Forward mean derivative $D\xi(t)$ of $\xi(t)$ at time t is an L_1-random variable of the form

$$D\xi(t) = \lim_{\Delta t \to +0} E_t^\xi\Big(\frac{\xi(t + \Delta t) - \xi(t)}{\Delta t}\Big) \tag{3.1}$$

where the limit is supposed to exists in $L_1(\Omega, \mathcal{F}, \mathsf{P})$ and $\Delta t \to +0$ means that Δt tends to 0 and $\Delta t > 0$, $t \in [0, T)$.

(ii) Backward mean derivative $D_*\xi(t)$ of $\xi(t)$ at t is an L_1-random variable

$$D_*\xi(t) = \lim_{\Delta t \to +0} E_t^\xi\Big(\frac{\xi(t) - \xi(t - \Delta t)}{\Delta t}\Big) \tag{3.2}$$

where the conditions and the notation are the same as in (i), but $t \in (0, T]$.

(iii) The derivative $D_S = \frac{1}{2}(D + D_*)$ is called symmetric mean derivative. The vector $v^\xi(t) = v^\xi(t, \xi(t)) = D_S\xi(t)$ is called current velocity of $\xi(t)$

Please note that the current velocities are natural analogues of physical velocities od deterministic processes.

From the properties of conditional expectation (see [9]) it follows that $D\xi(t)$ and $D_*\xi(t)$ can be represented as compositions of $\xi(t)$ and Borel measurable vector fields (regressions)

$$Y^0(t,x) = \lim_{\Delta t \to +0} E^\xi\Big(\frac{\xi(t+\Delta t) - \xi(t)}{\Delta t} | \xi(t) = x\Big)$$

$$Y^0_*(t,x) = \lim_{\Delta t \to +0} E^\xi\Big(\frac{\xi(t) - \xi(t-\Delta t)}{\Delta t} | \xi(t) = x\Big) \tag{3.3}$$

on $\mathbb{R}^n$. This means that $D\xi(t) = Y^0(t, \xi(t))$ and $D_*\xi(t) = Y^0_*(t, \xi(t))$.

Definition 3.2. For an L^1-stochastic process $\xi(t)$, $t \in [0, T]$, its quadratic mean derivative $D_2\xi(t)$ is defined by the formula

$$D_2\xi(t) = \lim_{\triangle t \to +0} E^\xi_t\Big(\frac{(\xi(t+\triangle t) - \xi(t))(\xi(t+\triangle t) - \xi(t))^*}{\triangle t}\Big), \tag{3.4}$$

where $(\xi(t+\triangle t) - \xi(t))$ is a column vector and $(\xi(t+\triangle t) - \xi(t))^*$ is its conjugate, i.e., the row vector, the limit is supposed to exist in $L^1(\Omega, \mathcal{F}, \mathsf{P})$.

One can easily derive that for an Ito process $\xi(t) = \int_0^t a(s)ds + \int_0^t A(s)dw(s)$ its quadratic mean derivative takes the form $D_2\xi(t) = AA^*$.

Let $Z(t,x)$ be a C^2-smooth vector field, and $\xi(t)$ be a stochastic process.

Definition 3.3. The forward $DZ(t, \xi(t))$ and the backward $D_*Z(t, \xi(t))$ mean derivatives of Z along $\xi(\cdot)$ at t are the L^1-limits of the form

$$DZ(t, \xi(t)) = \lim_{\Delta t \to +0} E^\xi_t\Big(\frac{Z(t+\Delta t, \xi(t+\Delta t)) - Z(t, \xi(t))}{\Delta t}\Big) \tag{3.5}$$

$$D_*Z(t, \xi(t)) = \lim_{\Delta t \to +0} E^\xi_t\Big(\frac{Z(t, \xi(t)) - Z(t-\Delta t, \xi(t-\Delta t))}{\Delta t}\Big) \tag{3.6}$$

Remark 3.4. We understand the second mean derivative, say, $DD_*\xi$ as the mean derivative D of the regression (i.e., vector field) of $D_*\xi$. Below we will deal with DD_* and D_*D.

Since $w(t)$ is a martingale, $Dw(t) = 0$, $t \in [0, l)$ (see above).

Lemma 3.5 (See, e.g., [10, 11])**.** *For $t \in (0, l]$ the equality $D_*w(t) = \frac{w(t)}{t}$ holds.*

Corollary 3.6. $D_Sw(t) = \frac{w(t)}{2t}$.

Let us turn to calculation of higher orders mean derivatives of $w(t)$. Taking into account the system of notation from [10, 11], we look for the k-th derivative as D^w, D^w_* or D^w_S of the $(k-1)$-th derivatives. This notation emphasizes that we always use the σ-algebra "present" of $w(t)$.

Lemma 3.7 (See, e.g., [10, 11]). *(i)* $D^w \frac{w(t)}{t} = -\frac{w(t)}{t^2}$ *for* $t \in (0, l)$.

(ii) $D^w_* \frac{w(t)}{t} = 0$ *for* $t \in (0, l]$.

(iii) $D^w_S \frac{w(t)}{t} = -\frac{w(t)}{2t^2}$ *for* $t \in (0, l]$.

4. Main result

Consider a controlled Leontief-type system with feedback of the form

$$\begin{cases} LD_S\xi(t) = M\xi(t) + u(t), \\ D_2\xi(t) = \Lambda, \\ u(t) \in U(t). \end{cases} \tag{4.1}$$

Here L is a singular real square matrix, M is a non-singular matrix, $\xi : [0, T] \times \Omega \to \mathbb{R}^n$, $\Lambda = QLQ^*$, $U : [0, T] \to \mathsf{K}(\mathbb{R}^m)$, and $u : [0, T] \to \mathbb{R}^m$. Here $\mathsf{K}(\mathbb{R}^m)$ denotes the family of compact subsets of $\mathbb{R}^m$.

A solution to the controlled Leontief-type system (4.1) is defined as a pair $\{\xi(t), u\}$ consisting of a process $\xi(t)$ and a control u. Here, $\xi(t)$ is a diffusion-type process that satisfies (4.1) almost everywhere on $[0, T]$, and u is a Borel measurable function satisfying the inclusion in (4.1) everywhere on $[0, T]$.

For convenience, we introduce the multivalued mapping $\mathbf{a}(t, x) = Mx(t) + U(t)$. We pass from the controlled system (4.1) to the associated Leontief-type differential inclusion

$$\begin{cases} LD_S\xi(t) \in M\xi(t) + U(t), \\ D_2\xi(t) = \Lambda, \end{cases} \tag{4.2}$$

or, taking into account the introduced notation,

$$\begin{cases} LD_S\xi(t) \in \mathbf{a}(t, \xi(t)), \\ D_2\xi(t) = \Lambda. \end{cases} \tag{4.3}$$

It is obvious that any solution of the system (4.1) also satisfies the inclusion (4.2). But we can consider the converse statement.

First, we state an auxiliary lemma [10, Lemma 19.1].

Lemma 4.1. *Let a multivalued mapping* $F : [0, T] \times C^0 \times \mathbb{R}^m \to C(\mathbb{R}^n)$ *be closed, let a mapping* $g : [0, T] \times C^0 \to \mathbb{R}^n$ *be continuous, and for any* $t \in [0, T]$ *there exists* $u(t) \in \mathbb{R}^m$ *such that*

$$g(t, x(\cdot)) \in F(t, x(\cdot), u(t)).$$

Then the multivalued mapping $\Gamma : [0, T] \to C(\mathbb{R}^m)$,

$$\Gamma(t) = \{u(t) \mid u(t) \in \mathbb{R}^m, g(t, x(\cdot)) \in F(t, x(\cdot), u(t))\}$$

is closed.

Theorem 4.2. *Let L be a singular matrix and M be a non-singular matrix. Assume that they form a regular pencil satisfying the rank-degree condition. Furthermore, let U be an upper semicontinuous multivalued mapping with convex values, and let $\xi(t)$ satisfy the inclusion (4.2).*

Then there exists a measurable selector $u \in S_U$, where S_U is the set of selectors of U, such that

$$\begin{cases} LD_S\xi(t) = M\xi(t) + u(t), \\ D_2\xi(t) = \Lambda, \\ u(t) \in U(t). \end{cases}$$

Proof. First, note that by assumption, the images $U(t)$ are compact subsets of $\mathbb{R}^m$, and hence closed sets. Thus, U is an upper semicontinuous multivalued mapping with closed convex values, which means that ε-approximations exist for it. These approximations, given the other conditions of the theorem, will generate perfect solutions of the inclusion (4.2); that is, a suitable ξ exists.

Denote $LD_S\xi(t) = \bar{a}(t, \xi(\cdot))$. We introduce the mapping

$$\Gamma(t) = \{u(t) \mid u(t) \in \mathbb{R}^m, \bar{a}(t, x(\cdot)) \in Mx(\cdot) + U(t)\}.$$

It is easy to see that the mapping $\Phi = \Gamma \cap U : [0, T] \to \mathsf{K}(\mathbb{R}^m)$ is well-defined, and obviously, its measurable selector is the desired function u. Therefore, it is necessary to prove the measurability of the multivalued mapping Φ.

From theorem 2.3 [10, Theorem 43.3], it follows that the multivalued mapping $\mathbf{a}$ is closed. Then, by lemma 4.1 [10, Lemma 19.1], the multivalued mapping Γ is closed. Consequently, by theorem 2.4 [10, Theorem 43.4], the multivalued mapping Φ is upper semicontinuous. Hence, due to the measurability of an upper semicontinuous multivalued mapping, Φ is a measurable mapping.

References

[1] Nelson E.: Derivation of the Schrödinger equation from Newtonian mechanics. *Phys. Reviews*, 1966, Vol. 150, No. 4, 1079 – 1085

[2] Nelson E.: *Dynamical theory of Brownian motion*, Princeton, Princeton University Press, 1967.

[3] Nelson E.: *Quantum fluctuations*, Princeton, Princeton University Press, 1985.

[4] Shestakov A.L., Sviridyuk G.A.: A new approach to measurement of dynamically perturbed signals, *Vestnik of South Ural State University*, 2010, Issue 16(192), 116 – 120 (Russian).

[5] Shestakov A.L., Sviridyuk G.A.: Optimal measurement of dynamically distorted signals, *Vestnik of South Ural State University*, 2011, Issue 17(234), 70 – 75.

[6] Gliklikh Y., Ryazantsev M. On completeness of flow generated by stochastic algebraic-differential equation with forward mean derivative satisfying the rank-degree condition // Journal of applied data analysis and modern stochastic modelling, 2024, Vol 1, no. 1, P. 1 – 6.

[7] Gliklikh Y. A stochastic algebraic-differential equation of Brownian motion type with mean derivative // Journal of Applied Data Analysis and Modern Stochastic Modelling. 2025. Vol. 2, No. 1, P. 1 - 5.

[8] Chistyakov V.F., Shcheglova A.A.: *Selected chapters of the theory of algebraic-differential systems*, Novosibirsk, Nauka, 2003 (Russian)

[9] Parthasarathy, K.R.: *Introduction to Probability and Measure*, New York, Springer-Verlag, 1978.

[10] Gliklikh Yu.E.: *Global Analysis in Mathematical Physics. Geometric and Stochastic Methods*. New York: Springer-Verlag,1997.

[11] Gliklikh Yu. E.: *Global and stochastic analysis with applications to mathematical physics*, London, Springer-Verlag, 2011.

Mikhail Yu. Ryazantsev, Voronezh State University, Voronezh 394018, Russian Federation

Email address: m.ryazancev20@yandex.ru