Global and Stochastic Analysis
Vol. 13 No. 4 (2026)

DOI: 10.64837/GSA.13.4.7

POSITIVE SOLUTIONS OF THE INITIAL-FINAL PROBLEM FOR A SOBOLEV-TYPE EQUATION WITH AN (L, p)-SECTORIAL OPERATOR*

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Abstract. For an abstract linear Sobolev-type equation with a degenerate operator acting on the derivative, the problem of the existence of positive solutions to the initial-final problem is investigated. A distinctive feature of such equations is the ill-posedness of the classical Cauchy problem, caused by the degeneracy of the operator acting on the time derivative. The approach is based on a synthesis of the theory of positive operator semigroups and the theory of degenerate holomorphic resolving semigroups. Sufficient conditions ensuring the existence of a positive solution are established, and the solution to the problem is obtained in explicit form.

1. Introduction

The study of Sobolev-type equations is of significant interest in the modern theory of differential equations, since such equations serve as adequate mathematical models for describing non-stationary processes in media with memory, viscoelastic materials, hydrodynamics, and other fields of natural science. A distinctive feature of these equations is the degeneracy of the operator acting on the derivative, which leads to the phenomenon of non-existence or non-uniqueness of solutions to classical initial value problems. In this regard, the study of initial-final problems [1] becomes particularly relevant, as they allow one to pose the problem correctly even under degeneracy conditions. An important aspect of the analysis of such problems is the existence of positive solutions that preserve non-negativity on the entire interval of definition [2]. Positive solutions have a direct interpretation in applied problems where the modelled quantities (concentrations, densities, pressures, etc.) cannot take negative values by their nature. The present work is based on the results presented in [3].

We consider an abstract linear inhomogeneous Sobolev-type equation

$$L\dot{u} = Mu + f \tag{1.1}$$

Date: Date of Submission 18th May 2026; Date of Acceptance 29 June 2026, Communicated by Yuri E. Gliklikh.

2020 *Mathematics Subject Classification.* Primary 35Q99; Secondary 60G65.

Key words and phrases. Sobolev type equations; degenerate positive holomorphic operator semigroups; positive solutions; Sobolev sequence spaces.

*This work was funded by Russian Science Foundation Grant No. 24-11-20037, https://rscf.ru/project/24-11-20037/.

with the initial-final condition

$$\lim_{t\to 0+} P_0(u(t)-u_0)=0, \quad P_1(u(\tau)-u_\tau)=0, \tag{1.2}$$

where P_0, P_1 are projectors associated with the decomposition of the phase space. We need to find conditions under which a positive solution exists.

It is convenient to start our considerations with the case where a continuous inverse operator L^{-1} exists. In this case, the linear homogeneous equation

$$L\dot{u} = Mu \tag{1.3}$$

trivially reduces to the standard equation $\dot{u} = Au$, where the operator A is sectorial. The description of conditions under which A generates a positive holomorphic semigroup is devoted to Section 2. Further, based on the theory of degenerate holomorphic operator semigroups [4], in the case of a relatively sectorial operator M, necessary and sufficient conditions for the existence of a degenerate positive holomorphic resolving semigroup for equation (1.1) are found (Section 4). The obtained results are applied to the study of the initial-final problem.

2. Positive analytic operator semigroups

Consider a Banach space $\mathfrak{V}$, an operator $A : \mathrm{dom}A \subset \mathfrak{V} \to \mathfrak{V}$ such that $A \in \mathcal{C}l(\mathfrak{V})$ (linear, closed and densely defined). Denote by $\rho(A) = \{\mu \in \mathbb{C} : (\mu I - A)^{-1} \in \mathcal{L}(\mathfrak{V})\}$ the resolvent set, and $\sigma(A) = \mathbb{C} \setminus \rho(A)$ the spectrum.

Definition 2.1. An operator $A \in \mathcal{C}l(\mathfrak{V})$ is called *sectorial* if

(1) there exist constants $a \in \mathbb{R}$ and $\Theta \in (\frac{\pi}{2}, \pi)$ such that

$$S_{a,\Theta} = \{\mu \in \mathbb{C} : \ |\arg(\mu - a)| < \Theta, \ \mu \neq a\} \subset \rho(A);$$

(2) there exists a constant $K \geq 1$ such that

$$\|(\mu I - A)^{-1}\|_{\mathcal{L}(\mathfrak{V})} \leq \frac{K}{|\mu - a|} \quad \text{for all } \mu \in S_{a,\Theta}.$$

Definition 2.2. A map $V^\bullet \in C^\infty(\overline{\mathbb{R}}_+; \mathcal{L}(\mathfrak{V}))$, where $\overline{\mathbb{R}}_+ = [0, +\infty)$, is called an *operator semigroup* if

$$V^s V^t v = V^{s+t} v, \quad \forall s, t \in \overline{\mathbb{R}}_+, \ \forall v \in \mathfrak{V}.$$

A semigroup is called *holomorphic* if it can be continuously extended to some sector containing the ray $\mathbb{R}_+$, while preserving the semigroup property.

Definition 2.3. An operator $A \in \mathcal{C}l(\mathfrak{B})$ is called the *infinitesimal generator* of a semigroup $V^\bullet = \{V^t : \ t \in \overline{\mathbb{R}}_+\}$ if

$$Av = \lim_{t \to 0+} (V^t v - v)t^{-1} \quad \forall v \in \mathrm{dom}A.$$

If A is the infinitesimal generator of a semigroup $V^\bullet$, we write $V^t = e^{At}$.

Theorem 2.4. *(Hille–Yosida) The following statements are equivalent:*

(1) *The operator $A \in \mathcal{C}\updownarrow(\mathfrak{B})$ is sectorial.*

(2) *The operator $A \in \mathcal{C}\updownarrow(\mathfrak{B})$ is the infinitesimal generator of a holomorphic semigroup $\{e^{tA} : t \in \overline{\mathbb{R}}_+\}$ of the form*

$$e^{At} = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_\Gamma (\mu I - A)^{-1} e^{\mu t} d\mu, & t \in \mathbb{R}_+, \\ I, & t = 0, \end{cases}$$

where the contour $\Gamma \subset S_{a,\Theta}$ is such that $|\arg \mu| \to \Theta$ as $|\mu| \to \infty$, $\mu \in \Gamma$.

For a detailed proof and further discussions on the Hille–Yosida theorem and its applications, we refer to [5].

Corollary 2.5. *Let $A \in \mathcal{C}l(\mathfrak{B})$ be a sectorial operator. Then*

$$e^{At} = s\text{-}\lim_{k\to\infty} \left(I - \frac{t}{k}A\right)^{-k}, \quad t \in \mathbb{R}_+.$$

Let us proceed to the construction of positive holomorphic operator semigroups. Initially, we consider the space $\mathfrak{B}$ as a vector space (i.e., its metric structure is not of interest to us yet). A convex subset $\mathfrak{B}_+ \subset \mathfrak{B}$ such that $\alpha\mathfrak{B}_+ + \beta\mathfrak{B}_+ \subseteq \mathfrak{B}_+$ for all $\alpha, \beta \in \overline{\mathbb{R}}_+$, will be called a *cone*. A cone $\mathfrak{B}_+$ will be called *proper* if $\mathfrak{B}_+ \cap \{-\mathfrak{B}_+\} = \{0\}$, and *generating* if $\mathfrak{B} = \mathfrak{B}_+ - \mathfrak{B}_+$, i.e., for each element $u \in \mathfrak{B}$ there exist elements $v, w \in \mathfrak{B}_+$ such that $u = v - w$.

In the vector space $\mathfrak{B}$ we can introduce an order relation "$\geq$" as follows

$$(u \geq v) \Leftrightarrow (u - v \in \mathfrak{B}_+).$$

This order relation satisfies the axioms of reflexivity, transitivity and antisymmetry. Moreover, this order relation is consistent with the vector structure of the space $\mathfrak{B}$, i.e., $\mathfrak{B}$ is a *Riesz space*. A Riesz space is called a *function Riesz space* if for each pair $u, v \in \mathfrak{B}$ there exist elements $u \vee v \in \mathfrak{B}$ and $u \wedge v \in \mathfrak{B}$, where $u \wedge v = \min\{u, v\}$, $u \vee v = \max\{u, v\}$. In a function Riesz space we define the elements $u_+ = \max\{u, 0\}$ and $u_- = \max\{-u, 0\}$, then $u = u_+ - u_-$. For any element $u \in \mathfrak{V}$ we define the absolute value (modulus) $|u| = \max\{u, -u\}$, then $|u| = u_+ + u_-$. Recall that $\mathfrak{B}$ is a normed space, and we require $(|u| \leq |v|) \Rightarrow (\|u\| \leq \|v\|)$. Thus, a *functional normed Riesz space* is defined. A complete functional normed Riesz space $(\mathfrak{B}, \mathfrak{B}_+, \|\cdot\|)$ is called a *Banach structure* [6, Ch. 2].

Definition 2.6. (1) Let $(\mathfrak{B}, \mathfrak{B}_+, \|\cdot\|)$ be a Banach structure. An operator $A \in \mathcal{L}(\mathfrak{B})$ is called *positive* if $A\mathfrak{B}_+ \subseteq \mathfrak{B}_+$.

(2) An operator semigroup $V^\bullet = \{V^t : t \in \overline{\mathbb{R}}_+\}$ is called *positive* if $V^t\mathfrak{V}_+ \subseteq \mathfrak{V}_+$ for all $t \in \mathbb{R}_+$.

The set of linear bounded positive operators $\mathcal{L}_+ = \{A \in \mathcal{L}(\mathfrak{V}) : A\mathfrak{V}_+ \subseteq \mathfrak{V}_+\}$, in turn, forms a closed convex cone in the ordered Banach space $(\mathcal{L}, \mathcal{L}_+, \|\cdot\|)$.

Theorem 2.7. *The following statements are equivalent:*

(1) *The operator $A \in \mathcal{C}l(\mathfrak{B})$ is sectorial and its resolvent $(\lambda I - A)^{-1}$ is a positive operator for sufficiently large $\lambda \in \mathbb{R}_+$.*
(2) *The operator $A \in \mathcal{C}l(\mathfrak{B})$ is the infinitesimal generator of a positive holomorphic operator semigroup.*

In the case when $A \in \mathcal{L}(\mathfrak{B})$ the following statements are equivalent.

(i)′ *The operator* $A \in \mathcal{L}(\mathfrak{B})$ *is positive.*

(ii)′ *The operator* $A \in \mathcal{L}(\mathfrak{B})$ *is the infinitesimal generator of a positive operator semigroup, holomorphic on the entire complex plane* $\mathbb{C}$.

The equivalence of the statements is based on the equality

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}, \; t \in \mathbb{R},$$

which is impossible in the case of an unbounded operator $A \in \mathcal{C}l(\mathfrak{B})$.

3. Degenerate positive holomorphic operator semigroups

Let us proceed to the construction of degenerate positive holomorphic operator semigroups. Let $\mathfrak{U}$, $\mathfrak{F}$ be Banach spaces, operators $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ (i.e., linear and continuous), $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$ (i.e., linear, closed and densely defined). The foundation of our research is the theory of degenerate operator semigroups and the phase space method, presented in [4, Ch. 3]. We provide the necessary information from the theory of degenerate operator semigroups in Banach spaces. We introduce the *L-resolvent set* $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})\}$ and the *L-spectrum* $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ *of the operator* M, as well as the operator functions $R^L_\mu(M) = (\mu L - M)^{-1} L$ and $L^L_\mu(M) = L(\mu L - M)^{-1}$, called respectively the *right* and *left L-resolvents of the operator* M (see [4, Ch. 1]). Let $\mu_q \in \rho^L(M)$, $q = 0, 1, \ldots, p$. The operator functions

$$R^L_{(\mu,p)}(M) = \prod_{k=0}^{p} R^L_{\mu_k}(M), \; L^L_{(\mu,p)}(M) = \prod_{k=0}^{p} L^L_{\mu_k}(M)$$

will be called respectively the *right* and *left* (L,p)*-resolvents of the operator* M.

Definition 3.1. An operator M is called *p-sectorial with respect to* the operator L (shortly, (L,p)*-sectorial*) with $p \in \{0\} \cup \mathbb{N}$, if

(1) there exist $a \in \mathbb{R}$, $\Theta \in (\pi/2, \pi)$ such that $S^L_{a,\Theta}(M) = \{\mu \in \mathbb{C} : |\arg(\mu - a)| < \Theta, \mu \neq a\} \subset \rho^L(M)$;

(2) there exists a constant $K \in \mathbb{R}_+$ such that

$$\max\left\{\|R^L_{(\mu,p)}(M)\|_{\mathcal{L}(\mathfrak{U})}, \|L^L_{(\mu,p)}(M)\|_{\mathcal{L}(\mathfrak{F})}\right\} \leq \frac{K}{\prod_{q=0}^{p} |\mu_q - a|}$$

for any $\mu_q \in S^L_{a,\Theta}(M)$, $q = 0, \ldots, p$.

Lemma 3.2. *Let* M *be* (L,p)*-sectorial,* $p \in \{0\} \cup \mathbb{N}$. *Then there exist degenerate holomorphic operator semigroups*

$$U^t = \frac{1}{2\pi i} \int_\Gamma R^L_\mu(M) e^{\mu t} d\mu, \qquad F^t = \frac{1}{2\pi i} \int_\Gamma L^L_\mu(M) e^{\mu t} d\mu,$$

where the contour $\Gamma \subset S^L_{a,\Theta}(M)$.

Recall [4, Ch. 3] that a semigroup $V^\bullet$ is called *degenerate* if $s\text{-}\lim_{t\to 0+} V^t \neq \mathbb{I}$. Here $t \in \mathbb{R}_+$, and the contour $\Gamma \subset S^L_{a,\Theta}(M)$ is such that $|\arg\mu| \to \Theta$ as $\mu \to \infty$, $\mu \in \Gamma$. If $V^\bullet \in C^\infty(\mathbb{R}_+;\mathcal{L}(\mathfrak{V}))$ is a degenerate holomorphic operator semigroup, then we can define $\ker V^\bullet = \{v \in \mathfrak{V} : V^t v = 0 \ \exists\, t \in \mathbb{R}_+\}$.

Define $\mathfrak{U}^0 = \ker U^\bullet$, $\mathfrak{F}^0 = \ker F^\bullet$, $\mathfrak{U}^1 = \operatorname{im} U^\bullet$, $\mathfrak{F}^1 = \operatorname{im} F^\bullet$. Let L_0, M_0 be the restrictions to $\mathfrak{U}^0$, and L_1, M_1 to $\mathfrak{U}^1$. We require the conditions

$$\mathfrak{U}^0 \oplus \mathfrak{U}^1 = \mathfrak{U}, \quad \mathfrak{F}^0 \oplus \mathfrak{F}^1 = \mathfrak{F}, \tag{A1}$$

$$\exists\, L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1;\mathfrak{U}^1). \tag{A2}$$

Due to the holomorphy of the semigroups $U^\bullet$ and $F^\bullet$, we set $\mathfrak{U}^0 = \ker U^\bullet$, $\mathfrak{F}^0 = \ker F^\bullet$ and denote by L_0 the restriction of L to $\mathfrak{U}^0$, and M_0 the restriction of M to $\mathfrak{U}^0 \cap \operatorname{dom} M$.

Theorem 3.3. [4, Ch. 3] *Let the operator M be (L,p)-sectorial, $p \in \{0\} \cup \mathbb{N}$. Then*

(i) *the operators $L_0 \in \mathcal{L}(\mathfrak{U}^0;\mathfrak{F}^0)$, and $M_0 : \mathfrak{U}^0 \cap \operatorname{dom} M \to \mathfrak{F}^0$;*

(ii) *there exists an operator $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0;\mathfrak{U}^0)$;*

(iii) *the operators $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{U}^0)$ $(G = L_0M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0))$ are nilpotent with degree less than or equal to p.*

Condition $(A1)$ is equivalent to the condition of strong (L,p)-sectoriality on the right and left of the operator M, respectively [4, Ch. 3]. Condition $(A2)$ is equivalent to the condition of strong (L,p)-sectoriality of the operator M [4, Ch. 3]. Note that in applications, proving the strong (L,p)-sectoriality of the operator M is more laborious than checking the conditions $(A1)$ and $(A2)$ in the case of an (L,p)-sectorial operator M, $p \in \{0\} \cup \mathbb{N}$.

Theorem 3.4. *Let M be (L,p)-sectorial and (A1), (A2) hold. Then*

(1) *there exist projectors $P = s\text{-}\lim_{t\to 0+} U^t \in \mathcal{L}(\mathfrak{U})$, $Q = s\text{-}\lim_{t\to 0+} F^t \in \mathcal{L}(\mathfrak{F})$;*

(2) *there exist sectorial operators $S = L_1^{-1}M_1 \in \mathcal{C}l(\mathfrak{U}^1)$ and $T = M_1L_1^{-1} \in \mathcal{C}l(\mathfrak{F}^1)$.*

Corollary 3.5. *Under the conditions of the theorem,*

$$U^t = \begin{cases} e^{tS}P = s\text{-}\lim\limits_{k\to\infty}\big[(L - \tfrac{t}{k(p+1)}M)^{-1}L\big]^{k(p+1)}, & t \in \mathbb{R}_+, \\ P, & t = 0, \end{cases}$$

$$F^t = \begin{cases} e^{tT}Q = s\text{-}\lim\limits_{k\to\infty}\big[L(L - \tfrac{t}{k(p+1)}M)^{-1}\big]^{k(p+1)}, & t \in \mathbb{R}_+, \\ Q, & t = 0. \end{cases}$$

Definition 3.6. Let degenerate semigroups $U^\bullet$ and $F^\bullet$ with projectors P, Q (splitting the spaces into direct sums) be given on Banach spaces $\mathfrak{U}$ and $\mathfrak{F}$. We say that the pair (L,M) *generates* the pair of degenerate semigroups $(U^\bullet, F^\bullet)$ if

(1) the actions split: $L \in \mathcal{L}(\mathfrak{U}^{0(1)};\mathfrak{F}^{0(1)})$, $M \in \mathcal{C}l(\mathfrak{U}^{0(1)};\mathfrak{F}^{0(1)})$, and there exist $L_1^{-1} \in \mathcal{L}(\mathfrak{F}^1;\mathfrak{U}^1)$, $M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0;\mathfrak{U}^0)$;

(2) there exist $S = L_1^{-1}M_1 \in \mathcal{C}l(\mathfrak{U}^1)$, $T = M_1L_1^{-1} \in \mathcal{C}l(\mathfrak{F}^1)$ such that $U^t = e^{tS}P$, $F^t = e^{tT}Q$;

(3) the operators $H = M_0^{-1}L_0 \in \mathcal{L}(\mathfrak{U}^0)$, $G = L_0M_0^{-1} \in \mathcal{L}(\mathfrak{F}^0)$ are nilpotent of degree at most p.

Theorem 3.7. *Let $L \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, $M \in \mathcal{C}l(\mathfrak{U};\mathfrak{F})$. The following statements are equivalent:*

(1) *M is (L,p)-sectorial and (A1), (A2) hold.*
(2) *The pair (L,M) generates a pair of degenerate holomorphic semigroups $(U^\bullet, F^\bullet)$.*

Next, we introduce the condition

$$\begin{array}{c} \sigma^L(M) = \sigma_1^L(M) \cup \sigma_0^L(M), \text{ moreover} \\ \sigma_1^L(M) \subset D \subset \mathbb{C} \text{ (bounded domain)}, \\ \gamma \cap \sigma^L(M) = \emptyset, \end{array} \tag{A3}$$

where γ is a piecewise smooth boundary of D.

Let conditions (A1) – (A3) be satisfied. We construct the relative spectral projector [7]

$$P_1 = \frac{1}{2\pi i}\int_\gamma R_\mu^L(M)d\mu,$$

and it turns out that under the condition of (L,p)-sectoriality of M, we have $P_1P = PP_1 = P_1$. Hence, in this case there exists a projector $P_0 = P - P_1$. Then there exists a group $\{U_1^t : t \in \mathbb{R}\}$, where $U_1^t = \frac{1}{2\pi i}\int_\gamma R_\mu^L(M)e^{\mu t}d\mu$ such that $P_1 = U_1^0$. We construct an analytic semigroup $\{U_0^t : t \in \mathbb{R}_+\}$, where $U_0^t = U^t - U_1^t$. Obviously, $P_0 = s - \lim\limits_{t\to 0+} U_0^t$. Set $\operatorname{im} P_{0(1)} = \mathfrak{U}_{0(1)}^1$; obviously, $\mathfrak{U}^1 = \mathfrak{U}_0^1 \oplus \mathfrak{U}_1^1$.

In the subspace $\mathfrak{U}^1$, we define a generating proper cone $\mathfrak{U}_+^1$ so that the subspace $(\mathfrak{U}^1, \mathfrak{U}_+^1, \|\cdot\|_{\mathfrak{U}})$ forms a Banach structure. Similarly, in the subspace $\mathfrak{F}^1$, we define a generating proper cone $\mathfrak{F}_+^1$ so that the subspace $(\mathfrak{F}^1, \mathfrak{F}_+^1, \|\cdot\|_{\mathfrak{F}})$ forms a Banach structure.

Theorem 3.8. *The semigroups $U^\bullet$, $U_{0(1)}^\bullet$; $F^\bullet$, $F_{0(1)}^\bullet$ are positive on the subspaces $(\mathfrak{U}^1, \mathfrak{U}_+^1, \|\cdot\|_{\mathfrak{U}})$; $(\mathfrak{F}^1, \mathfrak{F}_+^1, \|\cdot\|_{\mathfrak{F}})$, respectively, if and only if the pair (L,M) generates a pair of degenerate holomorphic semigroups and the L-resolvent of M is positive for sufficiently large μ. In this case, $U^t = e^{St}P$, $F^t = e^{Tt}Q$,*

$$e^{St} = \frac{1}{2\pi i}\int_\Gamma (\mu I - S)^{-1}e^{\mu t}d\mu = s\text{-}\lim_{n\to\infty}\big(I - \frac{St}{n}\big)^{-n}, \quad t \in \mathbb{R}_+.$$

In the subspace $\mathfrak{U}^0$, we define a generating proper cone $\mathfrak{U}_+^0$ so that the subspace $(\mathfrak{U}^0, \mathfrak{U}_+^0, \|\cdot\|_{\mathfrak{U}})$ forms a Banach structure. Then $\mathfrak{U}_+ = \mathfrak{U}_+^0 \oplus \mathfrak{U}_+^1$ forms a generating proper cone in the space $\mathfrak{U}$ and the subspace $(\mathfrak{U}, \mathfrak{U}_+, \|\cdot\|_{\mathfrak{U}})$ forms a Banach structure. Similarly, in the subspace $\mathfrak{F}^0$, we define a generating proper cone $\mathfrak{F}_+^0$ so that the subspace $(\mathfrak{F}^0, \mathfrak{F}_+^0, \|\cdot\|_{\mathfrak{F}})$ forms a Banach structure. Then $\mathfrak{F}_+ = \mathfrak{F}_+^0 \oplus \mathfrak{F}_+^1$ forms a generating proper cone in the space $\mathfrak{F}$ and the subspace $(\mathfrak{F}, \mathfrak{F}_+, \|\cdot\|_{\mathfrak{F}})$ forms a Banach structure.

If the operator semigroup $U^\bullet$ is positive in the subspace $(\mathfrak{U}^1, \mathfrak{U}_+^1, \|\cdot\|)$, then it is also positive in the subspace $(\mathfrak{U}, \mathfrak{U}_+, \|\cdot\|)$. Indeed, each element $u \in \mathfrak{U}_+$ can be represented as $u = u^1 + u^0$, where $u^1 \in \mathfrak{U}^1_+$, $u^0 \in \mathfrak{U}^0_+ \subset \ker U^\bullet$, hence we obtain

$$U^t u = U^t u^1 + U^t u^0 = U^t u^1.$$

Similarly, we can extend the action of the semigroup $F^\bullet$ to the subspace $(\mathfrak{F}, \mathfrak{F}_+, \|\cdot\|_{\mathfrak{F}})$. Then the following holds:

Corollary 3.9. *The operator semigroups $U^\bullet$, $F^\bullet$ are positive on the subspaces $(\mathfrak{U}, \mathfrak{U}_+, \|\cdot\|)$ and $(\mathfrak{F}, \mathfrak{F}_+, \|\cdot\|_{\mathfrak{F}})$, respectively, if and only if the pair of operators (L, M) generates a pair of degenerate holomorphic semigroups $(U^\bullet, F^\bullet)$ and the L-resolvent of the operator M is a positive operator for sufficiently large μ.*

Remark 3.10. In [8], the case where the operator M is (L, p)-bounded was considered. In this case, the operator $S \in \mathcal{L}(\mathfrak{U})$ and the following statements are equivalent.

(i)″ *The operator $S \in \mathcal{L}(\mathfrak{U})$ is positive.*

(ii)″ *The operator $S \in \mathcal{L}(\mathfrak{U})$ is the infinitesimal generator of a positive operator semigroup, holomorphic on the entire complex plane $\mathbb{C}$.*

In the case where the operator M is (L, p)-sectorial, the operator $S \in \mathcal{C}l(\mathfrak{U})$, and statements (i)″ and (ii)″ are not equivalent.

4. Positive solution of the abstract equation

Let the operator M be (L, p)-sectorial and conditions $(A1)$–$(A3)$ be satisfied. Fix $\tau \in \mathbb{R}_+$, u_0, $u_\tau \in \mathfrak{U}$. Consider the abstract Sobolev-type equation [9] with the initial-final condition (1.2)

Definition 4.1. A vector function $u \in C^\infty(\mathbb{R}_+; \mathfrak{U})$ satisfying equation (1.1) is called its solution. A solution $u(t)$ is called a solution of the initial-final problem if it satisfies condition (1.2).

We define Banach structures on $\mathfrak{U}$ and $\mathfrak{F}$ with cones $\mathfrak{U}_+ = \mathfrak{U}^0_+ \oplus \mathfrak{U}^1_+$, $\mathfrak{F}_+ = \mathfrak{F}^0_+ \oplus \mathfrak{F}^1_+$.

Definition 4.2. Let (A1) – (A3) be satisfied and the L-resolvent of M be positive for sufficiently large μ. Then the (L, p)-sectorial operator M is called *strongly positive* if

(1) $L_1 : \mathfrak{U}^1_+ \to \mathfrak{F}^1_+$ is a topological isomorphism;

(2) $-H^q M_0^{-1} : \mathfrak{F}^0_+ \to \mathfrak{U}^0_+$ for all $q = 0, \ldots, p$.

In this case, the Banach structures are called *consistent*.

Theorem 4.3. *Let M be a strongly positive (L, p)-sectorial operator and conditions (A1)–(A3) be satisfied. Then for any vector function $f : (0, \tau) \to \mathfrak{F}$ such that $f^0 \in C^{p+1}((0, \tau); \mathfrak{F}^0)$, $\frac{d^q f^0}{dt^q}(t) \in \mathfrak{F}^0_+$ ($q = 0, \ldots, p$), $f^1 \in C((0, \tau); \mathfrak{F}^1_+)$, and for any $u_0, u_\tau \in \mathfrak{U}$ with $u^1_0, u^1_\tau \in \mathfrak{U}^1_+$, there exists a unique positive solution of problem (1.1), (1.2), representable in the form*

$$u(t) = -\sum_{q=1}^{p} G^q M_0^{-1} \frac{d^q f^0}{dt^q}(t) + U_0^t u_0 + \int_0^t e^{(t-s)S_0} L_{10}^{-1} Q_0 f(s)\, ds +$$

$$+ U_1^{t-\tau} u_\tau + \int_\tau^t e^{(t-s)S_1} L_{11}^{-1} Q_1 f(s)\, ds.$$

5. Acknowledgments

The authors express their gratitude to the Russian Science Foundation for financial support (grant No. 24-11-20037).

6. Conclusions

In this paper, a theory of positive and degenerate positive holomorphic operator semigroups is constructed and applied to abstract Sobolev-type equations. The obtained criteria for the existence of positive solutions of initial-final problems can be used in the study of hydrodynamic processes modelled by stochastic systems (see, e.g., [10]). Further research will be directed towards the numerical implementation of the proposed algorithms.

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