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ON THE KHINCHIN TRANSFORMATION FOR FEJER PAIRS

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ABSTRACT. The article presents a method for constructing new Fejer pairs from adjoint probability densities using the Khinchin transformation. Their main properties are described, examples of constructing such Fejer pairs for the main positive definite probability densities are given, and relationships between them are found. The article shows the connection between the probability error function and the characteristic function from such a pair.

Introduction

In 1938, A.Ya. Khinchin, in his article "On unimodal distributions" [8] proved that for any characteristic function (ch.f.) $\hat{u}\,(x)$ of a unimodal [13, 9] distribution, the following representation holds:

$$\hat{u}\,(x) = \frac{1}{x}\int\limits_0^x \hat{f}\,(t)\,\mathrm{d}t\,,$$

in which $\hat{f}\,(t)$ is the characteristic function of some probability distribution.

This representation can be viewed as a transformation [5, 15] of characteristic functions. Interesting results are obtained by applying the Khinchin transformation to a certain subclass of even characteristic functions of unimodal distributions. The main classical distributions [6, 13] in probability theory belong to this class: the standard normal distribution, the standard Cauchy distribution, the continuous uniform distribution on an interval, the triangular distribution on an interval (Simpson's distribution), the Rademacher distribution (uniform Bernoulli distribution at two points), the Laplace distribution (double exponential distribution), the Linnik distribution, and others. Many properties of such distributions are known [11]. In the monograph [3, p. 28] many of them are combined into pairs, which the author calls Fejer pairs (kernels): the Fejer pair, the Abel-Poisson pair, and the Gauss pair. Thus, the Fejer pair [3, p. 29] consists of the characteristic function of the symmetric triangular distribution on the interval $[-1, 1]$ and its density; the Abel-Poisson pair can be associated with the Cauchy

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distribution and its characteristic function, or with the standard Laplace distribution (double exponential distribution) and its characteristic function; from a probabilistic point of view, the Gauss pair corresponds to the density of the standard normal distribution and its characteristic function. (Note that there does not seem to be a specific name for the distribution from this class whose characteristic function is a triangular function).

The classical Dirichlet integrals, Gauss integrals, or otherwise the Euler-Poisson integrals, are integrals of the characteristic functions of the triangular and normal distributions, respectively. One of the methods for finding the Dirichlet integral, using the Laplace transform, reduces [1, p. 459] it to finding the integral of the density of the Cauchy distribution, which, up to normalization, is itself a characteristic function.

The class of Fejer pairs (f, p) of functions in the monograph [3, p. 28] is defined by the following properties:

$$\begin{array}{cccccc}
f \in L_1(-\infty,+\infty) & , & & & p_0(x) \in L_1(0,+\infty) & , \\
f \in C(U_\varepsilon(0)) & , & |p(x)| \leqslant p_0(|x|) & , & p_0(x) \geqslant 0 & , \\
f(x) = f(-x) & , & p(x) = p(-x) & , & & \\
f(0) = 1 & , & \int\limits_{-\infty}^{+\infty} p(x)\,dx = 1 & , & &
\end{array}$$

where $p_0(x)$ is a non-increasing function on the positive half-axis $(0, +\infty)$ and

$$p(x) = \sqrt{2\pi}\mathscr{F}[f](x) = \int\limits_{-\infty}^{\infty} e^{-itx} f(t)\,dt\,.$$

Since further the pairs from this class are considered in a probabilistic context [12], the functions $p(x)$ will everywhere correspond to densities $p_\xi(x)$ of an even probability distribution, and $f(x)$ to characteristic functions $f_\xi(t)$ of the same distribution. Consequently, the conditions on the function $p(x)$ (in particular $p(x) = p_0(x)$), are modified so that it becomes a probability density decreasing on the positive half-axis $(0, +\infty)$ and

$$\begin{array}{cccc}
f_\xi \in L_1(-\infty,+\infty) & , & p_\xi(x) \in L_1(0,+\infty) & , \\
f_\xi \in C(U_\varepsilon(0)) & , & p_\xi(x) \geqslant 0 & , \\
f_\xi(x) = f_\xi(-x) & , & p_\xi(x) = p_\xi(-x) & , \\
f_\xi(0) = 1 & , & \int\limits_{-\infty}^{+\infty} p_\xi(x)\,dx = 1 & ,
\end{array}$$

$$p_\xi(x) = \frac{1}{\sqrt{2\pi}}\mathscr{F}[f_\xi](x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-itx} f_\xi(t)\,dt\,,$$

$$f_\xi(t) = \mathsf{E}e^{it\xi} = \int\limits_{-\infty}^{\infty} e^{itx} p_\xi(x)\,dx = \mathsf{E}\cos(t\xi) + i\mathsf{E}\sin(t\xi)\,.$$

A probabilistic approach to the study of such pairs can be found in the articles [11, 12]. Since in the case under consideration the density coincides, up to a constant, with some characteristic function, then, by the Bochner- Khinchin theorem, such a density is positive definite (in [9, p. 61] non-negative definite) and the following theorem holds for them [9, p. 62].

Theorem 0.1. *Let $g(t)$ be a positive definite function. Then: $g(0)$ is real and*

$$g(0) \geqslant 0\,,\ g(-t) = \overline{g(t)}\,,\ |g(t)| \leqslant g(0)\ .$$

Since in our case $g(t)$ s, up to a constant, a density, it is real-valued and even: $g(-t) = g(t)$. Also, since it is a characteristic function, $g(0) > 0$.

By positive definiteness, $p(x) \leqslant p(0)$ for all x. The equality $p(x) = p(0)$ for some $x \neq 0$ would imply periodicity, which is impossible for an integrable density on $\mathbb{R}$. Hence, for all $x \neq 0$

$$0 \leqslant p(x) < p(0)\ .$$

In paper [11], to every positive definite probability density function $p(x)$ there corresponds an adjoint probability density function $q(x)$. And for them,
in particular, it is proved that a probability density function p is positive definite if and only if its characteristic function f satis es the conditions $f \geqslant 0$ and $f \in L^1(\mathbb{R})$. Thus, if we follow [11], we obtain some correspondence between four functions. More precisely, the following theorem holds.

Theorem 0.2. *(H.-J. Rossberg, Theorem 2.1) Let $p(t)$ be a positive definite probability density with the characteristic function $\hat{p}$, and let $p_0 := p(0)$. Then there exists an adjoint positive definite probability density function q with the characteristic function $\hat{q}$ such that*

$$\hat{q}(x) = \frac{p(x)}{p_0}, \quad \hat{p}(x) = \frac{q(x)}{q_0} \tag{0.1}$$

where the value of the adjoint probability density function at the origin is defined as $q_0 := q(0)$, and the normalization constants are strictly bound by the equation:

$$2\pi p_0 q_0 = 1.$$

Moreover, the conjugation operation is reciprocal, and the adjoint probability density function q can be explicitly expressed via the characteristic function of the initial distribution by means of the inverse Fourier transform:

$$2\pi q(t) = \int\limits_{-\infty}^{+\infty} e^{itx}\hat{q}(x)dx = p_0^{-1}\hat{p}(t)\,.$$

The last equalities from the last theorem can be written in more detail:

$$q(t) = \frac{1}{2\pi p_0}\hat{p}(t) = \frac{1}{\pi \cdot p_0}\int\limits_{0}^{+\infty} p(x)\cos(tx)\ dx\,,$$

$$p(t) = \frac{1}{2\pi q_0}\hat{q}(t) = \frac{1}{\pi \cdot q_0}\int\limits_{0}^{+\infty} q(x)\cos(tx)\ dx\,.$$

In concluding the description of these functional classes, it is worth noting that among them, a special role is assigned to the self-reciprocal pairs [4]. For such dual systems, the initial probability density function coincides exactly with its own adjoint density up to a constant scaling factor, which is most prominently exempli ed by the standard Gaussian distribution where the functional form remains completely invariant under the Fourier-transform conjugation.

Next, we will consider the class of pairs $\mathscr{K}$, whose density is positive definite. Pairs $(p, \hat{p})$ from $\mathscr{K}$ will be called *probabilistic Fejer pairs*, if p a non-negative denite function. For each pair $(p, \hat{p})$ there exists an adjoint pair $(q, \hat{q})$.

We will further show that when considering such pairs, another Fejer pair $(u, \hat{u})$ and its corresponding adjoint pair $(v, \hat{v})$. naturally arise. The use of special techniques involving the Khinchin transformation and its generalizations, obtained via convolution with various types of distribution functions from the above class, allows us to find relations both between characteristic functions and between distribution functions and characteristic functions, as well as between probability densities and characteristic functions. It also allows us to show that a given function is a characteristic function. Next, we define some general representations for functions from the class $\mathscr{K}$ and consider examples with already classical functions [3, p. 29] from this class.

Representation and some properties of one class of characteristic functions

The initial idea is that the distribution function $F_p(x)$ of the pair can be expressed in terms of the characteristic function $\hat{q}$ of the adjoint pair $(q, \hat{q})$.

Theorem 0.3. *If the probability density $p(x)$ belongs to the densities of the class $\mathscr{K}$ and $F_p(x)$ is its distribution function, then the function*

$$\hat{u}(x) = \frac{F_p(x) - F_p(0)}{p(0) \cdot x} \tag{0.2}$$

will be a characteristic function of some unimodal distribution.

Proof. By Khinchin's theorem, for $x > 0$

$$\hat{u}(x) = \frac{1}{x} \int\limits_0^x \hat{q}(y)\, \mathrm{d}y \tag{0.3}$$

is a characteristic function of some unimodal distribution.

Since by assumption $p \in \mathscr{K}$, we have $p(y) = p(0) \cdot \hat{q}(y)$ and for $x > 0$ the following transformations hold:

$$F_p(x) = F_p(0) + \int\limits_0^x p(y)\, \mathrm{d}y = F_p(0) + c \int\limits_0^x \hat{q}(y)\, \mathrm{d}y =$$

$$= F_p(0) + cx \frac{1}{x} \int\limits_0^x \hat{q}(y)\, \mathrm{d}y = F_p(0) + x \cdot \hat{u}(x) \cdot p(0).$$

Whence for $x > 0$ we obtain (0.2). From the symmetry about zero of the original distribution, it follows that

$$F_p(x) = 1 - F_p(-x)\,, \qquad F_P(0) = \frac{1}{2}\,.$$

Therefore

$$\hat{u}(-x) = \frac{F_p(-x) - F_p(0)}{p(0) \cdot (-x)} = \frac{1 - F_p(x) - F_p(0)}{p(0) \cdot (-x)} = \frac{F_p(x) - F_p(0)}{p(0) \cdot x} = \hat{u}(x) \ .$$

The last equality shows that the definition $\hat{u}(x)$ as an even and real-valued function is correct, and also that the representation (0.2) holds for any real x. An obvio consequence of the obtained representation is the non-negativity of $\hat{u}(x)$.

Remark 0.4. Thus, if $p \in \mathscr{K}$, $F_p(x)$ can be represented as

$$F_p(x) = F_p(0) + x \cdot \hat{u}(x) \cdot p(0) \ ,$$

and since the representation (0.3), then according to [8]

$$F_q(x) = F_u(x) - x \cdot u(x) \ .$$

For the characteristic function $\hat{u}(x)$ other representations are possible through the components of the original reciprocal pairs. Let us consider some of them.

Theorem 0.5. *If the density $p(y)$ belongs to the densities of the class $\mathscr{K}$ and $\hat{p}(t)$ is its characteristic function, then the function*

$$\hat{u}(x) = \frac{1}{\pi p(0)} \int_0^{+\infty} \frac{\sin(xt)}{xt} \hat{p}(t) \ dt \, . \tag{0.4}$$

will be a characteristic function of some unimodal distribution.

Proof. Let $p(x)$ be a symmetric probability density function and $\hat{p}(t)$ be its corresponding real-valued characteristic function. We aim to derive the canonical integral representation for the characteristic tail function $\hat{u}(t)$.

Substituting the inverse Fourier cosine transform

$$p(y) = \frac{1}{\pi} \int_0^{+\infty} \hat{p}(t) \cos(ty) \ dt$$

and applying Fubini's theorem to change the order of integration, we obtain:

$$F_p(x) - F_p(0) = \int_0^x p(y) \ dy = \int_0^x \left(\frac{1}{\pi} \int_0^{+\infty} \hat{p}(t) \cos(ty) \ dt \right) dy =$$

$$= \frac{1}{\pi} \int_0^{+\infty} \hat{p}(t) \left(\int_0^x \cos(ty) \, dy \right) dt = \frac{1}{\pi} \int_0^{+\infty} \hat{p}(t) \, \frac{\sin(tx)}{t} \, dt \, .$$

Consequently, dividing both sides by $p(0) \cdot x$, we arrive at the desired canonical form:

$$\hat{u}(x) = \frac{F_p(x) - F_p(0)}{p(0) \cdot x} = \frac{1}{\pi p_0} \int_0^{+\infty} \hat{p}(t) \, \frac{\sin(xt)}{xt} \, dt \, .$$

This completes the proof.

Remark 0.6. The representation (0.4) can be written as

$$\hat{u}(x) = \frac{1}{\pi p(0)} \int\limits_{0}^{+\infty} \hat{p}(t)\, \hat{v}_x(t)\; dt\,, \tag{0.5}$$

where

$$\hat{v}_x(t) = \operatorname{sinc}(xt) = \frac{\sin(xt)}{(xt)}\,.$$

Corollary 0.7. *The representation* (0.4) *holds*

$$\hat{u}(x) = 2 \int\limits_{0}^{+\infty} \hat{v}_x(t)\; dF_q(t)\;, \tag{0.6}$$

where probability density function $q(t)$ *from* (0.1).

Proof. Since $\hat{p}(x) = \frac{q(x)}{q_0}$ and $2\pi p_0 q_0 = 1$, then

$$\hat{u}(x) = 2 \int\limits_{0}^{+\infty} \hat{v}_x(t)\; dF_q(t) = \frac{1}{\pi q_0 p_0} \int\limits_{0}^{+\infty} q(t)\, \frac{\sin(xt)}{xt}\, dt = 2 \int\limits_{0}^{+\infty} q(t)\, \frac{\sin(xt)}{xt}\, dt\,.$$

□

The representation (0.6) can be obtained from (0.6), using the evenness property of q. The following representation also relies on the evenness of the density q.

Theorem 0.8. *If the characteristic function* $\hat{u}$ *is given by*

$$\hat{u}(s) = 2 \int\limits_{0}^{+\infty} q(y)\, \frac{\sin(sy)}{sy}\, dy$$

then its corresponding density function u *satis es the integral representation:*

$$u(x) = \int\limits_{|x|}^{+\infty} \frac{q(y)}{y}\, dy\,. \tag{0.7}$$

Proof. Since u is a symmetric secondary density function, it can be reconstructed from its characteristic function $\hat{u}$ via the inverse Fourier cosine transform:

$$u(x) = \frac{1}{\pi} \int\limits_{0}^{+\infty} \hat{u}(s) \cos(xs)\; ds\,.$$

Substituting the given integral representation for $\hat{u}(s)$ into this equation yields:

$$u(x) = \frac{1}{\pi} \int\limits_{0}^{+\infty} \left(2 \int\limits_{0}^{+\infty} q(y)\, \frac{\sin(sy)}{sy}\, dy \right) \cos(xs)\; ds\,.$$

Factoring out the constants and grouping the terms dependent on x, we apply Fubini's theorem to invert the order of integration over the first quadrant ($s > 0, x > 0$):

$$u(x) = \frac{2}{\pi} \int\limits_0^{+\infty} \left(\frac{q(y)}{y} \int\limits_0^{+\infty} \frac{\sin(sy)\cos(xs)}{s} \, ds \right) dy \, .$$

The inner integral with respect to the variable x is a canonical Dirichlet integral:

$$\int\limits_0^{+\infty} \frac{\sin(sy)\cos(xs)}{s} \, \mathrm{d}x = \frac{\pi}{2} \cdot \theta(y - |x|)$$

where $\theta(y - |x|)$ is the Heaviside step function, which equals 1 if $y > |x|$ and 0 if $y < |x|$. Substituting this indicator kernel back into the outer integral directly truncates the lower bound of integration:

$$u(x) = \frac{2}{\pi} \int\limits_0^{+\infty} \frac{q(y)}{y} \cdot \left(\frac{\pi}{2} \cdot \theta(y - |x|) \right) dy = \int\limits_{|x|}^{+\infty} \frac{q(y)}{y} \, dy \, .$$

One of the expressions for the distribution function $F_u(x)$ in terms of the distribution function $F_q(x)$ can be obtained using (0.7) and the formula

$$F_q(x) = F_u(x) - x \cdot u(x)$$

from [8]. Thus,

$$F_u(x) = F_q(x) + x \cdot u(x) = F_q(x) + x \int\limits_{|x|}^{+\infty} \frac{dF_q(x)}{y} \, .$$

It is clear that for the characteristic function

$$\hat{v}(x) = \frac{F_q(x) - F_q(0)}{q(0) \cdot x}$$

all the previous results hold with the replacement of p by q, q by p and u by v. Next, we consider several examples.

Examples of probabilistic Fejer pairs

1. Probabilistic Gauss pair

Let us consider in more detail, as an example, the distribution function $\Phi(x)$ of the standard normal distribution [13] with density $\varphi(x)$ and characteristic function $\hat{\varphi}(x)$. The density of the normal distribution is self-reciprocal in the sense of the corresponding definition from the paper [4].

1. For $x > 0$ the distribution function $\Phi(x)$ of the standard normal distribution can be represented as

$$\Phi(x) = \Phi(0) + \int\limits_0^{x} \varphi(y) \, \mathrm{d}y \, ,$$

where $\varphi(y) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ is the density of the standard normal distribution. Moreover, this representation can be rewritten as

$$\Phi(x) - \Phi(0) = \frac{x}{\sqrt{2\pi}} \frac{1}{x} \int\limits_0^x \hat{\varphi}(t)\,\mathrm{d}t\,,$$

where $\hat{\varphi}(t) = e^{-t^2/2}$ is the characteristic function of the standard normal distribution. By Khinchin's theorem, the quantity

$$\hat{u}(x) = \frac{1}{x} \int\limits_0^x \hat{\varphi}(t)\,\mathrm{d}t$$

is a characteristic function of some unimodal distribution. Consequently, the distribution function $\Phi(x)$ of the standard normal distribution is expressed in terms of the characteristic function of a unimodal distribution:

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} x \cdot \hat{u}(x)\ . \tag{1.1}$$

2. Since

$$\Phi(-x) = \int\limits_{-\infty}^{-x} \varphi(y)\,\mathrm{d}y = 1 - \Phi(x) = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} x\hat{u}(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} (-x)\,\hat{u}(-x)\ ,$$

and $\hat{u}(x)$ is an even function, the representation (1.1) holds for any x.

3. It is well known [2, 6], that the function $\Phi(x)$ can be expressed in terms of the error function:

$$\Phi(x) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\ .$$

Therefore, taking into account (1.1) the error function, up to multiplication $\sqrt{\frac{2}{\pi}}x$ represents [14] a characteristic function:

$$\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) = \sqrt{\frac{2}{\pi}} x \cdot \hat{u}(x)\ . \tag{1.2}$$

The error function can also be expressed in terms of the gamma function.

Therefore,

$$\hat{u}(x) = \frac{1}{x}\sqrt{\frac{\pi}{2}} \cdot \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) = \frac{1}{x}\sqrt{\frac{\pi}{2}} \cdot \left(1 - \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}, \frac{x^2}{2}\right)\right)\ ,$$

$$\hat{u}(x) = \frac{1}{x}\sqrt{\frac{\pi}{2}} - \frac{1}{x\sqrt{2}}\Gamma\left(\frac{1}{2}, \frac{x^2}{2}\right)\ .$$

Whence

$$\Gamma\left(\frac{1}{2}, \frac{x^2}{2}\right) = 1 - \sqrt{\frac{2}{\pi}} x \cdot \hat{u}(x)\ .$$

4. In turn, the well-known expansion of the error function in a series allows one to write the characteristic function $\hat{u}(x)$ found above as an alternating series:

$$\hat{u}(x) = \frac{1}{x}\sqrt{\frac{\pi}{2}}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\,(2n+1)} \left(\frac{x}{\sqrt{2}}\right)^{2n} =$$

$$=\sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{n!\,(2n+1)\,2^n} = \sum_{n=0}^{+\infty} (-1)^n \frac{(2n-1)!!}{(2n+1)!} \frac{x^{2n}}{(2n)!} =$$

$$= \sum_{n=0}^{+\infty} H_{2n}(0) \frac{x^{2n}}{(2n+1)!},$$

where

$$H_{2n}(0) = \frac{(-1)^k \varphi^{(2n)}(0)}{\varphi(0)} = (-1)^n \frac{(2n)!}{2^n n!} = (-1)^n (2n-1)!! = (-1)^n \frac{(2n-1)!}{(2n-2)!!}$$

are the values at zero of the Chebyshev-Hermite polynomials

$$H_k(x) = (-1)^k \frac{\varphi^{(k)}(x)}{\varphi(x)}.$$

It is seen that the last representation

$$\hat{u}(x) = \frac{1}{x}\int_0^x \hat{\varphi}(t)\,\mathrm{d}t = \frac{\sqrt{2\pi}}{x}\int_0^x \varphi(t)\,\mathrm{d}t = \frac{1}{\varphi(0)} \sum_{n=0}^{+\infty} (-1)^k \varphi^{(2n)}(0) \frac{x^{2n}}{(2n+1)!}$$

holds by virtue of the equality $\hat{\varphi}(t) = \sqrt{2\pi} \cdot \varphi(t)$ can be obtained by integrating the expansion of the density of the standard normal distribution at zero. This is an alternating series in even powers. 5. According to [7] analogously

$$\hat{u}(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{n!\,(2n+1)\,2^n}$$

one can write

$$\hat{u}(x) = \sum_{n=0}^{+\infty} \frac{\theta_{2n}\left(\frac{x}{2}\right)}{2n+1},$$

where

$$\theta_n(x) = \frac{x^n}{n!} H_n(x)$$

and the values of $\theta_n(x)$ can be computed by the recurrence formula

$$\theta_n(x) = x^2 \frac{\theta_n(x) - \theta_{n-1}(x)}{n+1}.$$

6. The representation

$$\Phi(x) = \Phi(0) + \varphi(x) \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!!},$$

allows us to write another expansion for the characteristic function defined above:

$$\hat{u}(x) = \varphi(x)\sqrt{2\pi} \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n+1)!!} = \hat{\varphi}(x) \cdot \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n+1)!!}.$$

Hence it immediately follows that

$$\hat{u}(x) \geqslant \hat{\varphi}(x).$$

7. Since $\hat{u}(x)$ and $\hat{\varphi}(x)$ are characteristic functions, and the series

$$f(x) = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n+1)!!}$$

up to multiplication by x already appears in Laplace's work [10], t is appropriate to ask what properties the function $f(x)$, possesses, which, through the relation

$$\hat{u}(x) = \hat{\varphi}(x) \cdot f(x)$$

connects the characteristic functions $\hat{u}(x)$ and $\hat{\varphi}(x)$.

8. The function $f(x)$ is even and unbounded. It is also everywhere continuous, since for it the representation

$$f(x) = \frac{\Phi(x) - \Phi(0)}{x \cdot \varphi(x)}$$

holds and as $x \to 0$ there exists the limit

$$\frac{\Phi(x) - \Phi(0)}{x \cdot \varphi(x)} \to 1\,,$$

which coincides with the value $f(0) = 1$, while the radius of convergence of the series is infinite.

It is strictly increasing and convex downward for $x > 0$, because its first and second derivatives are positive for those values; moreover, for $x > 0$ the inequality holds

$$f(x) = \frac{\Phi(x) - \Phi(0)}{x\varphi(x)} \leqslant e^{x^2/2}\,.$$

Thus, it is separated from being a characteristic function only by unboundedness, since othewise the Titchmarsh-Polyas theorem [9, p. 70].

2. Probabilistic Fejer pair and the integral sine

The Fejer pair from [3, p. 29] corresponds to the triangular characteristic function

$$\hat{c}(y) = \max\left\{0, \left(1 - \frac{|y|}{\alpha}\right)\right\} = \alpha \cdot \tau(y)$$

and the density containing the square of the cardinal sine,

$$c(y) = \frac{\alpha}{2\pi} \cdot \frac{\sin^2 \frac{\alpha y}{2}}{\left(\frac{\alpha y}{2}\right)^2} = \frac{\alpha}{2\pi} \cdot \operatorname{sinc}^2\left(\frac{\alpha y}{2}\right) = \frac{\alpha}{2\pi} \cdot \hat{\tau}(x)\ ,$$

where respectively

$$\hat{\tau}(y) = \operatorname{sinc}^2\left(\frac{\alpha y}{2}\right) = \frac{\sin^2\left(\frac{\alpha y}{2}\right)}{\left(\frac{\alpha y}{2}\right)^2} = 2 \cdot \frac{1 - \cos(\alpha y)}{\alpha^2 y^2}$$

is the characteristic function, and $\tau(y)$ is the density of the symmetric triangular distribution. The distribution function $\mathcal{C}(x)$ for $x \geqslant 0$ can be written as

$\mathcal{C}(x) = \mathcal{C}(0) + \int_0^x c(y)\,\mathrm{d}y$. Then for $x > 0$

$$\mathcal{C}(x) - \mathcal{C}(0) = \frac{\alpha x}{2\pi} \cdot \frac{1}{x} \int_0^x \hat{\tau}(y)\,\mathrm{d}y = \frac{\alpha x}{2\pi} \cdot \hat{u}(x) ,$$

where $\hat{u}(x)$ is the characteristic function of a unimodal distribution. Consequently, after integration by parts,

$$\hat{u}(x) = \frac{1}{x} \int_0^x \hat{\tau}(y)\,\mathrm{d}y = 2 \cdot \frac{\mathrm{Si}(\alpha x)}{\alpha x} - 2 \cdot \frac{1-\cos(\alpha x)}{\alpha^2 x^2} = \left(2 \cdot \frac{\mathrm{Si}(\alpha x)}{\alpha x} - \hat{\tau}(x)\right) .$$

Hence it follows that

$$\frac{\mathrm{Si}(\alpha x)}{\alpha x} = \frac{\hat{u}(x) + \hat{\tau}(x)}{2}$$

is a characteristic function as a mixture of two characteristic functions.

It is interesting that these three characteristic functions, for quite obvious reasons (due to the simplest relationship between them), have almost identical series representations:

$$\hat{u}(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n (\alpha x)^{2n}}{(2n+1)!\,(n+1)\,(2n+1)} ,$$

$$\hat{\tau}(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n (\alpha x)^{2n}}{(2n+1)!\,(n+1)} ,$$

$$\frac{\mathrm{Si}(\alpha x)}{\alpha x} = \sum_{n=0}^{+\infty} \frac{(-1)^n (\alpha x)^{2n}}{(2n+1)!\,(2n+1)} .$$

3. Probabilistic triangular pair

The distribution function $\mathcal{T}(x)$ of the symmetric triangular distribution on $[-\alpha, \alpha]$ with density

$$\tau(y) = \max\left\{0, \frac{1}{\alpha}\left(1 - \frac{|y|}{\alpha}\right)\right\} = \frac{1}{\alpha} \cdot \hat{c}(y)$$

and characteristic function expressed by the square of the cardinal sine (the squared the sinc function)

$$\hat{\tau}(y) = \mathrm{sinc}^2\left(\frac{\alpha y}{2}\right) = \frac{\sin^2\left(\frac{\alpha y}{2}\right)}{\left(\frac{\alpha y}{2}\right)^2} = 2 \cdot \frac{1-\cos(\alpha y)}{\alpha^2 y^2}$$

for $x > 0$ can be written as $\mathcal{T}(x) = \mathcal{T}(0) + \int_0^x \tau(y)\,\mathrm{d}y = \mathcal{T}(0) + \frac{1}{\alpha} \int_0^x \hat{c}(y)\,\mathrm{d}y$. This representation allows us to notice that

$$\mathcal{T}(x) - \mathcal{T}(0) = \frac{x}{\alpha} \cdot \frac{1}{x} \int_0^x \hat{c}(y)\,\mathrm{d}y = \frac{x}{\alpha} \hat{u}(x)$$

where

$$\hat{u}(x) = \begin{cases} 1 - \frac{|x|}{2\alpha} & , \quad |x| \leqslant \alpha \\ \frac{|x|}{2\alpha} & , \quad \alpha \leqslant |x| \end{cases}$$

with

$$u(x) = \frac{\alpha}{2\pi}\left(\operatorname{sinc}(\alpha y) + \frac{1}{2}\operatorname{sinc}^2\left(\frac{\alpha y}{2}\right) - \operatorname{Ci}(\alpha y)\right).$$

4. Probabilistic Abel-Poisson pair

For $x > 0$ the distribution function $\mathcal{K}(x)$ of the standard Cauchy distribution can be represented as

$$\mathcal{K}(x) = \mathcal{K}(0) + \int_0^x k(y)\,\mathrm{d}y\,,$$

where $k(y) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$ is the density of the standard Cauchy distribution. Its characteristic function is $\hat{k}(t) = e^{-|t|}$. Then

$$\mathcal{K}(x) - \mathcal{K}(0) = \frac{1}{\pi}\int_0^x \hat{l}(t)\,\mathrm{d}t = k(0)\int_0^x \hat{l}(t)\,\mathrm{d}t\,,$$

where $\hat{l}(y) = \frac{1}{1+x^2}$ is the characteristic function of the Laplace distribution (double exponential distribution). And since

$$\mathcal{K}(x) - \mathcal{K}(0) = \frac{1}{\pi}\operatorname{arctg}(x)\,,$$

then

$$\hat{u}(x) = \pi \cdot \frac{\mathcal{K}(x) - \mathcal{K}(0)}{x} = \frac{\operatorname{arctg}(x)}{x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1}.$$

The density of this distribution is the exponential integral function:

$$u(x) = \frac{1}{2}\int_{|x|}^{+\infty} y^{-1}e^{-y}dy = \frac{1}{2} \cdot \mathrm{E}_1(|x|)\,.$$

5. Probabilistic Laplace-Cauchy pair

For $x > 0$ the distribution function $\mathcal{L}(x)$ of the Laplace distribution can be expressed through the density $l(y) = \frac{\alpha}{2} \cdot e^{-\alpha|y|}$ as

$$\mathcal{L}(x) = \mathcal{L}(0) + \int_0^x l(y)\,\mathrm{d}y = 1 - \frac{1}{2}e^{-\alpha x}.$$

This representation for $x > 0$ can be written as

$$\mathcal{L}(x) - \mathcal{L}(0) = \frac{\alpha}{2}\int_0^x \hat{k}(t)\,\mathrm{d}t = \frac{1}{2}\left(1 - e^{-\alpha x}\right),$$

where $\hat{k}(t) = e^{-\alpha|t|}$ is the characteristic function of the Cauchy distribution.

Consequently, according to (0.2) for $x > 0$

$$\hat{u}(x) = \frac{2}{\alpha} \cdot \frac{\mathcal{L}(x) - \mathcal{L}(0)}{x} = \frac{1 - e^{-\alpha x}}{\alpha x} = \sum_{n=0}^{\infty} (-1)^n \frac{(\alpha x)^n}{(n+1)!}.$$

and for any real x

$$\hat{u}(x) = \frac{1 - e^{-\alpha |x|}}{\alpha |x|} = \sum_{n=0}^{\infty} (-1)^n \frac{|\alpha x|^n}{(n+1)!}.$$

6. Conclusion

As can be seen from the above, the relationships between characteristic functions allow as to obtain related characteristic functions and densities to prove the the relationships between them and their corresponding distributions.

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