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THE STOCHASTIC SYSTEM OF WENTZELL MOISTURE FILTRATION EQUATIONS ON A HEMISPHERE AND ON ITS EDGE

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Abstract. The deterministic and stochastic Wentzell systems of the Barenblatt – Zheltov – Kochina equations, which describing the process of moisture filtration in a hemisphere and on its boundary are studied. In the deterministic case, the unambiguous solvability of the initial problem for the Wentzel system in a specific constructed Hilbert space is established. In the case of the stochastic system, the Nelson – Glicklich derivative theory is used and a stochastic solution is constructed, which allows us to determine the quantitative change in the geochemical regime of groundwater under pressureless filtration.

Introduction

Moisture filtration as well as its flow, evaporation, falling, etc. is one of the moisture transfer processes. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a manifold with edge Γ. The system of two Barenblatt – Zheltov – Kochina equations [1], which describing the moisture filtration process is defined on the compact $\Omega \cup \Gamma$

$$(\lambda - \Delta)u_t = \alpha\Delta u + \beta u,\ u = u(t,x),\ (t,x) \in \mathbb{R} \times \Omega, \tag{0.1}$$

$$(\lambda - \Delta)v_t = \gamma\Delta v + \frac{\partial u}{\partial \nu} + \delta v,\ v = v(t,x),\ (t,x) \in \mathbb{R} \times \Gamma, \tag{0.2}$$

$$\mathrm{tr}\ u = v,\ \text{on } \mathbb{R} \times \Gamma. \tag{0.3}$$

Here the symbol Δ in (0.1) denotes the Laplace – Beltrami operator on the smooth Riemannian manifold Ω, and in (0.2) the same symbol denotes the Laplace – Beltrami operator on the smooth Riemannian manifold Γ. The symbol $\nu = \nu(t,x)$, $(t,x) \in \mathbb{R} \times \Gamma$, denotes the external to $\mathbb{R} \times \Gamma$ normal to $\mathbb{R} \times \Omega$. The parameters $\alpha, \lambda, \beta, \gamma, \delta \in \mathbb{R}$ characterize the medium.

We will study the solvability of the system (0.1), (0.2) in the case: $\Omega = \{(\theta,\varphi) : \theta \in [0, \frac{\pi}{2}], \varphi \in [0, 2\pi]\}$ is a hemisphere in $\mathbb{R}^3$, and $\Gamma = \{\varphi : \varphi \in [0, 2\pi)\}$ is the edge of the hemisphere. In this case (0.1), (0.2) is transformed to the form

$$(\lambda - \Delta_{\theta,\varphi})u_t = \alpha\Delta_{\theta,\varphi}u + \beta u,\ u = u(t,\theta,\varphi),\ (t,\theta,\varphi) \in \mathbb{R} \times \Omega, \tag{0.4}$$

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$$(\lambda - \Delta_\varphi)v_t = \gamma\Delta_\varphi v + \partial_\theta u + \delta v,\ v = v(t,\varphi),\ (t,\varphi) \in \mathbb{R}\times\Gamma, \tag{0.5}$$

where the Laplace – Beltrami operator $\Delta_{\theta,\varphi}$ on the hemisphere and the Laplace – Beltrami operator Δ_φ on the edge of the hemisphere have the following form

$$\begin{aligned}\Delta_{\theta,\varphi} &= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2},\\ \Delta_\varphi &= \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2},\quad \partial_\theta = \frac{\partial}{\partial\theta}\bigg|_{\theta=\frac{\pi}{2}}.\end{aligned} \tag{0.6}$$

To this system we add the matching condition (0.3) and equip it with initial conditions

$$u(0,\theta,\varphi) = u_0(\theta,\varphi),\quad v(0,\varphi) = v_0(\varphi). \tag{0.7}$$

Let us call the solution of the problem (0.3) – (0.7) the deterministic solution of the Wentzell system. If we replace the functions u and v, defined by Ω and Γ respectively, on $\eta = \eta(t)$ and $\kappa = \kappa(t)$ are a stochastic processes on the interval $(0,\tau)$, we obtain stochastic Wentzell system, where the derivative of stochastic processes is understand by the Nelson – Gliklikh derivative of the process. It associated with correct definition of "white noise" as one-dimensional Wiener process (see, for example., [8, 9, 12]). Let us call the solution of the corresponding problem the stochastic solution of the Wentzell system.

The paper, besides the introduction and the list of references, contains three parts. In the first part, the existence and uniqueness of the deterministic system of Wentzell equations in the hemisphere and on its edge are considered. The second part contains abstract reasoning consisting in constructing the space of ($\mathfrak{H}$*-valued)* **K**-*"noise"*. The third part contains the proof of the existence and uniqueness of the stochastic Wentzell system of equations in the hemisphere and on its edge.

1. Wentzell's deterministic system

If $\theta_k = k(k+1)$ eigenvalues of the Laplace – Beltrami operator $\Delta_{\theta,\varphi}$, then

$$Y_k^m(\varphi,\theta) = \begin{cases} P_k^m(\cos\theta)\cos m\varphi, & m = 0,\cdots,k;\\ \\ P_k^{|m|}(\cos\theta)\sin|m|\varphi, & m = -k,\cdots,-1\end{cases}$$

are the corresponding eigenfunctions orthonormalized with respect to the scalar product. Here,

$$P_k(t) = \frac{1}{2^k k!}\frac{d^k}{dt^k}dt^k(t^2-1)^k$$

is a Lejandre polynomial of degree k, and

$$P_k^{|m|}(t) = (1-t^2)^{\frac{|m|}{2}}\frac{d^{|m|}}{dt^{|m|}}P_k(t)$$

is the attached Lejandre polynomial. The scalar product is calculated using the following formula

$$\langle Y_{k_1}^{m_1}, Y_{k_2}^{m_2}\rangle = \int\limits_0^{2\pi}\cos m_1\varphi\cos m_2\varphi d\varphi\int\limits_{-1}^{1}P_{k_1}^{m_1}(t)P_{k_2}^{m_2}(t)dt.$$

Consider the following series

$$u=\sum_{k=1}^{\infty}\sum_{m=0}^{k}\exp\left(t\frac{\beta-\alpha k^2}{\lambda+k^2}\right)\Big(a_{m,k}\cos m\varphi+b_{m,k}\sin m\varphi\Big)P_k^m(\cos\theta), \qquad (1.1)$$

where

$$a_{m,k}=\int\limits_0^{2\pi}u_0(\theta,\varphi)\cos m\varphi d\varphi\int\limits_0^{\frac{\pi}{2}}P_k^m(0)\sin\theta d\theta,$$

$$b_{m,k}=\int\limits_0^{2\pi}u_0(\theta,\varphi)\sin m\varphi d\varphi\int\limits_0^{\frac{\pi}{2}}P_k^m(0)\sin\theta d\theta.$$

It is easy to see that the series constructed above is a formal solution of the equation (0.4). Moreover, if the series in (1.1) converge uniformly, then we have a solution to the problem (0.4), (0.7), where $\partial_\theta u=0$. Given this, we can construct a solution to the problem (0.5), (0.7)

$$v=\sum_{k=1}^{\infty}\exp\left(t\frac{\delta-\gamma k^2}{\lambda+k^2}\right)\Big(c_k\cos k\varphi+d_k\sin k\varphi\Big), \qquad (1.2)$$

where

$$c_k=\int\limits_0^{2\pi}v_0(\varphi)\cos k\varphi d\varphi, d_k=\int\limits_0^{2\pi}v_0(\varphi)\sin k\varphi d\varphi.$$

In the case of the matching condition (0.3) we obtain the following equation

$$\sum_{k=1}^{\infty}\sum_{m=0}^{k}\exp\left(t\frac{\beta-\alpha k^2}{\lambda+k^2}\right)\Big(a_{m,k}\cos m\varphi+b_{m,k}\sin m\varphi\Big)P_k^m(\cos\theta)\Big|_{\theta=\frac{\pi}{2}}$$
$$=\sum_{k=1}^{\infty}\exp\left(t\frac{\delta-\gamma k^2}{\lambda+k^2}\right)\Big(c_k\cos k\varphi+d_k\sin k\varphi\Big).$$

Considering that $\alpha=\gamma$, $\beta=\delta$ we obtain equalivent system of equations

$$\sum_{m=0}^{k}\Big(a_{m,k}\cos m\varphi+b_{m,k}\sin m\varphi\Big)P_k^m(0)=c_k\cos k\varphi+d_k\sin k\varphi,\ \text{where } m+n=2k.$$

Substituting the integral coefficients we obtain an equivalent system

$$\sum_{m=0}^{k}\Big(\int_0^{2\pi} u_0(\theta,\varphi)\cos m\varphi d\varphi \int_0^{\frac{\pi}{2}} P_k^m(0)\sin\theta d\theta \cos m\varphi$$

$$+\int_0^{2\pi} u_0(\theta,\varphi)\sin m\varphi d\varphi \int_0^{\frac{\pi}{2}} P_k^m(0)\sin\theta d\theta \sin m\varphi\Big) P_k^m(0)$$

$$=\int_0^{2\pi} v_0(\varphi)\cos k\varphi d\varphi \cos k\varphi + \int_0^{2\pi} v_0(\varphi)\sin k\varphi d\varphi \sin k\varphi.$$

Here the auxiliary integrals are calculated by the formula

$$\int_0^{\frac{\pi}{2}} P_k^m(0)\sin\theta d\theta = P_k^m(0)\int_0^{\frac{\pi}{2}} \sin\theta d\theta = P_k^m(0),$$

and system has the following form

$$\sum_{m=0}^{k}\Big(\int_0^{2\pi} u_0(\theta,\varphi)\cos m\varphi d\varphi \cos m\varphi + \int_0^{2\pi} u_0(\theta,\varphi)\sin m\varphi d\varphi \sin m\varphi\Big)\Big(P_k^m(0)\Big)^2$$

$$=\int_0^{2\pi} v_0(\varphi)\cos k\varphi d\varphi \cos k\varphi + \int_0^{2\pi} v_0(\varphi)\sin k\varphi d\varphi \sin k\varphi. \tag{1.3}$$

Thus in the case $\alpha=\gamma$, $\beta=\delta$ and the obtained condition (1.3) the solutions to the problem (0.5) – (0.7) will satisfy the (0.3) matching condition.

Lineal closure span$\{P_k^m(\cos\theta)\sin m\varphi,\ P_k^m(\cos\theta)\cos m\varphi\colon m,k\in\mathbb{N}\backslash\{1\},\theta\in[0,\frac{\pi}{2}],\varphi\in[0,2\pi)\}$ generated by the scalar product

$$\langle\varphi,\psi\rangle=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\varphi(\theta,\varphi)\psi(\theta,\varphi)\sin\theta d\theta d\varphi,$$

we denote by the symbol $A(\Omega)$. Next, the closure of the span$\{\sin k\varphi,\ \cos k\varphi\colon k\in\mathbb{N},\varphi\in[0,2\pi)\}$ by the norm, generated by the scalar product

$$\langle\xi,\psi\rangle=\int_0^{2\pi}\xi(\varphi)\psi(\varphi)d\varphi,$$

denote by the symbol $A(\Gamma)$.

Thus, the following theorem holds.

Theorem 1.1. *For any $u_0\in A(\Omega)$ and $v_0\in A(\Gamma)$ such that* (0.3) *is satisfied, and for the coefficients $\alpha,\beta,\gamma,\delta,\lambda\in\mathbb{R}$, such that the following condition is satisfied $\alpha=\gamma$, $\beta=\delta$, and $\lambda\neq k^2$, where $k\in\mathbb{N}$, and condition* (1.3) *is fulfilled, there exists a single solution $(u,v)\in C^\infty(\mathbb{R};A(\Omega)\oplus A(\Gamma))$ of the problem* (0.3) – (0.5).

2. The space of "noises"

Let $\Omega \equiv (\Omega, \mathcal{A}, P)$ be a full probability space; $\mathbb{R}$ be set of real numbers endowed with the Borel σ-algebra. By a *random variable* we mean measurable mapping $\xi\colon \Omega \to \mathbb{R}$. A set of random variables $\{\xi : E\xi = 0, D\xi \le +\infty\}$, the mathematical expectation of which is equal to zero, and the dispersion is finite, forms the Hilbert space $\mathbf{L_2}$ with the scalar product $(\xi_1, \xi_2) = E\xi_1\xi_2$ and the norm $\|\xi\|^2_{\mathbf{L_2}} = D\xi$.

Consider the set $\mathcal{J} \subset \mathbb{R}$ and the following two mappings. First, $f\colon \mathcal{J} \to \mathbf{L_2}$, associates each $t \in \mathcal{J}$ with a random variable $\xi \in \mathbf{L_2}$. Second, $g\colon \mathbf{L_2} \times \Omega \to \mathbb{R}$, associates each pair (ξ, ω) with a point $\xi(\omega) \in \mathbb{R}$.

A mapping $\eta\colon \mathcal{J} \times \Omega \to \mathbb{R}$, having the form $\eta = \eta(t,\omega) = g(f(t),\omega)$, is called an *(one-dimensional) stochastic process*. For each fixed $t \in \mathcal{J}$, the value of the stochastic process $\eta = \eta(t,\cdot)$ is a random value, i.e. $\eta = \eta(t,\cdot) \in \mathbf{L_2}$, which is called *a section of a stochastic process at $t \in \mathcal{J}$*. For each fixed $\omega \in \Omega$, the function $\eta = \eta(\cdot,\omega)$ is called *a (sample) path of a stochastic process* , corresponding to the elementary event result $\omega \in \Omega$. The paths are also called realizations or sample functions of a random process.

Usually, when this does not lead to ambiguity, the dependence of $\eta(t,\omega)$ on ω is not specified and a random process is denoted by $\eta(t)$.

Let be an interval $\mathcal{J} \subset \mathbb{R}$, then the stochastic process $\eta = \eta(t), t \in \mathcal{J}$ is called *continuous*, if all its paths are almost sure continuos.

The set of continuous stochastic processes forms a Banach space, which we denote by $\mathbf{CL_2}$, where

$$\|\eta\|^2_{\mathbf{CL_2}} = \sup D\eta(t,\omega).$$

Let $\mathcal{A}_0$ be a σ-subalgebra of the σ-algebra $\mathcal{A}$. Construct the subspace $\mathbf{L_2^0} \subset \mathbf{L_2}$ of random variables measurable with respect to $\mathcal{A}_0$. Denote by $\Pi\colon \mathbf{L_2} \to \mathbf{L_2^0}$ an orthoprojector.

For any $\xi \in \mathbf{L_2}$, a random value of $\Pi\xi$ is called a*conditional expectation* of a random value of ξ with respect to $\mathcal{A}_0$ and is denoted by $\mathbf{E}(\xi|\mathcal{A}_0)$.

Fix $\eta \in \mathbf{CL_2}$ and $t \in \mathcal{J}$. Denote by $\mathbf{N}_t^\eta$ a σ-algebra generated by a random value of $\eta(t)$, and denote by $\mathbf{E}_t^\eta = \mathbf{E}(\cdot|\mathbf{N}_t^\eta)$ a conditional expectation with respect to $\mathbf{N}_t^\eta$.

Let $\eta \in \mathbf{CL_2}$, the *Nelson–Gliklikh derivative $\overset{\circ}{\eta}$ of the stochastic process $\eta(t)$ at the point $t \in \mathcal{J}$* is called a random variable

$$\overset{\circ}{\eta}(t,\cdot) = \frac{1}{2}\{\lim_{\Delta t \to 0+} \mathbf{E}_t^\eta\left(\frac{\eta(t+\Delta t,\cdot) - \eta(t,\cdot)}{\Delta t}\right) + \lim_{\Delta t \to 0+} \mathbf{E}_t^\eta\left(\frac{\eta(t,\cdot) - \eta(t-\Delta t,\cdot)}{\Delta t}\right)\},$$

if the limits exist in the sense of the uniform metric on $\mathbb{R}$.

If the Nelson–Gliklikh derivatives $\overset{\circ}{\eta}(t,\cdot)$ of the stochastic process $\eta(t)$ exist in all (or almost all) points of the interval $\mathcal{J}$, then we say that the Nelson–Gliklikh derivative $\overset{\circ}{\eta}(t,\cdot)$ exist on $\mathcal{J}$ (almost sure on $\mathcal{J}$).

As an example, consider the Nelson–Gliklikh derivative for the Wiener process $\beta(t)$ (see, for example, [6]), describing Brownian motion in the Einstein-Smoluchowski model

$$\overset{\circ}{\beta}(t) = \frac{\beta(t)}{2t}, \quad t \in \mathbb{R}_+.$$

Note that the set of continuous stochastic processes having the derivative $\overset{\circ}{\eta}(t,\cdot)$ forms the Banach space $\mathbf{C^1L_2}$ with the norm

$$\|\eta\|^2_{\mathbf{C^1L_2}} = \sup_{\mathcal{J}} \left(D\eta(t,\omega) + D\overset{\circ}{\eta}(t,\omega) \right).$$

Introduce the space $\mathbf{C^lL_2}$, $l \in \{0\} \cup \mathbb{N}$, of random processes from $\mathbf{CL_2}$, whose paths are differentiable (almost sure) by Nelson–Gliklikh on $\mathcal{J}$ up to the order l inclusively, define the norm in the space by the following formula:

$$\|\eta\|^2_{\mathbf{C^lL_2}} = \sup_{\mathcal{J}} \left(\sum_{k=0}^{l} D\overset{\circ}{\eta}{}^{l}(t,\omega) \right).$$

By definition, we understand the Nelson–Gliklikh derivative of the order zero $\overset{\circ}{\eta}{}^{0}$ as the original stochastic process, by the space $\mathbf{C^lL_2}$, $l \in \{0\} \cup \mathbb{N}$ we understand *the space of* $\mathbf{K}$*–"noises"*.

Let us consider a real separable Hilbert space $\mathfrak{U}$ ($\mathfrak{F}$) with orthonormal basis $\{\varphi_k\}$ ($\{\psi_k\}$). Introduce a monotonic sequence $K = \{\lambda_k\} \subset \mathbb{R}$ such that $\sum\limits_{k=1}^{\infty} \lambda_k^2 < \infty$. Denote by $\mathbf{U_KL_2}$ ($\mathbf{F_KL_2}$) the Hilbert space, which is a completion of the linear span of $\mathbf{K}$-*random variables*

$$\xi = \sum_{k=1}^{\infty} \lambda_k \xi_k \varphi_k, \quad \xi_k \in \mathbf{L_2} \quad \left(\zeta = \sum_{k=1}^{\infty} \mu_k \zeta_k \psi_k \quad \zeta_k \in \mathbf{L_2} \right)$$

by the norm

$$\|\xi\|^2_{\mathbf{U}} = \sum_{k=1}^{\infty} \lambda_k^2 D\xi_k, \quad \left(\|\zeta\|^2_{\mathbf{F}} = \sum_{k=1}^{\infty} \mu_k^2 D\zeta_k \right).$$

Note that for existence of a $\mathbf{K}$-*random variable* $\xi \in \mathbf{U_KL_2}$ ($\zeta \in \mathbf{F_KL_2}$) it is enough to consider a sequence of random variables $\{\xi_k\} \subset \mathbf{L_2}$ ($\{\zeta_k\} \subset \mathbf{L_2}$) having uniformly bounded dispersions $D\xi_k \leq Const$ $(D\zeta_k \leq Const)$, $k \in \mathbb{N}$.

Construct the *space of differentiable* $\mathbf{K}$*–"noises"*. Consider the inteval $(\epsilon, \tau) \subset \mathbb{R}$. A mapping $\eta\colon (\epsilon,\tau) \to \mathbf{U_KL_2}$ given by the formula

$$\eta(t) = \sum_{k=1}^{\infty} \lambda_k \xi_k(t) \varphi_k,$$

where the sequence $\{\xi_k\} \subset \mathbf{CL_2}$, is called a $\mathfrak{U}$-*valued continuous stochastic* $\mathbf{K}$-*process*, if the series on the right converges uniformly on any compact in $\mathcal{J}$ by the norm $\|\cdot\|_{\mathbf{U}}$ and paths of the process $\eta = \eta(t)$ are almost sure continuous.

A continuous stochastic $\mathbf{K}$-process

$$\overset{\circ}{\eta}(t) = \sum_{k=1}^{\infty} \lambda_k \overset{\circ}{\xi_k}(t) \varphi_k, \tag{2.1}$$

is called *continously differentiable by Nelson–Gliklikh* on $\mathcal{J}$, if the series converges uniformly on any compact in $\mathcal{J}$ by the norm $\|\cdot\|_{\mathbf{U}}$ and paths of the process $\overset{\circ}{\eta} = \overset{\circ}{\eta}(t)$ are almost sure continuous.

Denote by $\mathbf{C^l}(\mathcal{J}, \mathbf{U_K L_2})$, $l \in \{0\} \cup \mathbb{N}$ the space of *differentiable* **K**–*"noises"*, whose paths almost sure differentiable by Nelson–Gliklikh on $\mathcal{J}$ up to the order l inclusively, with the following norm:

$$\|\eta\|^2_{\mathbf{C^l}(\mathcal{J},\mathbf{U_K L_2})} = \sup_{\mathcal{J}} \left(\sum_{k=0}^{\infty} \lambda_k^2 \sum_{j=1}^{l} D \overset{\circ}{\eta}^{j} \right).$$

An example of *continously differentiable by Nelson–Gliklikh* up to the order l inclusively **K**-*process* is Wiener **K**-*process* (see, for example, [6])

$$W_{\mathbf{K}}(t) = \sum_{k=1}^{\infty} \lambda_k \beta_k(t) \varphi_k$$

where $\{\beta_k\} \subset \mathbf{C^l L_2}$ is a sequence of Brownian motions on $\mathbb{R}_+$.

Similarly, the space of $\mathbf{C^l}(\mathcal{J}, \mathbf{F_K L_2})$, i.e. *differentiable* **K**–*"noises"* on $\mathbf{F_K L_2}$, are constructed.

3. Wentzell's stochastic system

Let $A : \mathfrak{U} \to \mathfrak{F}$ be a linear operator. By the formula

$$A\xi = \sum_{k=1}^{\infty} \lambda_k \xi_k A\varphi_k \tag{3.1}$$

set the linear operator $A : \mathbf{U_K L}_2 \to \mathbf{F_K L}_2$, and if the series in the right-hand side of (3.1) converges (in the metric $\mathbf{F_K L}_2$), then $\xi \in \mathrm{dom}A$, and if it diverges, then $\xi \notin \mathrm{dom}A$. Traditionally, the spaces of linear continuous operators $\mathcal{L}(\mathbf{U_K L}_2; \mathbf{F_K L}_2)$ and linear closed densely defined operators are defined. The following holds

Lemma 3.1. *(i) Operator $A \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ exactly when $A \in \mathcal{L}(\mathbf{U_K L}_2; \mathbf{F_K L}_2)$.*

As you can easily see,

$$\|A\xi\|_{\mathbf{F}} \leq \sum_{k=1}^{\infty} \lambda_k^2 \mathbf{D}\xi_k \|A\varphi_k\|^2 \mathfrak{F} \leq \mathrm{const} \sum_{k=1}^{\infty} \lambda_k^2 \mathbf{D}\xi_k = \mathrm{const}\, \|\xi\|\|_{\mathbf{U}}.$$

(ii) Operator $A \in Cl(\mathfrak{U}; \mathfrak{F})$ exactly when $A \in Cl(\mathbf{U_K L}_2; \mathbf{F_K L}_2)$.

For simplicity's sake, let $\mathfrak{U} = \{u \in W_2^2(\Omega) \oplus W_2^2(\Gamma) : \partial_\theta u = 0\}$, $\mathfrak{F} = L_2(\Omega) \oplus L_2(\Gamma)$. Next, following the algorithm above, construct the spaces *of random* **K**-*values* . *Random* **K**-*value* $\xi \in \mathbf{U_K L_2}$ has the form

$$\eta = \sum_{k=1}^{\infty} \lambda_k \xi_k \varphi_k, \quad \kappa = \sum_{k=1}^{\infty} \mu_k \xi_k \psi_k \tag{3.2}$$

where $\{\varphi_k\}$ is the family of eigenfunctions of the Laplace – Beltrami operator $\Delta_{\theta,\varphi} \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ orthonormalized in the sense of the scalar product $\langle \cdot, \cdot \rangle$ of $L_2(\Omega)$; $\{\psi_k\}$ is the family of eigenfunctions of the Laplace – Beltrami operator $\Delta_\varphi \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ orthonormalized in the sense of the scalar product $\langle \cdot, \cdot \rangle$ of $L_2(\Gamma)$. Consider the linear stochastic Wentzel system of the moisture filtration equation in the hemisphere and at its edge. In this case (0.3) – (0.5) is transformed to the form

$$(\lambda - \Delta_{\theta,\varphi})\eta_t = \alpha \Delta_{\theta,\varphi} \eta + \beta \eta, \;\; \eta \in C^\infty(\mathbb{R}_+; \mathbf{U_K L}_2), \tag{3.3}$$

$$(\lambda - \Delta_\varphi)\kappa_t = \gamma\Delta_\varphi\kappa + \partial_\theta\kappa + \delta\kappa,\ \kappa \in C^\infty(\mathbb{R}_+; \mathbf{U_K L}_2), \tag{3.4}$$

where

$$\Delta_{\theta,\varphi} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2},$$
$$\Delta_\varphi = \frac{\partial^2}{\partial\varphi^2}, \quad \partial_\theta = \frac{\partial}{\partial\theta}\bigg|_{\theta=\frac{\pi}{2}}.$$

We add a matching condition to this system and supply it with initial conditions

$$\eta(0) = \eta_0,\ \kappa(0) = \kappa_0. \tag{3.5}$$

Let us call the solution of the problem (3.3)-(3.5) the stochastic solution of the Wentzell system.

Thus, the following theorem holds.

Theorem 3.2. *For any* $u_0 \in A(\Omega)$ *and* $v_0 \in A(\Gamma)$ *such that* (0.3) *is satisfied, and for the coefficients* $\alpha, \beta, \gamma, \delta, \lambda \in \mathbb{R}$*, such that the following condition is satisfied* $\alpha = \gamma$*,* $\beta = \delta$*, and* $\lambda \neq k^2$*, where* $k \in \mathbb{N}$*, and condition* (1.3) *is fulfilled, there exists a single solution* $(\eta, \kappa) \in C^\infty(\mathbb{R}; \mathbf{U_K L_2})$ *of the stochastic Wentzell problem* (0.3) – (0.5).

Conclusion

We constructed an solution for the deterministic and stochastic Wentzell system of moisture filtration equation on the hemisphere and on its edge.For this purpose, we used a new approach to the study of the stochastic model with "white noise", which we understand as the Nelson–Gliklikh derivative of one-dimensional Wiener process. Further, we plan to continue the results of the paper by applying the Wentzell boundary conditions in directions related to [15].

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References

1. Barenblatt, G.I., Zheltov, Iu.P., Kochina, I.N. Basic Concepts in the Theory of Seepage of Homogeneous Liquids in Fissured Rocks [Strata]. Journal of Applied Mathematics and Mechanics, 1960, Vol. 24, №. 5, P. 1286–1303. DOI: 10.1016/0021-8928(60)90107-6
2. Favini, A. Multipoint Initial-Final Value Problem for Dynamical Sobolev-Type Equation in the Space of Noises / A. Favini, S.A. Zagrebina, G.A. Sviridyuk // Electron. J. Evol. Equ. – 2018. – V. 2018, № 128. – P. 1–10.
3. Favini, A.The multipoint initial-final value condition for the Hoff equations in geometrical graph in spaces of K-"noises" / A. Favini, S.A. Zagrebina, G.A. Sviridyuk // Mediterr. J. Math. – 2022. – V. 19, № 2. – P. 53
4. Favini, A. Linear Sobolev Type Equations with Relatively p-Sectorial Operators in Space of "Noises" / A. Favini, G.A. Sviridyuk, N.A. Manakova // Abstract and Applied Analysis. – 2015. – V. 2015. – 8 p.
5. Favini, A. One Class of Sobolev Type Equations of Higher Order with Additive "White Noise" / A. Favini, G.A. Sviridyuk, A.A. Zamyshlyaeva // Communications on Pure and Applied Analysis. – 2016. – V. 15, № 1. – P. 185–196.

6. Favini, A. Linear Sobolev Type Equations with Relatively-Sectorial Operators in Space of "Noises" / A. Favini, G.A. Sviridyuk, M. Sagadeeva // Abstract and Applied Analysis. – 2016.– V. 13. – p. 4607.
7. Goncharov, N.S. Non-Uniqueness of Solutions to Boundary Value Problems with Wentzell Condition / N.S. Goncharov, S.A. Zagrebina, G.A. Sviridyuk // Bulletin of the South Ural State University. Series: Mathematical Modeling, Programming and Computer Software. – 2021. – V. 14, № 4. – P. 102–105.
8. Gliklikh Yu.E.: *Global and Stochastic Analysis with Applications to Mathematical Physics*, London; Dordrecht; Heidelberg; N.-Y., Springer, 2011.
9. Goncharov, N.S. Stochastic Barenblatt – Zheltov – Kochina Model on the Interval with Wentzell Boundary Conditions / N.S. Goncharov // Global and Stochastic Analysis. – 2020. – V. 7, № 1. – P. 11–23.
10. Giuseppe M.Coclite., Favini A., Gal Ciprian G., Goldstein G.R.: The Role of Wentzell Boundary Conditions in Linear and Nonlinear Analysis, *Tubinger Berichte*, **132**, (2008) 279–292.
11. Manakova N.A., Gavrilova O.V. About nonuniqueness of solutions of the Showalter-Sidorov problem for one Mathematical model of nerve impulse spread in membrane / N. A. Manakova, O. V. Gavrilova // Bulletin of the South Ural State University. Series: Mathematical Modeling, Programming and Computer Software. – 2018. – V.11, № 4. – P. 161–168.
12. Shestakov A.L., Sviridyuk G.A., Hudyakov Yu.V. Dynamic measurement in spaces of "noise" / A.L. Shestakov, G.A. Sviridyuk, Yu.V. Hudyakov // Bulletin of the South Ural State University. Series: Computer Technologies, Automatic Control, Radio Electronics. – 2013. – V. 13, № 2. – P. 4–11.
13. Wentzell A.D.: Semigroups of operators corresponding to a generalized differential operator of second order, *Doklady Academii Nauk SSSR*, **111**, (1956) 269–272. (in Russian)
14. Wentzell A.D.: On boundary conditions for multidimensional diffusion processes, *Theory of Probability and its Applications*, **4**, (1959) 164–177.
15. Zagrebina S.A., Konkina A.S. The Multipoint Initial-Final Value Condition For The Navier – Stokes Linear Model / S.A. Zagrebina, A.S. Konkina // Bulletin of the South Ural State University. Series: Mathematical Modeling, Programming and Computer Software. – 2015. – V. 8, № 1. – P. 132–136.

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