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BACKWARD MEAN DERIVATIVES FOR DIFFUSION TYPE PROCESSES

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ABSTRACT. We compute second order backward mean derivatives for diffusion type processes. To do this we first study derivatives of such type for Itô diffusion processes and then go to the non-markovian case of diffusion type processes. Transition from Itô diffusion processes to diffusion type processes is carried out using Malliavin derivatives.

Introduction

Edward Nelson determined the velocity and acceleration of a particle moving along a random trajectory using mean derivatives (see [7]). Indeed, this approach is convenient due to its transparent construction. In regular cases we usually take the mean velocity for the resolution h in each interval of the partition and then tend its length to zero. But also we can take the mean value of measurements for each resolution h and then tend h to zero.

This approach could be applied to compute velocity and acceleration for the processes with irregular dynamics. Such as, for example, the motion of the fluid. The laws of the motion of incompressible viscous fluid could be described in terms of mean derivatives. To be more precise, backward mean derivatives of the second order. Since the most interesting case is consering to the viscous fluid, we need the diffusion type processes.

In the first section, we discuss definitions of the backward mean derivatives and two types of processes we need. Such as a Itô diffusion process and a diffusion type process. In the second section we prove formulas for the backward mean derivatives of the first and second order for the Itô diffision processes using techniques from [1, 8]. This case is more general than in [3, 4] since the diffusion coefficient isn't a constant. In the last section we suppose that a stochastic process is a diffusion type process. This process is more demanded in applications, but since it is non-Markovian we can't reverse the time to get the result.

1. Preliminaries

Let us consider a stochastic process $\xi(t)$ given on a certain probability space $(\Omega, \mathscr{F}, \mathsf{P})$ with values in $\mathbb{R}^n$. $\xi(t)$ is a L_1-variable for each $t \in [0,T]$. Let $\mathscr{N}_t^\xi$ be the σ-algebra generated by preimages of Borel sets for the mapping $\xi(t) : \Omega \to \mathbb{R}^n$ and $\mathscr{P}_t^\xi$ be the σ-algebra generated by preimages of Borel sets for the mappings $\xi(s) : \Omega \to \mathbb{R}^n$, $0 < s \le t$. We denote by $E_t^\xi(\eta(t))$ the conditional expectation of a process $\eta(t)$ with respect to $\mathscr{N}_t^\xi$.

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We call the backward mean derivative of the process $\xi(t)$ at the moment $t \in (0,T]$ the L_1 random variable

$$D_*\xi(t) = \lim_{\triangle t \to +0} E_t^{\xi}\left(\frac{\xi(t) - \xi(t-\triangle t)}{\triangle t}\right), \tag{1.1}$$

if the limit exists in $L_1(\Omega, \mathscr{F}, \mathsf{P})$.

These derivative can be considered as the composition of $\xi(t)$ and the Borel vector field $Z(t,x)$:

$$Z(t,x) = \lim_{\triangle t \to +0} E\left(\frac{\xi(t) - \xi(t-\triangle t)}{\triangle t}\Big| \xi(t) = x\right). \tag{1.2}$$

Let $Z(t,x)$ be a C^2-vector field in $\mathbb{R}^n$. The backward mean derivative of $Z(t,x)$ along $\xi(t)$ is the L_1-random process

$$D_*Z(t,\xi(t)) = \lim_{\triangle t \to +0} E_t^{\xi}\left(\frac{Z(t,\xi(t)) - Z(t-\triangle t, \xi(t-\triangle t))}{\triangle t}\right). \tag{1.3}$$

Second order backward mean derivative $D_*D_*\xi(t)$ of the process $\xi(t)$ is the backward mean derivative for the regression $D_*\xi(t)$ in the sense of definition (1.3). $D_*\xi(t)$ can be represented as a composition of Borel vector field (regression) and $\xi(t)$. This vector field we denote by $F_*(t,x)$.

Consider the Itô diffision process $\xi(t)$ as a solution of the stochastic differential Itô equation

$$\mathrm{d}\xi(t) = a(t,\xi(t))\mathrm{d}t + A(t,\xi(t))\mathrm{d}w(t), \tag{1.4}$$

where the drift $a(t,x)$ and the diffusion summand $A(t,x)$ are a vector field and a field of linear operators in $\mathbb{R}^n$ respectively, $w(t)$ is a Wiener process in $\mathbb{R}^n$. Denote $A(t,x)A^*(t,x)$ by $\sigma(t,x)$ and by p_t the distribution of $\xi(t)$. Let us mention that this process is Markov w.r.t. σ-algebra $\mathscr{N}_t^{\xi}$.

Let $\xi(t)$ be a diffusion type process (see [2]), i.e. a solution of the stochastic differential equation of diffusion type

$$\mathrm{d}\xi(t) = a(t,\xi(\cdot))\mathrm{d}t + A(t,\xi(\cdot))\mathrm{d}w(t), \tag{1.5}$$

where coefficients are a vector field $a(t,\gamma) : [0,T] \times C^0([0,T],\mathbb{R}^n) \to \mathbb{R}^n$ and a field of linear operators $A(t,\gamma) : [0,T] \times C^0([0,T],\mathbb{R}^n) \to L(\mathbb{R}^n,\mathbb{R}^n)$, adapted w.r.t. $\mathscr{P}_t^{\xi}$. By the definition it depends of the history of the process $\xi(t)$ up to the moment t.

2. Backward mean derivatives for a diffusion process

Lemma 2.1. *Assume that there exists a constant K such that $a(t,x)$ and $A(t,x)$ satisfy the properties*

$$\sup_t (||A(t,x) - A(t,y)|| + ||a(t,x) - a(t,y)||) \le K||x-y||. \tag{2.1}$$

$$\sup_t (||A(t,x)|| + ||a(t,x)||) \le K(1 + ||x||). \tag{2.2}$$

Assume that the regression $F_*(t,x)$ of $D_*\xi(t)$ is $C^{1,2}$-smooth.

The backward mean derivatives of the first and the second order of the diffusion process $\xi(t)$ are equal to

$$D_*\xi(t) = a(t,\xi(t)) - \frac{1}{p_t(\xi(t))}\nabla_j(\sigma(t,\xi(t))p_t(\xi(t))), \tag{2.3}$$

$$D_*D_*\xi(t) = \left(\frac{\partial F_*}{\partial t} + \nabla F_* \cdot F_* - \frac{1}{2}\nabla^2 F_* \sigma\right)(t,\xi(t)). \tag{2.4}$$

Proof. We'll find the formula for the first order mean derivative using the techniques from [1, 5].

By the definition of conditional expectation we should check the property for all measurable functions g of $\xi(t)$ ([11]). Together with the Itô formula for smooth function f we get

$$\begin{aligned} &E\left((f(\xi(t)) - f(\xi(s)))g(\xi(t))\right) = \\ &\quad E\left(\left[\int_s^t (\nabla f A)(u,\xi(u))\mathrm{d}w + \int_s^t (\nabla f a + \frac{1}{2}\nabla^2 f\sigma)(u,\xi(t))\mathrm{d}u\right] g(\xi(t))\right) = \\ &E\left[\left(\int_s^t (\nabla f A)(u,\xi(u))\mathrm{d}w \cdot g(\xi(t))\right) + \int_s^t (\nabla f a + \frac{1}{2}\nabla^2 f\sigma)(u,\xi(t))\mathrm{d}u \cdot g(\xi(t))\right] \end{aligned} \tag{2.5}$$

Now apply the Ito formula to g in the first summand

$$\begin{aligned} &E\left(\int_s^t (\nabla f A)(u,\xi(t))\mathrm{d}w \cdot g(\xi(t))\right) = \\ &\quad E\left(\int_s^t (\nabla f A)(u,\xi(t))\mathrm{d}w \cdot (g(\xi(t)) \pm g(\xi(s)))\right) = \\ &E\left(\int_s^t (\nabla f A)(u,\xi(t))\mathrm{d}w \cdot (g(\xi(t)) - g(\xi(s)))\right) + E\left(\int_s^t \nabla f A \mathrm{d}w \cdot g(\xi(s))\right) = \\ &E\left(\int_s^t (\nabla f A)(u,\xi(t))\mathrm{d}w \left(\int_s^t (\nabla g A)(u,\xi(u))\mathrm{d}w + \int_s^t \left[\nabla g a + \frac{1}{2}\nabla^2 g\sigma\right](u,\xi(u))\mathrm{d}u\right)\right) + \\ &\qquad E\left(\int_s^t (\nabla f A)(u,\xi(u))\mathrm{d}w \cdot g(\xi(s))\right). \end{aligned} \tag{2.6}$$

By the property of conditional expectation

$$E\left(\int_s^t (\nabla f A)(u,\xi(u))\mathrm{d}w \cdot g(\xi(s))\right) = E\left(g(\xi(s)) \cdot E\left(\int_s^t (\nabla f A)(u,\xi(u))\mathrm{d}w \middle| \xi(s)\right)\right)$$

Since the stochastic integral is a martingale we get

$$E\left(g(\xi(s)) \cdot E\left(\int_s^t (\nabla f A)(u,\xi(u))\mathrm{d}w \middle| \xi(s)\right)\right) = 0.$$

Put $I=\int\limits_s^t(\nabla fA)(u,\xi(u))\mathrm{d}w$. So in (2.6) we get

$$E\left(I\cdot g(\xi(t))\right)=\\ E\left(I\cdot\left(\int\limits_s^t(\nabla gA)(u,\xi(u))\mathrm{d}w+\int\limits_s^t(\nabla ga+\frac{1}{2}\nabla^2 g\sigma)(u,\xi(u))\mathrm{d}u\right)\right). \quad (2.7)$$

By the Schwarz inequality in $L_2(\Omega,\mathscr{F},P)$, we get:

$$E\left(I\cdot\int\limits_s^t\left(\nabla ga+\frac{1}{2}\nabla^2 g\sigma\right)(u,\xi(u))\mathrm{d}u\right)\leq\\ \sqrt{E\left(\int\limits_s^t(\nabla fA)(u,\xi(u))\mathrm{d}w\right)^2}\cdot\sqrt{E\left(\int\limits_s^t\left(\nabla ga+\frac{1}{2}\nabla^2 g\sigma\right)(u,\xi(u))\mathrm{d}u\right)^2}=\\ \sqrt{E\left(\int\limits_s^t(\nabla fA)^2(u,\xi(u))\mathrm{d}u\right)}\cdot\sqrt{E\left(\int\limits_s^t\left(\nabla ga+\frac{1}{2}\nabla^2 g\sigma\right)(u,\xi(u))\mathrm{d}u\right)^2}.$$

Since the integrands are bounded, we get that the last expression is less then

$$C(t-s)^{3/2}$$

which tends to 0 if $t-s$ tends to 0.

So formula (2.5) by the Lebesgue theorem turns into

$$\lim_{t-s\to 0}\frac{1}{t-s}E\left((f(\xi(t))-f(\xi(s)))g(\xi(t))\right)=\\ E\left[\left((\nabla fA)+\frac{1}{2}\nabla^2 f\sigma\right)(t,\xi(t))g(\xi(t))+\nabla f(\xi(t))\sigma(t,\xi(t))\nabla g(\xi(t))\right] \quad (2.8)$$

Applying the integration by parts formula for the second summand in (2.5) we get:

$$\int\limits_{\mathbb{R}^n}\nabla f(x)\sigma(t,x)\nabla g(x)p_t(x)\mathrm{d}x=-\int\limits_{\mathbb{R}^n}\nabla(\nabla f(x)\sigma(t,x)p_t(x))g(x)\mathrm{d}x=$$

$$-E(\nabla^2 f(\xi(t))\sigma(t,\xi(t))g(\xi(t)))-E(\nabla f(\xi(t))\frac{1}{p_t(\xi(t))}\nabla(\sigma(t,\xi(t))p_t(\xi(t)))g(\xi(t)))$$

Let's put $f(x)=x$. So we get $D_*\xi(t)=a(t,\xi(t))-\frac{1}{p_t(\xi(t))}\nabla(\sigma(t,\xi(t))p_t(\xi(t)))$.

Let's consider a smooth function $f(t,x)$ instead of $f(x)$ and repeat all the steps. The only one difference is the application of Itô formula to $f(t,\xi(t))-f(s,\xi(s))$. So,

$$D_*f(t,\xi(t))=\left(\frac{\partial f}{\partial t}+\nabla f a-\frac{1}{2}\nabla^2 f\sigma-\nabla f\frac{1}{p_t(\xi(t))}\nabla(\sigma p_t(\xi(t)))\right)(t,\xi(t))=$$
$$\left(\frac{\partial f}{\partial t}+\nabla f\left(a-\frac{1}{p_t(\xi(t))}\nabla(\sigma p_t(\xi(t))\right)-\frac{1}{2}\nabla^2 f\sigma\right)(t,\xi(t))=$$
$$\left(\frac{\partial f}{\partial t}+\nabla f\cdot D_*\xi(t)-\frac{1}{2}\nabla^2 f\sigma\right)(t,\xi(t)). \quad (2.9)$$

Put $f(t,x)=F_*(t,x)$, then

$$D_*D_*\xi(t)=\left(\frac{\partial F_*}{\partial t}+\nabla F_*\cdot F_*-\frac{1}{2}\nabla^2 F_*\sigma\right)(t,\xi(t)).$$

□

3. Backward mean derivatives for diffusion type Itô process

Theorem 3.1. *Let $\xi(t)$ be a diffusion type process.*

Assume that there exists a constant K such that $a(t,x)$ and $A(t,x)$ satisfy the properties

$$\sup_t(||A(t,x)-A(t,y)||+||a(t,x)-a(t,y)||)\le K||x-y||. \quad (3.1)$$

$$\sup_t(||A(t,x)||+||a(t,x)||)\le K(1+||x||). \quad (3.2)$$

Assume that $F_*(t,x)$ is a $C^{1,2}$-vector field.

Then the backward mean derivatives of the first and the second order are equal to

$$D_*\xi(t)=E_t^{\xi}(a(t,\xi(\cdot)))-E_t^{\xi}\left(\frac{1}{p_t}\nabla_j(A(t,\xi(\cdot))p_t)\right) \quad (3.3)$$

$$D_*D_*\xi(t)=E_t^{\xi}\left(\frac{\partial F_*(t,\xi(t))}{\partial t}+(\nabla F_*\cdot F_*)(t,\xi(t))-\frac{1}{2}\nabla^2 F_*(t,\xi(t))\sigma(t,\xi(t))\right). \quad (3.4)$$

Proof. By the definition of mean derivative we have

$$D_*\xi(t)=\lim_{\triangle t\to+0}E_t^{\xi}\left(\frac{1}{\triangle t}\left(\int_{t-\triangle t}^{t}a(s,\xi(\cdot))\mathrm{d}s+\int_{t-\triangle t}^{t}A(s,\xi(\cdot))\mathrm{d}w(s)\right)\right)$$

By the property of conditional expectation we have that for any $\mathscr{N}_t^{\xi}$-measurable function g the last expression is equal to

$$\lim_{\triangle t\to+0}E\left(\frac{1}{\triangle t}\left(\int_{t-\triangle t}^{t}a(s,\xi_s(\cdot))\mathrm{d}s+\int_{t-\triangle t}^{t}A(s,\xi_s(\cdot))\mathrm{d}w(s)\right)g(\xi(t))\right)$$

Let's apply the Lebegue theorem for the first summand. We don't have the martingale property anymore. So since the Ito integral is a particular case of the Skhorohod integral

([13, 8]) we have that the last expression is equal to

$$E_t^{\xi}\left(a(t,\xi_t(\cdot))\right)+\lim_{\triangle t\to+0}\frac{1}{\triangle t}E\left(\int\limits_{t-\triangle t}^{t}A(s,\xi_s(\cdot))D_s g(\xi(t))\mathrm{d}s\right)$$

By the chain rule for the Malliavin derivatives ([13, 8]) and the Lebegue theorem again we get

$$E_t^{\xi}a(t,\xi(\cdot))+E\left(A(t,\xi(\cdot))\nabla_j g(\xi(t))D_t\xi(t)\right)$$

By the integration by parts and lemmas (A1, A2) from [5] the last expression transforms into

$$E_t^{\xi}a(t,\xi(\cdot))-\int\limits_{\mathbb{R}^n}\nabla_j(\sigma(t,\xi(\cdot))p_t(x))g(x)\mathrm{d}x.$$

Thus

$$E_t^{\xi}a(t,\xi(\cdot))-\int\limits_{\mathbb{R}^n}\frac{1}{p_t}\nabla_j\left(\sigma(t,\xi(\cdot))p_t\right)g(\xi(t))p_t(x)\mathrm{d}x=$$

$$=E_t^{\xi}a(t,\xi(\cdot))-E_t^{\xi}\frac{1}{p_t}\nabla_j\left(\sigma(t,\xi(\cdot))p_t\right)$$

So, we get

$$D_*\xi(t)=E_t^{\xi}\left(a(t,\xi(\cdot))\right)-E_t^{\xi}\left(\frac{1}{p_t}\nabla_j(A(t,\xi(\cdot))p_t)\right)$$

Now consider $D_*D_*\xi(t)$ for the diffusion type processes. Let $F_*(t,x)$ be the regression for the $D_*\xi(t)$. We will find the backward mean derivative for $F_*(t,x)$.

$$D_*D_*\xi(t)=D_*F_*(t,\xi(t))=\lim_{\triangle t\to+0}E_t^{\xi}\left(\frac{F_*(t,\xi(t))-F_*(t-\triangle t,\xi(t-\triangle t))}{\triangle t}\right). \quad (3.5)$$

Since we assume that Borel vector field $F_*(t,x)$ correspoinding to $D_*\xi(t)$ is C^2, we can repeat all steps from the proof of Lemma 2.1 except the one concerning the martingale property. But we have the property for Malliavin derivatives ([8]):

$$D_t\xi(s)=0,\quad t>s.$$

Then

$$E\left(\int\limits_s^t(\nabla fA)(u,\xi(\cdot))\mathrm{d}w\cdot g(\xi(s))\right)=E\left(\int\limits_s^t(\nabla fA)(u,\xi(\cdot))\cdot D_u g(\xi(s))\mathrm{d}u\right)=0.$$

We get the following formula

$$D_*D_*\xi(t)=E_t^{\xi}\left(\frac{\partial F_*(t,\xi(t))}{\partial t}+(\nabla F_*\cdot F_*)(t,\xi(t))-\frac{1}{2}\nabla^2F_*(t,\xi(t))\sigma(t,\xi(t))\right). \quad (3.6)$$

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