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GENERALIZING STOCHASTIC BASES AND MARTINGALES USING STOPPING TIMES

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Abstract. $\mathcal{T}$-generalized stochastic bases $\{Q_\tau^\nu, 0 \le \tau \le \nu \le \infty\}$, where τ and ν are stopping times, were axiomatically introduce. If τ and ν are constant stopping times, we obtain the notion from [1]. We discribe some important properties of $\mathcal{T}$-generalized stochastic bases and of $\mathcal{T}$-generalized martingales. The article explains the rationale for introducing generalized stochastic bases.

1. Introduction

Let $(\Omega, (\mathcal{F}_t)_{t=0}^\infty)$ be a filtration with continuous (or discrete) time. Denote by $\mathcal{F}_\infty$ the least σ-field containing all $\mathcal{F}_t$ and by $\mathcal{T}$ the set of all stopping times τ $(0 \le \tau \le \infty)$. For $\tau, \nu \in \mathcal{T}$ we write $\tau \le \nu$ if $\tau(\omega) \le \nu(\omega)\ \forall \omega \in \Omega$. If $(\Omega, \mathcal{F}, P)$ is a probability space and $\mathcal{G}$ is a sub-σ-field of the σ-field $\mathcal{F}$, then we use the notation

$$E_{\mathcal{G}}^P f = E^P[f|\mathcal{G}].$$

Consider a family $\mathbf{Q} = (Q_\tau^\nu, \mathcal{F}_\nu)_{\{0 \le \tau \le \nu \le \infty\}}$ of probability measures Q_τ^ν on $\mathcal{F}_\nu$ and generated by them the family of operators $\mathbf{E} = (E_\tau^\nu)_{\{0 \le \tau \le \nu \le \infty\}}$ of conditional expectation

$$E_\tau^\nu f := E_{\mathcal{F}_\tau}^{Q_\tau^\nu} f,$$

where f is a non-negative $\mathcal{F}_\nu$-measurable random variable (r.v.).

In what follows, we denote the absolute continuity of measures by the symbol $\ll$ and the equivalence of measures by the symbol $\sim$.

Definition 1.1. A triplet

$$(\Omega, \mathbf{Q}, \mathbf{E}) \tag{1.1}$$

is called $\mathcal{T}$-generalized stochastic basis ($\mathcal{T}$-GSB) if $\forall \tau, \mu, \nu \in \mathcal{T}$ such that $0 \le \tau \le \mu \le \nu \le \infty$

$$Q_\tau^\mu \ll Q_\tau^\nu|_{\mathcal{F}_\mu}, \tag{1.2}$$

$$Q_\tau^\nu|_{\mathcal{F}_\mu} \ll Q_\mu^\nu|_{\mathcal{F}_\mu}, \tag{1.3}$$

and

$$E_\tau^\mu E_\mu^\nu f = E_\tau^\nu f \tag{1.4}$$

Q_τ^μ-almost surely (a.s.) for any $\mathcal{F}_\nu$-measurable r.v. $f \ge 0$.

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Corollary 1.2. *From* (1.2) *it follows that* $\forall 0 \le \tau \le \nu \le \infty$

$$Q_\tau^\tau \ll Q_\tau^\nu|_{\mathcal{F}_\tau}. \tag{1.5}$$

From (1.3) *it follows that* $\forall 0 \le \tau \le \nu \le \infty$

$$Q_\tau^\nu \ll Q_\nu^\nu. \tag{1.6}$$

From (1.5) *and* (1.6) *it follows that* $\forall 0 \le \tau \le \mu \le \nu \le \infty$

$$Q_\tau^\mu \ll Q_\mu^\nu|_{\mathcal{F}_\mu}. \tag{1.7}$$

From (1.7) *it follows that* $\forall 0 \le \tau_1 \le \nu_1 \le \tau_2 \le \nu_2 \le \infty$

$$Q_{\tau_1}^{\nu_1} \ll Q_{\tau_2}^{\nu_2}|_{\mathcal{F}_{\nu_1}}. \tag{1.8}$$

From (1.4) *it follows that* $\forall 0 \le \tau \le \mu_1 \le \mu_2 \le \dots \le \mu_{n-1} \le \nu \le \infty$ *and for any non-negative* $\mathcal{F}_\nu$*-mesurable r.v.* f *the equality*

$$E_\tau^\nu f = E_\tau^{\mu_1} E_{\mu_1}^{\mu_2} \dots E_{\mu_{n-1}}^\nu f. \tag{1.9}$$

holds $Q_\tau^{\mu_1}$ *-a.s.*

2. Examples of $\mathcal{T}$-generalized stochastic bases

Example 2.1. Let $(\Omega, (\mathcal{F}_t)_{t=0}^\infty, P)$ be an usuel stochastic basis. Define $\forall\, 0 \le \tau \le \nu \le \infty$ $Q_\tau^\nu = P|_{\mathcal{F}_\nu}$ and corresponding operators E_τ^ν. It is easy to see that $(\Omega, \mathbf{Q}, \mathbf{E})$ is a $\mathcal{T}$-GSB. We will say that such $\mathcal{T}$-GSB is generated by probability P.

Example 2.2. Let $(\Omega, (\mathcal{F}_t)_{t=0}^\infty, P)$ be a stochastic basis. Let us fixe $\sigma \in \mathcal{T}$ such that $P(\{\sigma = 0\}) > 0$. Denote $A_\mu := \{\sigma \le \mu\}$ for all $\mu \in \mathcal{T}$.

Define $\forall\, 0 \le \tau \le \nu \le \infty$

$$Q_\tau^\nu(A) := \frac{P(AA_\tau)}{P(A_\tau)} \quad (\forall A \in \mathcal{F}_\nu). \tag{2.1}$$

Prove that $\mathbf{Q} = (Q_\tau^\nu)$ with the corresponding $\mathbf{E}$ form a $\mathcal{T}$-DSB (we say that such $\mathcal{T}$-DSB is generated by conditional expectations).

1) Suppose that $A \in \mathcal{F}_\mu$ $(\tau \le \mu \le \nu)$. Then

$$Q_\tau^\mu(A) = \frac{P(AA_\tau)}{P(A_\tau)} = Q_\tau^\nu|_{\mathcal{F}_\mu}(A). \tag{2.2}$$

Hence $Q_\tau^\mu = Q_\tau^\nu|_{\mathcal{F}_\mu}$ and relation (1.2) is obtained.

2) Now prove that property (1.3) is fulfilled. Taking into account the embedding $A_\tau \subset A_\mu$ we have for $A \in \mathcal{F}_\mu$:

$$Q_\mu^\nu|_{\mathcal{F}_\mu}(A) = \frac{P(AA_\mu)}{P(A_\mu)} \ge \frac{P(AA_\tau)}{P(A_\mu)}.$$

On the other side

$$Q_\tau^\nu|_{\mathcal{F}_\mu}(A) = \frac{P(AA_\tau)}{P(A_\tau)}.$$

Hence

$$Q_\mu^\nu|_{\mathcal{F}_\mu}(A) \ge \frac{P(A_\tau)}{P(A_\mu)} Q_\tau^\nu|_{\mathcal{F}_\mu}(A),$$

i.e. $Q_\mu^\nu|_{\mathcal{F}_\mu} \gg Q_\tau^\nu|_{\mathcal{F}_\mu}$.

3) Finally, prove (1.4). It suffi to establish the equality

$$E^{Q^\mu_\tau}\left[E^{Q^\nu_\mu}[f|\mathcal{F}_\mu] \mid \mathcal{F}_\tau\right] = E^{Q^\nu_\tau}[f|\mathcal{F}_\tau] \quad Q^\mu_\tau - a.s.$$

for all $\mathcal{F}_\nu$-mesurable r.v. $f \geq 0$. It is clear that $Q^\nu_\tau << P|_{\mathcal{F}_\nu}$ and $\frac{dQ^\nu_\tau}{dP|_{\mathcal{F}_\nu}} = \frac{I_{A_\tau}}{P(A_\tau)}$ P-a.s. By Bayes formula we obtain Q^ν_τ-a.s. ($\Rightarrow Q^\mu_\tau$-a.s. on $\mathcal{F}_\tau$ by the point 1) of this proof):

$$E^{Q^\nu_\tau}[f|\mathcal{F}_\tau] = \frac{E^P\left[\frac{I_{A_\tau}}{P(A_\tau)}f|\mathcal{F}_\tau\right]}{E^P\left[\frac{I_{A_\tau}}{P(A_\tau)}|\mathcal{F}_\tau\right]} = E^P\left[fI_{A_\tau}|\mathcal{F}_\tau\right].$$

Similarly we get Q^ν_μ-a.s.($\Rightarrow Q^\mu_\tau$-a.s. on $\mathcal{F}_\mu$ by the points 1) and 2) of this proof):

$$E^{Q^\nu_\mu}[f|\mathcal{F}_\mu] = E^P\left[fI_{A_\mu}|\mathcal{F}_\mu\right].$$

Denoting $h = E^P\left[fI_{A_\mu}|\mathcal{F}_\mu\right]$ we obtain by the same way:

$$E^{Q^r_s}[h|\mathcal{F}_s] = E^P\left[hI_{A_\tau}|\mathcal{F}_\tau\right] = E^P\left[E^P[fI_{A_\mu}|\mathcal{F}_\mu]I_{A_\tau}|\mathcal{F}_\tau\right] = E^P\left[fI_{A_\tau}|\mathcal{F}_\tau\right].$$

The required formula has been proven.

Definition 2.3. 1) We say that $\mathcal{T}$-GSB $(\Omega, \mathbf{Q}, \mathbf{E})$ satisfies the **regularity condition** if $\forall\tau, \mu, \nu \in \mathcal{T}$ such that $0 \leq \tau \leq \mu \leq \nu \leq \infty$

$$Q^\mu_\tau|_{\mathcal{F}_\tau} = Q^\nu_\tau|_{\mathcal{F}_\tau}. \tag{2.3}$$

2) $\mathcal{T}$-GSB $(\Omega, \mathbf{Q}, \mathbf{E})$ is called **strictly regular** if $\forall\tau, \mu, \nu \in \mathcal{T}$ such that $0 \leq \tau \leq \mu \leq \nu \leq \infty$

$$Q^\mu_\tau = Q^\nu_\tau|_{\mathcal{F}_\mu}. \tag{2.4}$$

From example 2.2 it follows that $\mathcal{T}$-GSB generated by conditional expectations is strictly regular.

3. Results on $\mathcal{T}$-generalized stochastic bases

Proposition 3.1. *If $\mathcal{T}$-GSB $(\Omega, \mathbf{Q}, \mathbf{E})$ satisfies the regularity condition, then*

a) the equalities (1.4) *and* (1.9) *are satisfied Q^t_s-a.s.;*

b) $\forall 0 \leq \tau \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \nu \leq \infty$ and $\forall A \in \mathcal{F}_\nu$

$$Q^\nu_\tau(A) = E^{Q^{\mu_1}_\tau} E^{\mu_2}_{\mu_1} \dots E^\nu_{\mu_{n-1}} I_A; \tag{3.1}$$

c) $\forall 0 \leq \tau \leq \mu \leq \nu \leq \infty \qquad Q^\mu_\tau = Q^\nu_\tau|_{\mathcal{F}_\mu}$.

Proof. a) It is suffice to apply (1.4) and Definition 2.3.

b) Using (1.9) and regularity property in the form $Q^{\mu_1}_\tau|_{\mathcal{F}_\tau} = Q^\nu_\tau|_{\mathcal{F}_\tau}$ we have:

$$Q^\nu_\tau(A) = E^{Q^\nu_\tau} E^\nu_\tau(I_A) = E^{Q^\nu_\tau} E^{\mu_1}_\tau E^{\mu_2}_{\mu_1} \dots E^\nu_{\mu_{n-1}} I_A =$$
$$= E^{Q^\nu_\tau} E^{Q^{\mu_1}_\tau}[E^{\mu_2}_{\mu_1} \dots E^\nu_{\mu_{n-1}} I_A|\mathcal{F}_\tau] = E^{Q^{\mu_1}_\tau} E^{Q^{\mu_1}_\tau}[E^{\mu_2}_{\mu_1} \dots E^\nu_{\mu_{n-1}} I_A|\mathcal{F}_\tau] =$$
$$= E^{Q^{\mu_1}_\tau} E^{\mu_2}_{\mu_1} \dots E^\nu_{\mu_{n-1}} I_A.$$

c) This property follows from b):

$$A \in \mathcal{F}_\mu \subset F_\nu \Longrightarrow Q^\nu_\tau(A) = E^{Q^\mu_\tau} E^\nu_\mu I_A = E^{Q^\mu_\tau}(I_A) = Q^\mu_\tau(A).$$

Theorem 3.2. *For $\mathcal{T}$-GSB $(\Omega, \mathbf{Q}, \mathbf{E})$ the following conditions are equivalent:*
1) $\mathcal{T}$-GSB $(\Omega, \mathbf{Q}, \mathbf{E})$ satisfies regularity condition;
2) equalities

$$Q_\tau^\nu(A) = E^{Q_\tau^\mu} E_\mu^\nu I_A \tag{3.2}$$

are fulfilled $\forall 0 \leq \tau \leq \mu \leq \nu \leq \infty$ and $\forall A \in \mathcal{F}_\nu$;
3) $\mathcal{T}$-GSB $(\Omega, \mathbf{Q}, \mathbf{E})$ is strictly regular.

Proof. 1) $\Rightarrow$ 2) is proved in Proposition 3.1.
2) $\Rightarrow$ 3). This is also contained in Proposition 3.1 (proof of the implication *b*) $\Rightarrow$ *c*)).
3) $\Rightarrow$ 1) is trivial. □

Proposition 3.3. *If a triplet* (1.1) *satisfies the properties* (1.2) *and* (1.3) *and the equalities* (3.2) *are fulfilled $\forall 0 \leq \tau \leq \mu \leq \nu \leq \infty$ and $\forall A \in \mathcal{F}_\nu$, then* (1.1) *is a strictly regular $\mathcal{T}$-GSB.*

Proof. Since from (1.2) and (1.3) it follows the property (1.7), then the right part of (3.2) is defined correctly. We have ($\forall\, 0 \leq \tau \leq \mu \leq \nu \leq \infty$, $\forall A \in \mathcal{F}_\nu$, $\forall\, \mathcal{F}_\nu - mesurable\, f$, $\forall B \in \mathcal{F}_\tau$):

$$Q_\tau^\nu(A) = E^{Q_\tau^\mu} E_\mu^\nu I_A \Leftrightarrow E^{Q_\tau^\nu} E_\tau^\nu(I_A) = E^{Q_\tau^\mu} E_\mu^\nu(I_A) \Leftrightarrow$$
$$E^{Q_\tau^\nu} E_\tau^\nu(fI_B) = E^{Q_\tau^\mu} E_\mu^\nu(fI_B) \Leftrightarrow$$
$$\Leftrightarrow E^{Q_\tau^\nu} E_\tau^\nu(fI_B) = E^{Q_\tau^\mu} E_\tau^\mu E_\mu^\nu(fI_B). \tag{3.3}$$

From (3.2) it follows as in Proposition 3.1 that $\forall 0 \leq \tau \leq \mu \leq \nu \leq \infty \quad Q_\tau^\mu = Q_\tau^\nu|_{\mathcal{F}_\mu}$. Using this equality, let us continue our chain of equivalences

$$(3.3) \Leftrightarrow E^{Q_\tau^\mu}(I_B E_\tau^\nu f) = E^{Q_\tau^\mu}(I_B E_\tau^\mu E_\mu^\nu f) \Leftrightarrow E_\tau^\nu f = E_\tau^\mu E_\mu^\nu f \quad Q_\tau^\mu - a.s.$$

So, (1.1) is a $\mathcal{T}$-GSB. From Theorem 3.2 this $\mathcal{T}$-GSB is strictly regular. □

4. Generalized martingales

In the sequel we suppose that σ-algebras $\mathcal{F}_\tau$ are complete and $\mathcal{F}_\tau$-mesurables r.v. M_τ are defined Q_τ^τ-a.s.

Definition 4.1. We say that a process $\mathbf{M} = (M_\tau, \mathcal{F}_\tau)_{\tau \in \mathcal{T}}$ on $\mathcal{T}$-GSB $(\Omega, \mathbf{Q}, \mathbf{E})$ is $\mathcal{T}$-generalized martingale ($\mathcal{T}$-GM) if $\forall\, 0 \leq \tau \leq \nu \leq \infty$

$$E^{Q_\tau^\nu}(|M_\nu|) < \infty \tag{4.1}$$

and

$$E^{Q_\tau^\nu} M_\nu = M_\tau \;\; Q_\tau^\tau - a.s. \tag{4.2}$$

Remark 4.2. Since M_ν is defined Q_ν^ν-a.s. and taking into account that $Q_\tau^\nu << Q_\nu^\nu$ (c.f. relation (1.6)) the condition (4.1) is correct.

Since $Q_\tau^\tau << Q_\tau^\nu|_{\mathcal{F}_\tau}$ (c.f. relation (1.5)), we obtain that $E^{Q_\tau^\nu} M_\nu$ is defined Q_τ^τ-a.s. Therefore, the requirement that equality (4.2) be satisfied almost surely is correct.

Proposition 4.3. *If $\mathbf{M} = (M_\tau, \mathcal{F}_\tau)_{\tau \in \mathcal{T}}$ on $\mathcal{T}$-GSB $(\Omega, \boldsymbol{Q}, \boldsymbol{E})$ is $\mathcal{T}$-GM, then $\forall \tau \in \mathcal{T}$*

$$M_\tau = E^{Q_\tau^\infty} M_\infty. \tag{4.3}$$

Proof. The proof follows from Definition 4.1.

Definition 4.4. A $\mathcal{T}$-GSB is called bounded if $\forall\, 0 \le \tau \le \nu \le \infty$ $\frac{dQ_\tau^\nu}{dQ_\nu^\infty}$ is bounded Q_ν^∞-a.s.

Proposition 4.5. *Let* $\mathcal{T}$-GSB *under consideration is bounded, and* $\mathcal{F}_\infty$*-mesurable r.v.* f *is such that* $E^{Q_\tau^\infty}(|f|) < \infty$ $\forall \tau \in \mathcal{T}$. *Then* $\mathbf{M} = (M_\tau, \mathcal{F}_\tau)_{\tau \in \mathcal{T}}$, *where* $M_\tau = E^{Q_\tau^\infty} f$, *is a* $\mathcal{T}$-GM.

Proof. Let $\tau \le \nu$. We have: $M_\tau = E^{Q_\tau^\infty} f$ Q_τ^τ-a.s. and $M_\nu = E^{Q_\nu^\infty} f$ Q_ν^ν-a.s.Let us calculate $E^{Q_\tau^\nu} M_\nu$. Since $f \in L_1(\Omega, \mathcal{F}_\infty, Q_\nu^\infty)$, we have $E_\nu^\infty f \in L_1(\Omega, \mathcal{F}_\infty, Q_\nu^\infty)$. Denote $g := E_\nu^\infty f$. Then

$$\int\limits_\Omega |g| dQ_\tau^\nu = \int\limits_\Omega |g| h dQ_\tau^\nu \le c \int\limits_\Omega |g| dQ_\tau^\nu < \infty.$$

Hence $E_\nu^\infty f \in L_1(\Omega, \mathcal{F}_\infty, Q_\nu^\nu$ and $E^{Q_\tau^\nu} M_\nu = E^{Q_\tau^\nu} E^{Q_\nu^\infty} f = E^{Q_\tau^\infty} f = M_\tau$.

5. Conclusion

This article lais the foundation for obtaining results similar to Doob's optional sampling theorem within the framework of generalized stochastic bases. One of the initial results in this is Proposition 4.5. The autors are aware of two possible methods for studying the generalized martingales introduced in Definition 4.1. The first method was tested in papers [3]-[13]. The results obtained there relate to generalized martingales with descrete time (they were called deformed martingales there). The developed theory has the peculliarity that, in proving certain results (including optimal sampling theorem), we were forced to move away from probability measures and consider deformed stochastic bases consisting of sigma-finite measures. This circumstance has limited the applicability of this tipe of research to financial market modeling and related disciplines.

The second methodology, the foundations of which are laid in this paper, makes it possible to obtain martingale (in the generalized sense) results without resorting to sigma-finite measures. Indeed, suppose we work on a $\mathcal{T}$-generalized stochastic basis and study generalized martingales in the sense of Definition 4.1, but with all stopping times replaced by constant times. The problems described above reduce to how to write the corresponding relations for arbitrary stopping times. This procedure allows us to remain within the framework of probabilistic considerations. A paper on this topic will soon be submitted for publication by the authors.

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