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Dedicated to the memory of V.V. Senatov.

ON ESTIMATES IN NEW ASYMPTOTIC EXPANSIONS IN THE CENTRAL LIMIT THEOREM

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ABSTRACT. This paper presents a new asymptotic expansion for the probability density function of normalized sums of independent and identically distributed random variables in the central limit theorem. By establishing sharp bounds for the remainder term, we obtain explicit estimates that avoid the use of Zolotarev's ideal metric. The theoretical results are validated through a numerical example with an exponential model distribution, demonstrating a significant improvement in accuracy compared to previously known bounds.

Introduction

Direct computation of the distributions of sums of identically distributed random variables involves well-known difficulties: these distributions, which are convolutions of the component distributions, can be explicitly calculated only in exceptional cases.

The Central Limit Theorem (CLT) of probability theory states that under fairly general conditions, the sum of a large number of independent or weakly dependent random variables is approximately standard normally distributed. The question of approximation accuracy in the CLT has remained relevant for many decades. Extensive research has focused on sharpening the Berry-Esseen bound, a problem conclusively resolved by I.G. Shevtsova [21]. However, due to its overly general approach and minimal information about the underlying distribution, this theorem yields rather conservative estimates. Constructing asymptotic expansions offers another way to improve accuracy in the CLT. Until recently, however, most such expansions only provided information on the orders of approximation accuracy and were unsuitable for numerical computation. In papers [2, pp. 129–130] and [3, 4], the simplest variants of asymptotic expansions with explicit accuracy estimates were presented. An asymptotic expansion for the binomial distribution is given in [2, pp. 129–130] (see also [13, p. 184]). The expansions in [3, 4] consist of the normal distribution function and one additional term decreasing as $n^{-1/2}$, which guarantees remainder accuracy estimates of order n^{-1}.

An important step in this direction is the monograph by V.V. Senatov [14], which first appeared in 2009 and was subsequently reprinted. It provides a fairly comprehensive presentation of methods for constructing asymptotic expansions in the CLT with explicit accuracy estimates, most of which were discovered by the author for the first time. For

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instance, Sections 8 and 9 present the Edgeworth-Cram?r expansions, while Sections 11–13 discuss various Gram-Charlier expansions. These expansions are constructed using the method of accompanying signed measures, which imposes certain moment conditions on the underlying distribution. Section 14 constructs a Gram-Charlier expansion without using accompanying signed measures.

One of the most recent works providing asymptotic expansions of arbitrary length with an explicit remainder estimate is the paper by V.V. Senatov and V.N. Sobolev [16] on new types of asymptotic expansions in the CLT. As V.V. Senatov notes, "apparently, the form of the principal part of the asymptotic expansion specified in [16] is one of the simplest among those known." Improving the remainder estimate in expansions of this type is one of the main results of the present paper. Let $X_1, X_2, \dots$ be independent and identically distributed (i.i.d.) random variables with zero mean and unit variance. Let P denote the distribution of X_1, and let P_n denote the distribution of the normalized sum

$$(X_1 + \cdots + X_n)\, n^{-\frac{1}{2}},$$

while $\Phi(x)$ denotes the standard normal distribution function with density $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

Furthermore, we assume that the random variable X_1 possesses an absolute moment of order $m+2$, where $m \in \mathbb{N}$. Consequently, for any $k \leqslant m+2$, there exist the quantities

$$\alpha_k = \frac{\mathbb{E}X_1^k}{k!}, \quad \beta_k = \frac{\mathbb{E}\,|X_1|^k}{k!},$$

which we shall call normalized moments. It follows from the initial assumptions on the distribution P that $\alpha_0 = 1$, $\alpha_1 = 0$, and $\alpha_2 = \frac{1}{2}$. Let $\alpha_k(\varphi)$ and $\beta_k(\varphi)$ denote the corresponding normalized moments of the standard normal distribution Φ. For instance, $\alpha_{2j}(\varphi) = \frac{1}{2^j j!}$ for $j \in \mathbb{N}_0$.

If the distribution P_n possesses a continuous density $p_n(x)$, then, as is well known [14], the local form of the central limit theorem holds:

$$p_n(x) \to \varphi(x)$$

as $n \to +\infty$ for all $x \in \mathbb{R}$. For this case, this paper presents an asymptotic expansion of the density $p_n(x)$. Although the principal part of the expansion is largely similar to that previously obtained in [16, 19], a new, sharper bound is proven for the remainder term. In particular, this bound does not involve V.M. Zolotarev's ideal metric [8]. Furthermore, the algebraic and probabilistic aspects of the proof are clearly delineated here, which we hope makes it simpler and more transparent.

1. Preliminaries and Notation

Let $f(t)$ be the characteristic function of the distribution P. We assume that for some $\nu > 0$, the function $|f(t)|^\nu$ is integrable over the entire real line. Let us define

$$\alpha(T) = \sup\{|f(t)| : t \geqslant T\},$$

$$\Lambda_n(T) = \frac{\sqrt{n}}{\pi} \alpha^{n-\nu}(T) \int_T^{+\infty} |f(t)|^\nu dt, \quad n > \nu. \tag{1.1}$$

As noted in [14, p. 150], the quantities $\Lambda_n(T)$ decrease exponentially fast as $n \to +\infty$.

The convergence of the integral

$$\int_T^{+\infty} |f(t)|^\nu dt < \infty$$

and the validity of the inequality $\alpha(T) < 1$ for any positive $T > 0$ [14, p. 43] imply the existence [14, p. 42] of a continuous density $p_n(x)$ for the distribution P_n whenever $n \geqslant \nu$. This density can be represented via the inverse Fourier transform [14, pp. 42, 147] of the characteristic function f^n:

$$p_n(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx} f^n\left(\frac{t}{\sqrt{n}}\right) dt\,. \tag{1.2}$$

2. Densities of Convolutions of the Normal Distribution

In deriving the expansions for the density $p_n(x)$, we will require the characteristic functions $g(t) = e^{-t^2/2}$ and $g^n\left(\frac{t}{\sqrt{n}}\right) = e^{-t^2/2}$ of the standard normal distribution Φ and its n-fold normalized convolution Φ_n. For any $n \geqslant 1$, the density $q_n(x)$ of the distribution Φ_n can be computed by the formula

$$q_n(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx} g^n\left(\frac{t}{\sqrt{n}}\right) dt\,,$$

which, for $g(t) = e^{-t^2/2}$, takes the form

$$q_n(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\frac{t^2}{2} - itx} dt\,.$$

Along with these characteristic functions, it will be convenient to consider the quantities

$$B_{l,n-k} = \frac{1}{2\pi}\int_{-T\sqrt{n}}^{+T\sqrt{n}} |t|^l \mu^{n-k}\left(\frac{t}{\sqrt{n}}\right) dt\,, \quad B_l = \frac{\beta_l(\varphi)}{\sqrt{2\pi}} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |t|^l e^{-\frac{t^2}{2}} dt\,, \tag{2.1}$$

where the non-negative even function $\mu(t)$ is defined as

$$\mu(t) = \max\{|f(t)|, g(t)\}\,. \tag{2.2}$$

Henceforth, we always assume (see [17, p. 295]) that the parameter $T > 0$ is chosen such that

$$B_{l,n-k} \to B_l$$

as $n \to \infty$ for any fixed l and k.

The use of local expansions for characteristic functions leads to the appearance of the quantities

$$L_l(u) = \frac{1}{2\pi}\int_{|t|\geqslant u} |t|^l e^{-\frac{t^2}{2}} dt \tag{2.3}$$

in the remainder term of the asymptotic expansions in the CLT considered below. These quantities are obviously related to B_l. As $u \to +\infty$, the functions $L_l(u)$ decrease exponentially fast.

3. Chebyshev-Hermite Moments and Related Quantities

The Chebyshev-Hermite polynomials $H_k(x)$ and the moments θ_k of the underlying distribution P are defined by the formulas

$$H_k(x) = (-1)^k \varphi^{(k)}(x)/\varphi(x)\,, \quad \theta_k = \frac{1}{k!}\int_{-\infty}^{\infty} H_k(x)\, d\mathrm{P}(x)\,, \quad k \geqslant 0\,.$$

The following relations hold for them:

$$\frac{H_k(x)}{k!} = \sum_{j=0}^{\left[\frac{k}{2}\right]} (-1)^j \alpha_{2j}(\varphi)\frac{x^{k-2j}}{(k-2j)!}\,,$$

$$\theta_k = \sum_{j=0}^{\left[\frac{k}{2}\right]} (-1)^j \alpha_{k-2j}\alpha_{2j}(\varphi)\,. \tag{3.1}$$

Since $\alpha_0 = 1$, $\alpha_1 = 0$, and $\alpha_2 = \frac{1}{2}$, the first few Chebyshev-Hermite moments can be easily found:

$$\theta_1 = 0\,, \quad \theta_2 = 0\,, \quad \theta_3 = \alpha_3\,, \quad \theta_4 = \alpha_4 - \frac{1}{8}\,,$$

while the equality $\theta_0 = 1$ holds for any distribution P.

As can be seen from (3.1), if moments α_s of order higher than $m+2$ do not exist for the underlying distribution P, then the Chebyshev-Hermite moments of order $m+3$ and above cannot be fully constructed. In this connection, we define the incomplete Chebyshev-Hermite moments of order l as

$$\theta_k^{(l)} = \sum_{j=0;\,2j\geqslant k-l}^{\left[\frac{m+3}{2}\right]} (-1)^j \alpha_{k-2j}\alpha_{2j}(\varphi)\,. \tag{3.2}$$

Note that from (3.1) and (3.2), the equality $\theta_k^{(l)} = \theta_k$ follows whenever $l \geqslant k$.

Using the products of the Chebyshev-Hermite moments

$$\Theta_{k_1,k_2,\ldots,k_l} = \theta_{3+k_1}\theta_{3+k_2}\ldots\theta_{3+k_l}\,,$$

we define the polynomials

$$S_l(\omega) = \omega^{3l}\sum_{k_1,k_2,\ldots,k_l}^{m} \Theta_{k_1,k_2,\ldots,k_l}\,\omega^{k_1+k_2+\cdots+k_l}\,, \quad l \geqslant 1. \tag{3.3}$$

Here, for $l \geqslant 2$, the summation $\sum_{k_1,k_2,\ldots,k_l}^{m}$ is understood as

$$\sum_{0\leqslant k_1+k_2+\cdots+k_l\leqslant m-(1+l)} = \sum_{k_1=0}^{m-(1+l)}\;\sum_{k_2=0}^{m-(1+l+k_1)}\;\sum_{k_3=0}^{m-(1+l+k_1+k_2)}\cdots\sum_{k_l=0}^{m-(1+l+k_1+k_2+\cdots+k_{l-1})}\,, \tag{3.4}$$

and for $l = 1$ as

$$S_1(\omega) = \omega^3\sum_{k_1=0}^{m-2}\theta_{3+k_1}\,\omega^{k_1}\,. \tag{3.5}$$

We also assume that $S_0(\omega) = 1$. In addition to the polynomials $S_1(\omega)$ of degree m, we will require similar polynomials of lower degrees:

$$A_s(\omega) = \begin{cases} \sum_{k=3}^{s+1}\theta_k\omega^k & , \quad 2 \leqslant s \leqslant m+1\,, \\ 0 & , \quad s < 2\,. \end{cases} \tag{3.6}$$

These polynomials $A_s(\omega)$, constructed from the underlying distribution P (which possesses zero mean and unit variance), represent the initial part of the Chebyshev-Hermite moment expansion of the characteristic function f of this distribution P:

$$f = e^{-\frac{t^2}{2}} \left(1 + \sum_{k=3}^{s} \theta_k \omega^k + \rho_s\right) = g(t)\,(1 + A_s(\omega) + \rho_s)\,, \quad \omega = it\,. \tag{3.7}$$

An estimate for the functions ρ_s is obtained below in Lemma 6.1. In what follows, we will mostly assume that $\omega = it$, where $t \in \mathbb{R}$ and i is the imaginary unit. This is due to the fact, as noted in [18, p. 181], that "working with characteristic functions, it will sometimes be convenient for us to deal with the variable it."

The convenience of applying Chebyshev-Hermite moments is related to the following observation: the identities (see, e.g., [1, pp. 37, 38], [14, pp. 137, 149])

$$H_k(x)\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (it)^k \varphi(t) e^{-itx} dt\,, \quad x \in \mathbb{R}, \quad k \geqslant 0\,, \tag{3.8}$$

allow one to easily write down the inverse Fourier transform of the expansions of characteristic functions in terms of Chebyshev-Hermite moments. For instance, from (3.7) we obtain the following representation of the density of the underlying distribution P as a sum:

$$p(x) = \varphi(x) + \sum_{k=3}^{m} \theta_k H_k(x)\varphi(x) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \rho_s(it) e^{-itx - \frac{t^2}{2}}\, dt\,.$$

In [11], V.V. Senatov and A.E. Kondratenko used a signed measure with the characteristic function $g(t)\,(1 + A_s(\omega))$ to construct asymptotic expansions with explicit remainder estimates. Signed measures of this type were also used by Yu.V. Prokhorov in [12] and I.A. Ibrahimov in [9]. However, it was only in the works of V.V. Senatov (for details, see [14]) that these signed measures were first applied to obtain concrete values of the remainder estimates for expansions of arbitrary length. In constructing asymptotic expansions, the use of signed measures leads to well-known difficulties because the normalized convolutions of signed measures converge to the normal distribution only under certain conditions (for example, the inequality $|g(t)\,(1 + A_s(\omega))\,| < 1$ must hold for $t \neq 0$). This, in turn, imposes restrictions on the moments of the underlying distribution. In the present work, signed measures are not used, which allows the moments of the underlying distribution to be considered arbitrary.

4. On the Mutual Summation Operation

Henceforth, to shorten the notation, we will use the operation $*$, which we call *mutual summation*. Let (see, e.g., (3.4)) $A = \sum_{k_1,k_2,\dots,k_l}^{m} A_{k_1,k_2,\dots,k_l}$ and $B = B_{k_1,k_2,\dots,k_l}$. Then we define:

$$A * B \stackrel{\text{def}}{=} \sum_{k_1,k_2,\dots,k_l}^{m} A_{k_1,k_2,\dots,k_l} B_{k_1,k_2,\dots,k_l}\,.$$

For example,

$$S_l(\omega) * A_{m-(k_1+k_2+\dots+k_l)-l}(\omega) =$$

$$= \omega^{3l} \sum_{k_1,k_2,\dots,k_l}^{m} \Theta_{k_1,k_2,\dots,k_l} \omega^{k_1+k_2+\dots+k_l} A_{m-(k_1+k_2+\dots+k_l)-l}(\omega)\,. \tag{4.1}$$

We immediately note that the latter formula can be simplified as follows:

$$S_l(\omega) * A_{m-(k_1+k_2+\cdots+k_l)-l}(\omega) =$$
$$= \left(\omega^{3l} \sum_{0\leqslant k_1+k_2+\cdots+k_l\leqslant m-(1+l)} \Theta_{k_1,k_2,\ldots,k_l}\,\omega^{k_1+k_2+\cdots+k_l}\right) *$$
$$* \left(\sum_{0\leqslant k_{l+1}\leqslant m-(k_1+k_2+\cdots+k_l)-l-2} \theta_{3+k_{l+1}}\,\omega^{3+k_{l+1}}\right) =$$
$$= \omega^{3(l+1)} \sum_{0\leqslant k_1+k_2+\cdots+k_l+k_{l+1}\leqslant m-(1+l+1)} \Theta_{k_1,k_2,\ldots,k_l,k_{l+1}}\,\omega^{k_1+k_2+\cdots+k_l+k_{l+1}} = S_{l+1}(\omega)\,.$$

Whence,

$$S_{l+1}(\omega) = S_l(\omega) * A_{m-(k_1+k_2+\cdots+k_l)-l}(\omega)\,. \tag{4.2}$$

5. Auxiliary Lemmas

5.1. Two Identities for Binomial Coefficients. In constructing the main expansion, we will use the following two lemmas, where n, l, m, and k denote non-negative integers.

Lemma 5.1. *For binomial coefficients, the following identity holds for any $s \geqslant l$:*

$$\binom{s+1}{l+1} = \sum_{j=0}^{s-l} \binom{s-j}{l}\,.$$

Proof. Using the hockey-stick identity (summation over the lower index) for binomial coefficients, $\binom{n+1}{l+1} = \sum_{k=l}^{n} \binom{k}{l}$, we find

$$\sum_{j=0}^{s-l} \binom{s-j}{l} = \{k = s-j\} = \sum_{k=l}^{s} \binom{k}{l} = \binom{s+1}{l+1}\,.$$

□

Lemma 5.2. *For any function $z(j)$, the following identity holds for $s \geqslant l+1$:*

$$\sum_{j_0=1}^{s-l} \sum_{j_1=0}^{j_0-1} \binom{s-j_0}{l} z(j_1) = \sum_{j_1=0}^{s-(l+1)} \binom{s-j_1}{l+1} z(j_1)\,.$$

Proof. The assertion of the lemma follows from the obvious chain of equalities:

$$\sum_{j_0=1}^{s-l} \sum_{j_1=0}^{j_0-1} \binom{s-j_0}{l} z(j_1) = \sum_{j_0=1}^{s-l} \sum_{\substack{j_1=0 \\ j_1\leqslant j_0-1}}^{s-l-1} \binom{s-j_0}{l} z(j_1) =$$
$$= \sum_{j_1=0}^{s-l-1} \sum_{\substack{j_0=1 \\ j_1+1\leqslant j_0}}^{s-l} \binom{s-j_0}{l} z(j_1) = \sum_{j_1=0}^{s-l-1} \sum_{j_0=j_1+1}^{s-l} \binom{s-j_0}{l} z(j_1) =$$
$$= \{k = s-j_0\} = \sum_{j_1=0}^{s-l-1} z(j_1) \sum_{k=l}^{s-j_1-1} \binom{k}{l} = \sum_{j_1=0}^{s-l-1} \binom{s-j_1}{l+1} z(j_1)\,.$$

The validity of the last equality follows from Lemma 5.1.

6. Expansions of Characteristic Functions

To estimate the remainder ρ_s of the expansion of the characteristic function f of the underlying distribution P, we will require the following quantities:

$$\|\theta_s\| = \beta_{s+2} + \sum_{j=1}^{\left[\frac{s}{2}\right]-2} \alpha_{2j}(\varphi)|\alpha_{s-2j}| + \left| \sum_{j=\left[\frac{s}{2}\right]-1}^{\left[\frac{s}{2}\right]} (-1)^j \alpha_{2j}(\varphi)\alpha_{s-2j} \right| .$$

The notation $\|\theta_s\|$ was introduced by V.V. Senatov in [5, p. 135]. In our paper, the definitions of these quantities are slightly modified, since the values $\alpha_0 = 1$, $\alpha_1 = 0$, $\alpha_2 = \frac{1}{2}$, and $\alpha_{2l}(\varphi) = \frac{1}{2^l l!}$ are known, and their combinations can be computed immediately. For instance, for an even $s = 2l$:

$$\left| \sum_{j=\left[\frac{s}{2}\right]-1}^{\left[\frac{s}{2}\right]} (-1)^j \alpha_{2j}(\varphi)\alpha_{s-2j} \right| = \left| \sum_{j=l-1}^{l} (-1)^j \alpha_{2j}(\varphi)\alpha_{2l-2j} \right| =$$

$$= \left| (-1)^l \alpha_{2l}(\varphi)\alpha_0 - (-1)^{l-1}\alpha_{2l-2}(\varphi)\alpha_2 \right| = |\alpha_{2l}(\varphi)\alpha_0 - \alpha_{2l-2}(\varphi)\alpha_2| =$$

$$= \left| \frac{1}{2^l l!} - \frac{1}{2^l(l-1)!} \right| = \frac{1}{2^l(l-1)!}\left(1 - \frac{1}{l}\right) = \frac{1}{2^l(l-2)!\,l} ,$$

and for an odd $s = 2l+1$:

$$\left| \sum_{j=\left[\frac{s}{2}\right]-1}^{\left[\frac{s}{2}\right]} (-1)^j \alpha_{2j}(\varphi)\alpha_{s-2j} \right| = \left| \sum_{j=l-1}^{l} (-1)^j \alpha_{2j}(\varphi)\alpha_{2l+1-2j} \right| = \alpha_{2l-2}(\varphi)|\alpha_3| .$$

In a similar manner, we define the quantities $\|\theta_s^{(l)}\|$: they are defined by the same formulas as $\|\theta_s\|$, in which all α_k for $k > l$ should be set to zero.

Lemma 6.1. *Assume that the distribution* P *possesses an absolute moment of order* $m+2$*. Then, for its characteristic function* f*, the following representation holds for any* $2 \leqslant s \leqslant m$*:*

$$f = g(1 + A_s + \rho_s) .$$

Here, g *is the characteristic function of the standard normal distribution,* A_s *are the polynomials defined by identity* (3.6)*, and the following estimate holds for the functions* ρ_s*:*

$$|\rho_s| \leqslant \|\theta_{s+2}\| |t|^{s+2} g^{-1} + \|\theta_{s+3}^{(s+1)}\| |t|^{s+3} g^{-1} .$$

Proof. Let $t \in \mathbb{R}$ and $\omega = it$. The expansions for the characteristic function

$$f(t) = \sum_{k=0}^{s+1} \alpha_k \omega^k + r_s(t) , \quad |r_s(t)| \leqslant \beta_{s+2} t^{s+2}$$

and for the exponential function

$$e^{\frac{t^2}{2}} = e^{-\omega^2/2} = \sum_{j=0}^{\infty} (-1)^j \alpha_{2j}(\varphi)\omega^{2j}$$

are well known. We transform their product

$$f(t)e^{\frac{t^2}{2}} = \sum_{j=0}^{\infty} (-1)^j \alpha_{2j}(\varphi)\omega^{2j} \sum_{k=0}^{s+1} \alpha_k \omega^k + \rho_s(t)e^{\frac{t^2}{2}}$$

into the form

$$f(t)e^{\frac{t^2}{2}}=\sum_{k=0}^{s+1}\theta_k\omega^k+\sum_{k=s+2}^{\infty}\theta_k^{(s+1)}\omega^k+\rho_s(t)e^{\frac{t^2}{2}}.$$

Let us split the last sum into two parts:

$$\sum_{k=0}^{\infty}\theta_{s+2+k}^{(s+1)}\omega^{s+2+k}=\omega^{s+2}\sum_{k=0}^{\infty}\theta_{s+2+2k}^{(s+1)}\omega^{2k}+\omega^{s+3}\sum_{k=0}^{\infty}\theta_{s+3+2k}^{(s+1)}\omega^{2k}. \tag{6.1}$$

The incomplete moments from the right-hand side of (6.1) can be transformed as follows:

$$\theta_{s+2+2k}^{(s+1)}=\sum_{\substack{j=0\\2j\geqslant 2k+2}}^{\left[\frac{s+2k+2}{2}\right]}(-1)^j\alpha_{s+2k+2-2j}\alpha_{2j}(\varphi)=$$

$$=\sum_{j=k+1}^{[s/2]+k+1}(-1)^j\alpha_{s-2(j-k-1)}\alpha_{2j}(\varphi)=\sum_{j=1}^{\left[\frac{s+2}{2}\right]}(-1)^{k+j}\alpha_{s+2-2j}\alpha_{2(k+j)}(\varphi).$$

Since

$$\alpha_{2(k+j)}(\varphi)\leqslant\alpha_{2k}(\varphi)\alpha_{2j}(\varphi),$$

it follows that

$$\left|\theta_{s+2+2k}^{(s+1)}\right|\leqslant\sum_{j=1}^{\left[\frac{s-2}{2}\right]}\alpha_{2k+2j}(\varphi)\left|\alpha_{s-2j}\right|+\left|\sum_{j=\left[\frac{s}{2}\right]}^{\left[\frac{s+2}{2}\right]}(-1)^j\alpha_{2k+2j}(\varphi)\alpha_{s-2j}\right|\leqslant$$

$$\leqslant\alpha_{2k}(\varphi)\left(\sum_{j=1}^{\left[\frac{s-2}{2}\right]}\alpha_{2j}(\varphi)\left|\alpha_{s-2j}\right|+\left|\sum_{j=\left[\frac{s}{2}\right]}^{\left[\frac{s+2}{2}\right]}(-1)^j\alpha_{2j}(\varphi)\alpha_{s-2j}\right|\right),$$

$$\left|\theta_{s+2+2k}^{(s+1)}\right|\leqslant\alpha_{2k}(\varphi)\|\theta_{s+2}^{(s+1)}\|.$$

Next, taking into account the identity $\|\theta_{s+2}^{(s)}\|=\|\theta_{s+2}^{(s+1)}\|$, we obtain an estimate for the first term in (6.1):

$$\left|\omega^{s+2}\sum_{k=0}^{\infty}\theta_{s+2+2k}^{(s+1)}\omega^{2k}\right|\leqslant\|\theta_{s+2}^{(s)}\||t|^{s+2}\sum_{k=0}^{\infty}\alpha_{2k}(\varphi)|t|^{2k}=\|\theta_{s+2}^{(s)}\||t|^{s+2}e^{t^2/2}.$$

By virtue of the identity $\|\theta_{s+2}\|=\beta_{s+2}+\|\theta_{s+2}^{(s)}\|$, we combine this estimate with the one obtained above for r_s. This yields the first term in the bound for ρ_s.

Similarly, we find the representation for the moments:

$$\theta_{s+2k+3}^{(s+1)}=\sum_{\substack{j=0\\2j\geqslant 2k+2}}^{\left[\frac{s+2k+3}{2}\right]}(-1)^j\alpha_{s+2k+3-2j}\alpha_{2j}(\varphi)=$$

$$=\sum_{j=k+1}^{\left[\frac{s-1}{2}\right]+k+2}(-1)^j\alpha_{s-1+2k+4-2j}\alpha_{2j}(\varphi)=\sum_{j=1}^{\left[\frac{s+3}{2}\right]}(-1)^{k+j}\alpha_{s+3-2j}\alpha_{2(k+j)}(\varphi).$$

Whence,

$$\left|\theta_{s+2k+3}^{(s+1)}\right| \leqslant \alpha_{2k}(\varphi)\left(\sum_{j=2}^{\left[\frac{s-1}{2}\right]} \alpha_{2j}(\varphi)\left|\alpha_{s-2j}\right| + \left|\sum_{j=\left[\frac{s+1}{2}\right]}^{\left[\frac{s+3}{2}\right]} (-1)^j \alpha_{2j}(\varphi)\alpha_{s-2j}\right|\right) = \alpha_{2k}(\varphi)\|\theta_{s+3}^{(s+1)}\|$$

and

$$\left|\omega^{s+3}\sum_{k=0}^{\infty}\theta_{s+3+2k}^{(s+1)}\omega^{2k}\right| \leqslant \|\theta_{s+3}^{(s+1)}\||t|^{s+3}\sum_{k=0}^{\infty}\alpha_{2k}(\varphi)|t|^{2k} = \|\theta_{s+3}^{(s+1)}\||t|^{s+3}e^{t^2/2}.$$

The last estimate completes the proof. □

A corollary of Lemma 6.1 is the following representation for the difference of the powers of the characteristic functions:

$$f^{j_0} - g^{j_0} = (f-g)\sum_{j_1=0}^{j_0-1} f^{j_1} g^{j_0-j_1-1} = (A_s+\rho_s)\sum_{j_1=0}^{j_0-1} f^{j_1} g^{j_0-j_1}, \tag{6.2}$$

as well as the expression of the power of the function f in terms of its lower-order powers:

$$f^{j_0} = g^{j_0} + (A_s+\rho_s)\sum_{j_1=0}^{j_0-1} f^{j_1} g^{j_0-j_1}. \tag{6.3}$$

7. Recurrent Relations

Representation (6.2) allows us to derive a recurrence formula for the incomplete binomial sums $C_l(f)$, which are defined by the identity

$$C_l(f) = \sum_{j=0}^{n-1-l}\binom{n-1-j}{l} f^j g^{n-j}. \tag{7.1}$$

Lemma 7.1. *Let the inequalities $l < n$, $m < n$, and $2 \leqslant s \leqslant m$ hold. Then the following representation is valid:*

$$C_{l-1}(f) = g^n\binom{n}{l} + (A_s+\rho_s)C_l(f).$$

Proof. In identity (7.1), written as

$$C_{l-1}(f) = \sum_{j_0=0}^{n-l}\binom{n-1-j_0}{l-1} f^{j_0} g^{n-j_0},$$

we replace f^{j_0} using formula (6.3). This yields

$$C_{l-1}(f) = \sum_{j_0=1}^{n-l}\binom{n-1-j_0}{l-1}\left[g^{j_0} + (A_s+\rho_s)\sum_{j_1=0}^{j_0-1} f^{j_1} g^{j_0-j_1}\right] g^{n-j_0}.$$

Next, applying Lemmas 5.1 and 5.2, we find

$$C_{l-1}(f) = g^n\sum_{j_0=0}^{n-l}\binom{n-1-j_0}{l-1} + (A_s+\rho_s)\sum_{j_0=1}^{n-l}\binom{n-1-j_0}{l-1}\sum_{j_1=0}^{j_0-1} f^{j_1} g^{n-j_1} =$$

$$= g^n\binom{n}{l} + (A_s+\rho_s)\sum_{j_0=1}^{n-l}\sum_{j_1=0}^{j_0-1}\binom{n-1-j_0}{l-1} f^{j_1} g^{n-j_1} =$$

$$= g^n \binom{n}{l} + (A_s + \rho_s) \sum_{j_1=0}^{n-1-l} \binom{n-1-j_1}{l} f^{j_1} g^{n-j_1} = g^n \binom{n}{l} + (A_s + \rho_s) C_l(f).$$

This completes the proof. □

In turn, the representation for $C_l(f)$ enables us to find a recurrence relation for the quantities $S_l(\omega, f)$ specified by the formula

$$S_l(\omega, f) = S_l(\omega) C_{l-1}(f). \tag{7.2}$$

Lemma 7.2. *For $l < n$, the following identity holds:*

$$S_l(\omega, f) = S_{l+1}(\omega, f) + g^n \binom{n}{l} S_l(\omega) + R_l(\omega, f),$$

where

$$R_l(\omega, f) = S_l(\omega) * \rho_{m-(l+k_1+k_2+\cdots+k_l)} C_l(f). \tag{7.3}$$

Proof. Using Lemma 7.1 and representation (3.3), we obtain

$$S_l(\omega, f) = S_l(\omega) C_{l-1}(f) = S_l(\omega) \left(g^n \binom{n}{l} + (A_s + \rho_s) C_l(f) \right). \tag{7.4}$$

The last identity holds for any $s \leqslant m$.

We shall substitute s into (7.4) depending on l in the form

$$s = s(l) = m - (l + k_1 + k_2 + \cdots + k_l). \tag{7.5}$$

The reason for this choice will become clear later. For now, we only note that it guarantees the maximum possible order of the remainder estimate for the desired expansion with the minimum number of terms in the principal part.

By virtue of (7.5), identity (7.4) takes the form

$$S_l(\omega, f) = S_l(\omega) * \left(g^n \binom{n}{l} + A_{m-(l+k_1+k_2+\cdots+k_l)} C_l(f) + \rho_{m-(l+l+k_1+k_2+\cdots+k_l)} C_l(f) \right) =$$

$$= g^n \binom{n}{l} S_l(\omega) + S_l(\omega) * A_{m-(l+k_1+k_2+\cdots+k_l)} C_l(f) + S_l(\omega) * \rho_{m-(l+k_1+k_2+\cdots+k_l)} C_l(f) =$$

$$= g^n \binom{n}{l} S_l(\omega) + S_l(\omega) * A_{m-(l+k_1+k_2+\cdots+k_l)} C_l(f) + R_l(\omega, f).$$

To complete the proof, it remains to use identity (4.2) and definition (7.2). □

In conclusion, we provide a quite obvious estimate for the quantities $C_l(f)$ (taking into account Lemma 5.1):

$$|C_l(f)| = \left| \sum_{j=0}^{n-1-l} \binom{n-1-j}{l} f^j g^{n-j} \right| \leqslant \mu^n \sum_{j=0}^{n-1-l} \binom{n-1-j}{l} = \binom{n}{l+1} \mu^n, \tag{7.6}$$

where the quantities μ are defined by formula (2.2).

8. Representation of Powers of Characteristic Functions

The lemmas proven above allow us to express the powers f^n of the characteristic function f of the underlying distribution P in terms of the polynomials $S_l(\omega)$ defined by formula (3.3).

Theorem 8.1. *For $m < n$, the following relation holds:*

$$f^n = g^n \sum_{l=0}^{m-1} \binom{n}{l} S_l(\omega) + \sum_{l=0}^{m-2} R_l(\omega, f)\,, \tag{8.1}$$

where $R_l(\omega, f)$ are defined by identity (7.3)*.*

Proof. It follows from (3.5) and (3.6) that $S_1(\omega) = A_m(\omega)$. Together with identity (6.2), this allows us to write the following elementary transformations for the difference of powers $f^n - g^n$:

$$f^n - g^n = (f-g)\sum_{j_0=0}^{n-1} f^{j_0} g^{n-j_0-1} = (A_m + \rho_m)\sum_{j_0=0}^{n-1} f^{j_0} g^{n-j_0} =$$
$$= C_0(f)\,(A_m + \rho_m) = C_0(f)\,(S_1(\omega) + \rho_m)\,.$$

Next, setting $R_0(\omega, f) = C_0(f)\rho_m$ and applying Lemma 7.2, we find

$$f^n - g^n = S_1(\omega)C_0(f) + R_0(\omega, f) = S_1(\omega, f) + R_0(\omega, f) =$$
$$= S_2(\omega, f) + g^n \binom{n}{1} S_1(\omega) + R_1(\omega, f) + R_0(\omega, f) =$$
$$= S_3(\omega, f) + g^n \binom{n}{2} S_2(\omega) + R_2(\omega, f) + g^n \binom{n}{1} S_1(\omega) + R_1(\omega, f) + R_0(\omega, f)\,.$$

Applying Lemma 7.2 repeatedly in a similar manner, we arrive at the desired representation. $\square$

In what follows, we will require an upper bound for the sum $\sum_{l=0}^{m-2} R_l(\omega, f)$, which will subsequently be used to estimate the remainder of the sought-after asymptotic expansion.

Lemma 8.2. *For $m < n$, the following estimate holds:*

$$\left|\sum_{l=0}^{m-2} R_l(\omega, f)\right| \leqslant \mu^{n-1} \sum_{l=0}^{m-2} \binom{n-1}{l+1} \left(S_{l,2}(\Theta)|\omega|^{m+2+2l} + S_{l,3}(\Theta)|\omega|^{m+3+2l}\right),$$

where the quantities $S_{l,2}(\Theta)$ and $S_{l,3}(\Theta)$ are defined by formulas (8.2)*,* (8.3)*, and* (8.4)*.*

Proof. Using inequality (7.6), we find

$$\left|\sum_{l=1}^{m-2} R_l(\omega, f)\right| \leqslant \sum_{l=1}^{m-2} |R_l(\omega, f)| \leqslant \sum_{l=1}^{m-2} |C_l(f)| \left|S_l(\omega) * \rho_{m-(l+k_1+k_2+\cdots+k_l)}\right| \leqslant$$
$$\leqslant \mu^n \sum_{l=1}^{m-2} \binom{n}{l+1} \left|S_l(\omega) * \rho_{m-(l+k_1+k_2+\cdots+k_l)}\right|\,.$$

Applying the inequality from Lemma 6.1 for ρ_s with $s \leqslant m$, we obtain the following chain of inequalities:

$$\left|\sum_{l=1}^{m-2} R_l(\omega, f)\right| \mu^{-n} g \leqslant$$

$$\leqslant \sum_{l=1}^{m-2} \binom{n}{l+1} \left| S_l(\omega) * \left(\|\theta_{m+2-(l+k_1+k_2+\cdots+k_l)}\| |\omega|^{m+2-(l+k_1+k_2+\cdots+k_l)} \right) \right| +$$

$$+ \sum_{l=1}^{m-2} \binom{n}{l+1} \left| S_l(\omega) * \left(\|\theta^{(m+1-l-k_1-k_2-\cdots-k_l)}_{m+3-l-k_1-k_2-\cdots-k_l}\| |\omega|^{m+3-(l+k_1+k_2+\cdots+k_l)} \right) \right| \leqslant$$

$$\leqslant \sum_{l=1}^{m-2} \binom{n}{l+1} |\omega|^{3l} \sum_{k_1,k_2,\ldots,k_l}^{m} |\Theta_{k_1,k_2,\ldots,k_l}| |\omega|^{k_1+k_2+\cdots+k_l} \cdot$$

$$\cdot \|\theta_{m+2-(l+k_1+k_2+\cdots+k_l)}\| |\omega|^{m+2-(l+k_1+k_2+\cdots+k_l)} +$$

$$+ \sum_{l=0}^{m-2} \binom{n}{l+1} |\omega|^{3l} \sum_{k_1,k_2,\ldots,k_l}^{m} |\Theta_{k_1,k_2,\ldots,k_l}| |\omega|^{k_1+k_2+\cdots+k_l} \cdot$$

$$\cdot \|\theta^{(m+1-l-k_1-k_2-\cdots-k_l)}_{m+3-l-k_1-k_2-\cdots-k_l}\| |\omega|^{m+3-(l+k_1+k_2+\cdots+k_l)} \leqslant$$

$$\leqslant \sum_{l=1}^{m-2} \binom{n}{l+1} S_{l,2}(\Theta) |\omega|^{m+2+2l} + \sum_{l=1}^{m-2} \binom{n}{l+1} S_{l,3}(\Theta) |\omega|^{m+3+2l},$$

where for $l \geqslant 1$:

$$S_{l,2}(\Theta) = \sum_{k_1,k_2,\ldots,k_l}^{m} |\Theta_{k_1,k_2,\ldots,k_l}| \|\theta_{m+2-(l+k_1+k_2+\cdots+k_l)}\|, \tag{8.2}$$

and

$$S_{l,3}(\Theta) = \sum_{k_1,k_2,\ldots,k_l}^{m} |\Theta_{k_1,k_2,\ldots,k_l}| \|\theta^{(m+2-l-k_1-k_2-\cdots-k_l)}_{m+3-l-k_1-k_2-\cdots-k_l}\|. \tag{8.3}$$

Recall (see (3.4)) that the summation in the last two formulas is taken over the sets of non-negative integers $k_1, k_2, \ldots, k_l$ satisfying the inequality $0 \leqslant k_1 + k_2 + \cdots + k_l \leqslant m - l - 1$. Let us define

$$S_{0,2}(\Theta) = \|\theta_{m+2}\|, \qquad S_{0,3}(\Theta) = \|\theta^{(m+1)}_{m+3}\|. \tag{8.4}$$

Now, to complete the proof, it suffices to obtain the estimate for $l = 0$:

$$|R_0(\omega, f)| = |C_0(f)\rho_m| \leqslant \binom{n}{1} \mu^n |\rho_m| \leqslant \binom{n}{1} \mu^{n-1} \left(S_{0,2}(\Theta)|\omega|^{m+2} + S_{0,3}(\Theta)|\omega|^{m+3} \right).$$

□

Lemma 8.3. *For the integrated quantities $R_l\left(\frac{it}{\sqrt{n}}, f\right)$, the following estimate holds:*

$$\left| \frac{1}{2\pi} \int_{-T\sqrt{n}}^{T\sqrt{n}} e^{-itx} \sum_{l=0}^{m-2} R_l\left(\frac{it}{\sqrt{n}}, f\right) dt \right| \leqslant$$

$$\leqslant \frac{1}{(\sqrt{n})^{m+2}} \sum_{l=0}^{m-2} \frac{\binom{n}{l+1}}{n^l} \left(B_{m+2+2l,n-1} S_{l,2}(\Theta) + B_{m+3+2l,n-1} S_{l,3}(\Theta) \frac{1}{\sqrt{n}} \right),$$

where the quantities $S_{l,2}(\Theta)$ and $S_{l,3}(\Theta)$ are defined by formulas (8.2), (8.3), *and* (8.4), *and the values $B_{s,n-1}$ are given by formula* (2.1).

Proof. Using the estimates from Lemma 8.2, we find

$$\left|\frac{1}{2\pi}\int_{-T\sqrt{n}}^{T\sqrt{n}} e^{-itx}\sum_{l=0}^{m-2} R_l\left(\frac{it}{\sqrt{n}},f\right)dt\right| \leqslant \frac{1}{2\pi}\int_{-T\sqrt{n}}^{T\sqrt{n}}\sum_{l=0}^{m-2}\left|R_l\left(\frac{it}{\sqrt{n}},f\right)\right|dt \leqslant$$

$$\leqslant \frac{1}{2\pi}\int_{-T\sqrt{n}}^{T\sqrt{n}}\mu^{n-1}\left(\frac{t}{\sqrt{n}}\right)\sum_{l=0}^{m-2}\binom{n}{l+1}\left(S_{l,2}(\Theta)\left|\frac{t}{\sqrt{n}}\right|^{m+2+2l}+S_{l,3}(\Theta)\left|\frac{t}{\sqrt{n}}\right|^{m+3+2l}\right)dt =$$

$$= \sum_{l=0}^{m-2}\binom{n}{l+1}S_{l,2}(\Theta)\frac{1}{2\pi}\int_{-T\sqrt{n}}^{T\sqrt{n}}\mu^{n-1}\left(\frac{t}{\sqrt{n}}\right)\left|\frac{t}{\sqrt{n}}\right|^{m+2+2l}dt+$$

$$+\sum_{l=0}^{m-2}\binom{n}{l+1}S_{l,3}(\Theta)\frac{1}{2\pi}\int_{-T\sqrt{n}}^{T\sqrt{n}}\mu^{n-1}\left(\frac{t}{\sqrt{n}}\right)\left|\frac{t}{\sqrt{n}}\right|^{m+3+2l}dt =$$

$$= \frac{1}{(\sqrt{n})^{m+2}}\sum_{l=0}^{m-2}\binom{n}{l+1}\left(B_{m+2+2l,n-1}S_{l,2}(\Theta)+B+B_{m+3+2l,n-1}S_{l,3}(\Theta)\frac{1}{\sqrt{n}}\right)\frac{1}{n^l}\,.$$

This completes the proof of the lemma. □

9. Construction of Asymptotic Expansions

The lemmas proven above allow us to construct, in a fairly straightforward manner, the following asymptotic expansion in the central limit theorem with an explicit remainder estimate.

Theorem 9.1. *Let* P *be the distribution of independent and identically distributed (i.i.d.) random variables* X_i*,* $i=1,\dots,n$*, with zero mean and unit variance, and assume that there exists a finite absolute moment of order* $m+2$*. Suppose there exists a number* $\nu>0$ *such that the characteristic function* $f(t)$ *of the distribution* P *satisfies the condition* $\int_{-\infty}^{\infty}|f(t)|^{\nu}dt<\infty$*. Then, for any* $n\geqslant\max(\nu,m+1)$ *and for all real* $x\in\mathbb{R}$*:*

$$p_n(x)=\varphi(x)\sum_{l=0}^{m-1}\frac{\binom{n}{l}}{(\sqrt{n})^{3l}}\sum_{k_1,k_2,\dots,k_l}^{m}\frac{\Theta_{k_1,k_2,\dots,k_l}}{(\sqrt{n})^{k_1+k_2+\dots+k_l}}H_{3l+k_1+\dots+k_l}(x)+R_{n,m}(x)\,,$$

where

$$|R_{n,m}(x)|\leqslant\frac{1}{(\sqrt{n})^{m+2}}\sum_{l=0}^{m-2}\frac{\binom{n}{l+1}}{n^l}\left(B_{m+2+2l,n-1}S_{l,2}(\Theta)+B_{m+3+2l,n-1}S_{l,3}(\Theta)\frac{1}{\sqrt{n}}\right)+$$

$$+\Lambda_n(T)+\sum_{l=0}^{m-1}\binom{n}{l}\sum_{k_1,k_2,\dots,k_l}^{m}\frac{|\Theta_{k_1,k_2,\dots,k_l}|}{(\sqrt{n})^{3l+k_1+k_2+\dots+k_l}}L_{3l+k_1+k_2+\dots+k_l}\left(T\sqrt{n}\right)\,,$$

and the last two terms on the right-hand side of the remainder estimate, defined by formulas (1.1) *and* (2.3)*, decrease exponentially fast.*

Proof. We split integral (1.2) into two parts:

$$p_n(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-itx}f^n\left(\frac{t}{\sqrt{n}}\right)dt=$$

$$=\frac{1}{2\pi}\int_{-T\sqrt{n}}^{T\sqrt{n}}e^{-itx}f^n\left(\frac{t}{\sqrt{n}}\right)dt+\frac{1}{2\pi}\int_{|t|\geqslant T\sqrt{n}}e^{-itx}f^n\left(\frac{t}{\sqrt{n}}\right)dt\,.$$

We substitute f^n in the first integral on the right-hand side of the last identity using formula (8.1) from Theorem 1:

$$p_n(x) = \frac{1}{2\pi}\int_{-T\sqrt{n}}^{T\sqrt{n}} e^{-itx} g^n\left(\frac{t}{\sqrt{n}}\right)\sum_{l=0}^{m-1}\binom{n}{l} S_l\left(\frac{it}{\sqrt{n}}\right) dt +$$

$$+\frac{1}{2\pi}\int_{-T\sqrt{n}}^{T\sqrt{n}} e^{-itx}\sum_{l=0}^{m-2} R_l\left(\frac{it}{\sqrt{n}}, f\right) dt + \frac{1}{2\pi}\int_{|t|\geqslant T\sqrt{n}} e^{-itx} f^n\left(\frac{t}{\sqrt{n}}\right) dt\,,$$

recalling that $g^n\left(\frac{t}{\sqrt{n}}\right) = e^{-t^2/2}$. Next, we add and subtract terms in the first component on the right-hand side of the last representation so that the integration is performed over the entire real line:

$$p_n(x) = \sum_{l=0}^{m-1}\binom{n}{l}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx-\frac{t^2}{2}} S_l\left(\frac{it}{\sqrt{n}}\right) dt\right) -$$

$$-\sum_{l=0}^{m-1}\binom{n}{l}\left(\frac{1}{2\pi}\int_{|t|\geqslant T\sqrt{n}} e^{-itx-\frac{t^2}{2}} S_l\left(\frac{it}{\sqrt{n}}\right) dt\right) +$$

$$+\frac{1}{2\pi}\int_{-T\sqrt{n}}^{T\sqrt{n}} e^{-itx}\sum_{l=0}^{m-2} R_l\left(\frac{it}{\sqrt{n}}, f\right) dt + \frac{1}{2\pi}\int_{|t|\geqslant T\sqrt{n}} e^{-itx} f^n\left(\frac{t}{\sqrt{n}}\right) dt\,.$$

We have thus obtained a representation of the density $p_n(x)$ as a sum of four terms. An estimate for the penultimate term has already been found in Lemma 8.3. The last term decreases exponentially fast, which follows from the bound:

$$\left|\frac{1}{2\pi}\int_{|t|\geqslant T\sqrt{n}} e^{-itx} f^n\left(\frac{t}{\sqrt{n}}\right) dt\right| \leqslant \frac{1}{\pi}\int_{T\sqrt{n}}^{+\infty}\left|f\left(\frac{t}{\sqrt{n}}\right)\right|^n dt \leqslant \Lambda_n(T)\,.$$

The estimate for the second term (taking into account notation (2.3) and transformation (3.8)):

$$\frac{1}{2\pi}\left|\sum_{l=0}^{m-1}\binom{n}{l}\int_{|t|\geqslant T\sqrt{n}} e^{-itx-\frac{t^2}{2}} S_l\left(\frac{it}{\sqrt{n}}\right) dt\right| \leqslant \frac{1}{2\pi}\sum_{l=0}^{m-1}\binom{n}{l}\int_{|t|\geqslant T\sqrt{n}} e^{-\frac{t^2}{2}}\left|S_l\left(\frac{it}{\sqrt{n}}\right)\right| dt \leqslant$$

$$\leqslant \sum_{l=0}^{m-1}\binom{n}{l}\sum_{k_1,k_2,\dots,k_l}^{m}|\Theta_{k_1,k_2,\dots,k_l}|\frac{1}{2\pi}\int_{|t|\geqslant T\sqrt{n}} e^{-\frac{t^2}{2}}\left|\frac{t}{\sqrt{n}}\right|^{3l+k_1+k_2+\dots+k_l} dt =$$

$$= \sum_{l=0}^{m-1}\binom{n}{l}\sum_{k_1,k_2,\dots,k_l}^{m}\frac{|\Theta_{k_1,k_2,\dots,k_l}|}{(\sqrt{n})^{3l+k_1+k_2+\dots+k_l}} L_{3l+k_1+k_2+\dots+k_l}\left(T\sqrt{n}\right)$$

also decreases exponentially fast.

The first sum (after applying transformation (3.8)) yields the principal part of the desired asymptotic expansion:

$$\sum_{l=0}^{m-1}\binom{n}{l}\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx-\frac{t^2}{2}} S_l\left(\frac{it}{\sqrt{n}}\right) dt =$$

$$= \sum_{l=0}^{m-1}\binom{n}{l}\sum_{k_1,k_2,\dots,k_l}^{m}\Theta_{k_1,k_2,\dots,k_l}\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx-\frac{t^2}{2}}\left(\frac{it}{\sqrt{n}}\right)^{3l+k_1+k_2+\dots+k_l} dt =$$

$$= \varphi(x) \sum_{l=0}^{m-1} \binom{n}{l} \sum_{k_1,k_2,\dots,k_l}^{m} \frac{\Theta_{k_1,k_2,\dots,k_l}}{(\sqrt{n})^{3l+k_1+k_2+\cdots+k_l}} H_{3l+k_1+k_2+\cdots+k_l}(x).$$

Remark 9.2. Note that for large n, asymptotically $\binom{n}{l+1} \sim \frac{n^{l+1}}{(l+1)!}$. Therefore, using the notation $\lesssim$ from [18, p. 190] (which means that the left-hand side of the inequality does not exceed a quantity equivalent to the right-hand side), the remainder estimate can finally be written in a simpler form:

$$|R_{n,m}(x)| \lesssim \frac{1}{(\sqrt{n})^m} \sum_{l=0}^{m-2} \frac{1}{(l+1)!} \left(B_{m+2+2l,n} S_{l,2}(\Theta) + B_{m+3+2l,n} S_{l,3}(\Theta) \frac{1}{\sqrt{n}} \right).$$

10. On the Structure of the Asymptotic Expansion

Let us study the rate of decrease with respect to n of the terms in the principal part of the expansion in Theorem 9.1. To do this, we consider the groups of terms in the principal part associated with the same value of l in the expression for $\binom{n}{l}$.

The first group ($l = 0$) contains only a single term, $\varphi(x)$.

The terms of the next level ($l = 1$) take the form

$$\varphi(x) \frac{\binom{n}{1}}{(\sqrt{n})^3} \sum_{k_1=0}^{m-2} \frac{\Theta_{k_1}}{(\sqrt{n})^{k_1}} H_{3+k_1}(x) = \frac{\varphi(x)}{\sqrt{n}} \sum_{k_1=0}^{m-2} \frac{\Theta_{k_1}}{(\sqrt{n})^{k_1}} H_{3+k_1}(x).$$

Here, the orders of the terms with respect to n vary from $1/\sqrt{n}$ to $1/(\sqrt{n})^{m-1}$, and the indices of the Chebyshev-Hermite polynomials start from three.

In the next sum ($l = 2$), only polynomials of order six and higher are present:

$$\varphi(x) \frac{\binom{n}{2}}{(\sqrt{n})^6} \sum_{k_1=0}^{m-3} \sum_{k_2=0}^{m-(3+k_1)} \frac{\Theta_{k_1,k_2}}{(\sqrt{n})^{k_1+k_2}} H_{6+k_1+k_2}(x) \sim \frac{\varphi(x)}{2n} \sum_{k_1=0}^{m-3} \sum_{k_2=0}^{m-(3+k_1)} \frac{\Theta_{k_1,k_2}}{(\sqrt{n})^{k_1+k_2}} H_{6+k_1+k_2}(x).$$

We represent the terms appearing here in the form of a table.

k_1	Terms	Variation of order with respect to n
0	$\varphi(x) \frac{\binom{n}{2}}{(\sqrt{n})^6} \sum_{k_2=0}^{m-3} \frac{\Theta_{0,k_2}}{(\sqrt{n})^{k_2}} H_{6+k_2}(x)$	$1/(\sqrt{n})^2$ to $1/(\sqrt{n})^{m-1}$
1	$\varphi(x) \frac{\binom{n}{2}}{(\sqrt{n})^6} \sum_{k_2=0}^{m-4} \frac{\Theta_{1,k_2}}{(\sqrt{n})^{1+k_2}} H_{7+k_2}(x)$	$1/(\sqrt{n})^3$ to $1/(\sqrt{n})^{m-1}$
$\dots$	$\dots$	$\dots$
$m-2$	$\varphi(x) \frac{\binom{n}{2}}{(\sqrt{n})^6} \sum_{k_2=0}^{1} \frac{\Theta_{m-2,k_2}}{(\sqrt{n})^{m-2+k_2}} H_{m+4+k_2}(x)$	$1/(\sqrt{n})^{m-2}$ to $1/(\sqrt{n})^{m-1}$
$m-3$	$\varphi(x) \frac{\binom{n}{2}}{(\sqrt{n})^6} \frac{\Theta_{m-3,0}}{(\sqrt{n})^{m-3}} H_{m+3}(x)$	$1/(\sqrt{n})^{m-1}$

It can be seen that the sum of level $\binom{n}{2}$ contains terms whose rate of decrease with respect to n varies from $1/(\sqrt{n})^2$ to $1/(\sqrt{n})^{m-1}$.

The terms containing $\binom{n}{3}$ are written as follows:

$$\varphi(x) \frac{\binom{n}{3}}{(\sqrt{n})^9} \sum_{k_1=0}^{m-4} \sum_{k_2=0}^{m-(4+k_1)} \sum_{k_3=0}^{m-(4+k_1+k_2)} \frac{\Theta_{k_1,k_2,k_3}}{(\sqrt{n})^{k_1+k_2+k_3}} H_{9+k_1+k_2+k_3}(x).$$

Here, the asymptotics of the terms vary from $1/(\sqrt{n})^3$ to $1/(\sqrt{n})^{m-1}$. For $\binom{n}{m}$, there is only one term of order $1/(\sqrt{n})^{m-1}$.

Obviously, terms having the same rate of decrease with respect to n can be easily grouped together.

11. Numerical Example

The estimate of the remainder in the asymptotic expansion presented in Theorem 9.1, although appearing quite cumbersome, can be easily calculated. We illustrate the derivation of the numerical remainder estimate using the example of an exponential distribution P with zero mean and unit variance:

$$p(x) = \begin{cases} e^{-x-1} & , \quad x \geqslant -1\,, \\ 0 & , \quad x < -1\,. \end{cases}$$

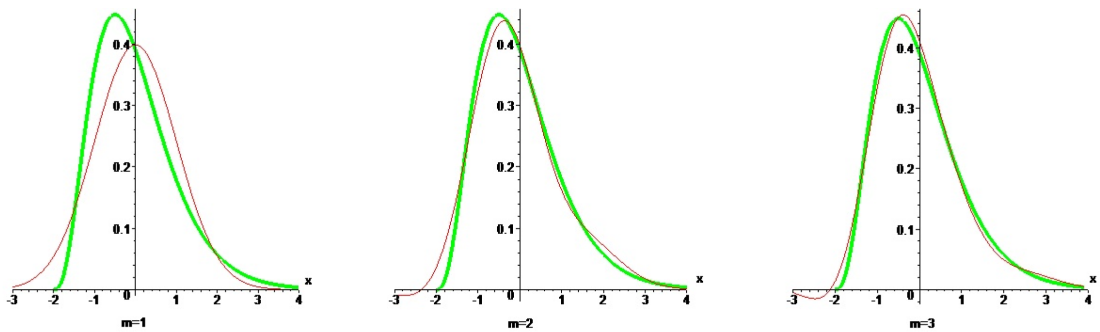

FIGURE 1. Plots of $p_4(x)$ (bold line) and the principal part of the approximation from Theorem 9.1 (thin line) for various values of m.

The choice of this distribution as a model is motivated by the following:

(1) it is obviously not the best distribution for approximation;
(2) the explicit formula for the density $p_n(x)$ of the sum distribution P_n is known, and the computations based on it are quite simple.

Thus, for $m = 3$, the estimate of the remainder term (in which the quantities $B_{k,n}$ are replaced by their bounding limiting values B_k) from Theorem 9.1 for the model distribution does not exceed

$$\frac{2.7}{n^{3/2}} + \frac{3.2}{n^2}\,, \tag{11.1}$$

which is significantly better than the result obtained in [7], where the estimate with the same rate of decrease takes the form

$$\frac{31.7}{n^{3/2}} + \frac{16.3}{n^2}\,. \tag{11.2}$$

After comparing these remainder estimates, it is clear that this trend will also persist for $m > 3$.

In conclusion, we present the plots of the density $p_4(x)$ and the principal part of the approximation from Theorem 9.1 for various values of m at $n = 4$ (see Fig. 1). It can be seen that the actual error is significantly less than $1/10$, and by no means of order 1, as might be concluded on the basis of (11.1). Therefore, the question of the actual accuracy of asymptotic expansions [15, 18] remains open.

12. Conclusion

In this paper, we have developed a novel approach to constructing asymptotic expansions in the central limit theorem without relying on accompanying signed measures or Zolotarev's ideal metric. The established explicit remainder estimates offer a more direct and mathematically transparent framework for evaluating approximation exactness. As demonstrated by the numerical example, the proposed bounds significantly reduce theoretical error constants, yielding highly practical results for non-optimal distributions.

References

[1] Cramér H.: On the composition of elementary errors. *Skandinavisk Aktuarietidskrift*. 1928. Vol. 11. P. 13–74, 141–180.
[2] Uspensky J.V.: *Introduction to Mathematical Probability*. New York: McGraw-Hill Book Company, 1937.
[3] Shimizu R.: On the remainder term for the central limit theorem. *Annals of the Institute of Statistical Mathematics*. 1974. Vol. 26. No. 2. P. 195–201.
[4] Dobric V., Ghosh B.K.: Some analogs of Berry-Esseen bound for first order Chebychev-Edgeworth expansions. *Statistics & Decisions*. 1996. Vol. 14. No. 4. P. 383–404.
[5] Senatov V.V.: On the application of Chebyshev-Hermite moments in asymptotic expansions. *Theory of Probability and Its Applications*. 2001. Vol. 46. No. 1. P. 190–192.
[6] Sobolev V.N.: On asymptotic expansions in the central limit theorem. *Theory of Probability and Its Applications*. 2007. Vol. 52. No. 3. P. 490–505.
[7] Senatov V.V., Sobolev V.N.: On new forms of asymptotic expansions in the central limit theorem. *Theory of Probability and Its Applications*. 2012. Vol. 57. No. 1. P. 124–140.
[8] Zolotarev V.M.: *Modern Theory of Summing Independent Random Variables*. Moscow: Nauka, 1986.
[9] Ibragimov I.A.: On the Chebyshev-Cramér asymptotic expansions. *Theory of Probability and Its Applications*. 1967. Vol. 12. No. 3. P. 596–619.
[10] Kondratenko A.E.: On the connection between the rate of convergence of moments of normalized sums and the Chebyshev-Hermite moments. *Theory of Probability and Its Applications*. 2001. Vol. 46. No. 2. P. 383–386.
[11] Kondratenko A.E., Senatov V.V.: On the error estimate of asymptotic expansions in the CLT. *Doklady Mathematics*. 2001. Vol. 378. No. 6. P. 748–750.
[12] Prokhorov Yu.V.: Some refinements of Lyapunov's theorem. *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya*. 1952. Vol. 16. No. 3. P. 281–292.
[13] Prokhorov Yu.V., Rozanov Yu.A.: *Probability Theory. Basic Concepts, Limit Theorems, Stochastic Processes*. Moscow: Nauka, 1987.
[14] Senatov V.V.: *Central Limit Theorem: Accuracy of Approximation and Asymptotic Expansions*. Moscow: LIBROCOM Publishing House, 2009.
[15] Senatov V.V.: On the real accuracy of approximations in the central limit theorem. *Siberian Mathematical Journal*. 2011. Vol. 52. No. 4. P. 913–935.
[16] Senatov V.V., Sobolev V.N.: On new forms of asymptotic expansions in the central limit theorem. *Theory of Probability and Its Applications*. 2012. Vol. 57. No. 1. P. 124–140.
[17] Senatov V.V.: On the construction of high-accuracy asymptotic expansions. *Theory of Probability and Its Applications*. 2015. Vol. 60. No. 2. P. 290–310.
[18] Senatov V.V.: On the real accuracy of approximations in the central limit theorem. II. *Mathematical Proceedings*. 2016. Vol. 19. No. 2. P. 170–199.
[19] Sobolev V.N.: On asymptotic expansions in the CLT. *Bulletin of Tver State University. Series: Applied Mathematics*. 2010. No. 18. P. 35–48.
[20] Sobolev V.N.: On an expansion of the characteristic function. *Obrenie Prikladnoi i Promyshlennoi Matematiki*. 2014. Vol. 21. No. 5. P. 533–534.

[21] Shevtsova I.G.: *Optimization of the Structure of Moment Estimates for the Accuracy of Normal Approximation for Distributions of Sums of Independent Random Variables*. Doctoral Dissertation in Physics and Mathematics. Moscow: Lomonosov Moscow State University, 2013.

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