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ON MARTINGALE SOLUTIONS OF FRACTIONAL STOCHASTIC EVOLUTION EQUATIONS

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Abstract. We study the Cauchy problem for a fractional stochastic evolution equation driven by a cylindrical Wiener process, in which the time derivative is understood in the Caputo sense of order $q \in (0,1)$ and the noise enters through a Riemann–Liouville fractional integral. After collecting the required notions on progressively measurable processes, martingales and Hilbert–Schmidt operators, we establish regularity properties of the fractional Itô integral and give sufficient conditions for its existence, together with a Burkholder-type inequality. The solution operator families generated by the Mittag-Leffler function are used to formulate the notion of a mild solution and then of a martingale solution. The main result asserts the existence of a martingale solution of the problem under compactness and measurability assumptions on the coefficients.

1. Introduction

Over recent decades fractional calculus has confirmed its effectiveness in handling nonlocal interactions and effects, including the modelling of aftereffect and the accounting of hereditary characteristics of dynamical systems [1]. These advances have broadened the field of application of mathematical modelling and have allowed a more accurate representation of complex phenomena influenced by long-term dependencies and historical factors. In physics they provide insight into the chaotic behaviour of turbulent flows [2, 3] and into the thermal properties of materials with memory [4, 6]. The financial sector benefits from their application in modelling market dynamics and in the valuation of financial derivative instruments, where long-term correlations and market volatility present significant problems [7, 8].

In biology, epidemiology, oncology and other areas of medicine, fractional models are used to understand the spread of diseases and the movement of biological populations, taking into account both the spatial distribution and random environmental influences [9, 10].

In recent years fractional stochastic partial differential equations have come to the forefront of mathematical innovation, owing to their sophisticated approach

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to describing the dynamics of processes that are often excessively simplified by traditional mathematical models [11]. They give the mathematical community a powerful toolkit for the study of modelling and for solving problems that involve not only randomness and extensive interactions, but also memory effects. This is, above all, fractional calculus, which extends the concept of derivatives and integrals with respect to time and spatial coordinates to non-integer orders, offering a more accurate and comprehensive representation of phenomena characterised by long-range dependencies spanning huge time periods and spatial distances, and by anomalous diffusion, which describes the propagation of particles in a manner deviating from classical Brownian motion [13, 14].

Solving such equations is by its nature a computationally laborious process, since the solution lives in an infinite-dimensional function space over sets of random variables. This complexity increases when fractional operators are included, which introduce memory effects into the model. Symmetry-based models are widespread in various applications, such as combustion models [15], CAR data analysis [16], autocatalytic networks [17] and stochastic models for the Navier–Stokes equations [18]. We note that studies of symmetries and martingales in models for the Navier–Stokes equations have revealed a certain invariance [19].

The present work concerns the proof of existence and uniqueness of a global martingale solution of a fractional stochastic equation of the form

$$\begin{cases} \partial_{0t}^q x(t) + Ax(t) = I_{0t}^{1-q}\big[\sigma(x)\dot{W}(t)\big], & t \in [0,T] \\ x(0) = x_0 \end{cases} \tag{1.1}$$

where ∂_{0t}^q is the Caputo fractional derivative with $q \in (0,1)$, $A = -\Delta$ is the Laplace operator with zero Dirichlet boundary condition, I_{0t}^{1-q} is the fractional integral operator defined below, $\dot{W}(t) = dW(t)/dt$ is white noise modelling the noise effects, and the function σ is globally Lipschitz continuous.

For any $q > 0$ we define the function $g_q(t)$ by

$$g_q(t) = \begin{cases} \dfrac{1}{\Gamma(q)}\, t^{q-1}, & t > 0, \\ 0, & t \le 0, \end{cases} \tag{1.2}$$

where $\Gamma(\cdot)$ is the Euler gamma function of the second kind.

The Riemann–Liouville fractional integral I_t^q is defined by

$$I_t^q X(t) = (g_q * X)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} X(s)\, ds, \qquad t > 0, \tag{1.3}$$

with $I_t^0 X(t) = X(t)$. For $q \in (0,1)$ and $t > 0$, the expression

$$\partial_{0,t}^q X(t) = \frac{d}{dt}\Big[I_t^{1-q}\big(X(t) - X(0)\big)\Big] = \frac{d}{dt}\left(\int_0^t \frac{(t-s)^{-q}}{\Gamma(1-q)}\big(X(s) - X(0)\big) ds\right), \tag{1.4}$$

is called the Caputo fractional derivative of the function X (see [20]).

We define the Laplace transform of the function X with respect to t by

$$\hat{X}(\lambda) = \mathcal{L}\{X(t)\} = \int_0^\infty e^{-\lambda t} X(t)\, dt,$$

then the Laplace transform of the Caputo derivative $\partial^q_{0,t}$ and of the Riemann–Liouville fractional integral operator I^q_t are given by

$$\begin{cases} \mathcal{L}\big[\partial^q_{0,t} X(t)\big] = \lambda^q \hat{X}(\lambda) - \lambda^{q-1} X(0), \\ \mathcal{L}\big[I^q_t X(t)\big] = \lambda^{-q} \hat{X}(\lambda). \end{cases} \tag{1.5}$$

2. Notation, definitions and preliminary results

Here we set out the concepts connected with stochastic processes in general, and with Wiener and martingale processes in particular.

2.1. Stochastic processes and martingales. Throughout, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assume that H is a separable Hilbert space with norm $|\cdot|_H$. The content of the present subsection is based on [20, 21].

Definition 2.1. An H-valued time-continuous stochastic process $\{\xi_t\}_{t\in T}$ is a family of H-valued random variables with time index t. Moreover, either $T = [0, T]$ or $T = [0, \infty)$.

For each $\omega \in \Omega$, the mapping

$$\xi(\omega): \ \mathbb{T} \ni t \to \xi_t(\omega) \in H$$

is called a trajectory of the process ξ. For simplicity we use the notation ξ instead of $\{\xi_t\}_{t\in T}$.

Definition 2.2. An H-valued stochastic process ξ on T is called a modification (or version) of a process ζ if

$$\mathbb{P}\big(\{\omega \in \Omega: \ \xi(t,\omega) \neq \zeta(t,\omega)\}\big) = 0 \qquad \text{for each } t \in T.$$

Definition 2.3. An H-valued stochastic process ξ is called measurable if the mapping $\xi : [0, T] \times \Omega \to H$ is $\mathcal{B}([0, T]) \otimes \mathcal{F}$-measurable.

Definition 2.4. A process ξ is called stochastically continuous at $t_0 \in [0, T]$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $t \in [t_0 - \rho, t_0 + \rho] \cap [0, T]$

$$\mathbb{P}\big(\{\omega \in \Omega: \ |\xi(t,\omega) - \xi(t_0,\omega)|_H \geq \varepsilon\}\big) \leq \delta.$$

If a process ξ is stochastically continuous for each $t \in [0, T]$, then it is called stochastically continuous on $[0, T]$.

Lemma 2.5 ([20], Proposition 3.2)**.** *If a process ξ is stochastically continuous on $[0, T]$, then it has a measurable modification on $[0, T]$.*

Definition 2.6. A family $\{\mathcal{F}_t\}_{t\geq 0}$ of σ-fields such that for all t, $\mathcal{F}_t \subset \mathcal{F}$, is called a filtration if for any $0 \leq s < t < \infty$, $\mathcal{F}_s \subset \mathcal{F}_t$.

Henceforth we assume that $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is a filtration.

Definition 2.7. A process ξ is called adapted to $\mathbb{F}$ if for each $t \in [0, T]$, ξ_t is $\mathcal{F}_t$-measurable.

Definition 2.8. A process ξ is called progressively measurable if for each $t \in [0, T]$ the mapping

$$\xi : [0, t] \times \Omega \to H, \qquad \xi(s, \omega) = \xi_s(\omega) \in H$$

is $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$-measurable.

Definition 2.9. A subset $P \subseteq [0,\infty)\times\Omega$ is called progressively measurable if the process $\xi_t(\omega) = \mathbf{1}_P(t,\omega)$ is progressively measurable. The σ-field generated by all such P from $[0,\infty)\times\Omega$ is called the progressively measurable σ-field.

Remark 2.10. If a process ξ is progressively measurable, then it is adapted to $\mathbb{F}$.

Lemma 2.11. *The limit of progressively measurable processes is progressively measurable.*

Lemma 2.12 ([20], Proposition 3.5)**.** *If a process ξ is stochastically continuous on $[0,T]$ and adapted to $\mathbb{F}$, then it has a progressively measurable modification.*

Definition 2.13. A random variable $\tau : \Omega \to (0,\infty)$, i.e. a random time, is called a stopping time (relative to the filtration $\mathbb{F}$) if for all $t \in T$

$$\{\tau \le t\} = \{\omega \in \Omega : \ \tau(\omega) \le t\} \in \mathcal{F}_t.$$

Remark 2.14.

(i) It is easy to see that if τ and σ are two stopping times, then $\tau\wedge\sigma$, $\tau\vee\sigma$ and $\tau+\sigma$ are also stopping times.

(ii) Every stopping time τ is $\mathcal{F}$-measurable, where

$$\mathcal{F}_\tau = \{B \in \mathcal{F} : \ B \cap \{\tau \le t\} \in \mathcal{F}_t \text{ for all } t \in T\}$$

is a σ-field.

(iii) A random variable ζ is called $\mathcal{F}$-measurable if for all $t \in T$, $\zeta\,\mathbf{1}\{\tau \le t\}$ is $\mathcal{F}$-measurable.

Lemma 2.15 ([22], Proposition 1.1.3)**.** *Let ζ be a progressively measurable process, and τ a stopping time. Then $\zeta_\tau\,\mathbf{1}\{\tau < \infty\}$ is $\mathcal{F}_\tau$-measurable, and the stopped process is also progressively measurable.*

Definition 2.16. An M-valued process ξ is called an $\mathbb{F}$-martingale if

(i) ξ is adapted to $\mathbb{F}$,

(ii) for each $t \in [0,T]$, $\mathbb{E}\big(|\xi(t)|_H\big) < \infty$,

(iii) for each $t,s \in [0,T]$ with $t \ge s$, $\mathbb{E}\big(\xi(t) \mid \mathcal{F}_s\big) = \xi(s)$, i.e. for any $A \in \mathcal{F}_s$

$$\int_A \xi(t)\,d\mathbb{P} = \int_A \xi(s)\,d\mathbb{P}.$$

Lemma 2.17 ([20], Proposition 3.9)**.** *Let M^2_T be the space of all H-valued, continuous and square-integrable martingales ξ on $[0,T]$. Then M^2_T is a Banach space with respect to the norm*

$$|\xi|M^2_T = \left(\mathbb{E}\sup_{t\in[0,T]}|\xi(t)|^2_H\right)^{1/2}, \qquad \xi \in M^2_T. \tag{2.1}$$

Theorem 2.18 (Burkholder–Davis–Gundy inequality; [22], Theorem 1.16)**.** *Let $1 \le p < \infty$. Then for all H-valued continuous martingales M_τ with $M_0 = 0$ and every stopping time τ there exist positive constants c_p and C_p such that*

$$c_p\mathbb{E}\Big[\langle M\rangle_\tau^{p/2}\Big] \le \mathbb{E}\left(\sup_{0\le t\le\tau}|M_t|^p\right) \le C_p\mathbb{E}\Big[\langle M\rangle_\tau^{p/2}\Big],$$

where $\langle M\rangle$ denotes the quadratic variation of M.

Further we shall need the following generalisation of Gronwall's lemma ([23], Lemma 3.9).

Lemma 2.19 (Generalised Gronwall lemma)**.** *Let X, Y, I and φ be nonnegative processes and Z a nonnegative integrable random variable. Assume that I is a nondecreasing process and that there exist nonnegative constants $C, \alpha, \beta, \gamma, \eta$ with the properties*

$$\int_0^T \varphi(s)\,ds \le C \quad \textit{almost surely,} \qquad \gamma\beta e^C \le \alpha, \tag{2.2}$$

and such that for $0 \le t \le T$,

$$X(t) + \alpha Y(t) \le Z + \int_0^t \varphi(z)X(z)\,dz + I(t) \qquad \textit{a.s.}, \tag{2.3}$$

$$\mathbb{E}\big(I(t)\big) \le \beta\,\mathbb{E}\big(X(0)\big) + \gamma\int_0^t \mathbb{E}\big(X(s)\big)ds + \eta\,\mathbb{E}\big(Y(t)\big) + \tilde{C}, \tag{2.4}$$

where $\tilde{C} \ge 0$ is a constant. If $X \in \mathcal{L}^\infty([0,T]\times\Omega)$, then

$$\mathbb{E}\big[X(t) + \alpha Y(t)\big] \le 2\exp\big(C + 2t\gamma e^C\big)\big(\mathbb{E}(Z) + \tilde{C}\big), \qquad t \in [0,T]. \tag{2.5}$$

2.2. Hilbert–Schmidt operators. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\ge0})$ be a probability space with normal filtration $\{\mathcal{F}_t\}_{t\ge0}$. We recall that Q is a positive bounded linear operator on some Hilbert space $\mathcal{U}$ with finite operator trace. Let $W = \{W(t),\, t \ge 0\}$ be a $\mathcal{U}$-valued Wiener process with covariance operator Q. We introduce the subspace $\mathcal{U}_0 = Q^{1/2}(\mathcal{U})$, equipped with the inner product

$$(u,v)_{\mathcal{U}_0} = \big(Q^{-1/2}u, Q^{-1/2}v\big), \qquad u, v \in \mathcal{U}_0$$

and induced by the norm $\|\cdot\|_{\mathcal{U}_0}$, where by $Q^{-1/2}$ we denote the quasi-inverse of $Q^{1/2}$. We also denote by $\mathcal{L}_2^0 = \mathcal{L}_2^0(\mathcal{U}_0, H)$ the space of Hilbert–Schmidt operators $T : \mathcal{U}_0 \to H$, equipped with the norm

$$\|\varphi\|^2_{\mathcal{L}_2^0} = \mathrm{Tr}\Big[(\varphi Q^{1/2})(\varphi Q^{1/2})^*\Big] < \infty,$$

for any $\varphi \in \mathcal{L}_2^0$. Let $\{v(t)\}_{t\in[0,T]}$ be an $\mathcal{L}_2^0$-valued predictable stochastic process. For stochastic integrals the Itô isometry is satisfied, i.e. the equality

$$\mathbb{E}\left\|\int_0^t v(s)\,dW(s)\right\|^2 = \int_0^t \mathbb{E}\|v(s)\|^2_{\mathcal{L}_2^0}\,ds, \qquad t \in [0,T]. \tag{2.6}$$

2.3. Function spaces of fractional order. Next we introduce into consideration fractional orders of spaces and norms. We define the linear operator $A = -\Delta$ with zero boundary condition on $D(A)$. We denote by $\{\varphi_k\}_{k\ge1}$ the complete orthonormal system of eigenfunctions in H for the operator A, i.e. for $k = 1, 2, \ldots$,

$$A\varphi_k = \lambda_k\varphi_k,$$

with $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots$.

For any $r > 0$, we denote by $\dot{H}^r$ the domain of definition of the fractional-order operator $A^{r/2}$, which can be defined in the form

$$A^{r/2}X = \sum_{k=1}^{\infty} \lambda_k^{r/2}(x, \varphi_k)\varphi_k,$$

$$\dot{H}^r = \mathcal{D}\big(A^{r/2}\big) = \left\{ x \in H : \ \|A^{r/2}x\|^2 = \sum_{k=1}^{\infty} \lambda_k^r (x, \varphi_k)^2 < \infty \right\}$$

with induced norm

$$\|x\|_r^2 = \|A^{r/2}x\|^2 = \sum_{k=1}^{\infty} \lambda_k^r (x, \varphi_k)^2.$$

Applying the Laplace transform to both sides of (1.1) and using (1.5) we obtain

$$z^q \hat{X}(z) - z^{q-1}X(0) + A\hat{X}(z) = z^{q-1}\hat{g}(z),$$

where $\hat{g}(z)$ is the Laplace transform of the term $\sigma(X)\dot{W}$, and thus

$$\hat{X}(z) = \frac{z^{q-1}}{z^q + A}\big[X(0) + \hat{g}(z)\big]. \tag{2.7}$$

Now we define the Mittag-Leffler function [21]

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}, \tag{2.8}$$

and using the Laplace transform relative to (2.8) we find

$$\mathcal{L}\big[E_{q,1}(-\lambda t^q)\big] = \int_0^{\infty} e^{-zt} E_{q,1}(-\lambda t^q)\, dt = \frac{z^{q-1}}{z^q + \lambda}. \tag{2.9}$$

Taking into account (2.9) and Duhamel's principle, from the representation (2.7) there follows the analogue of the classical variation-of-constants formula

$$X(t) = \mathcal{E}(t)X_0 + \int_0^t \mathcal{E}(t-s)\sigma\big(X(s)\big)\, dW(s), \tag{2.10}$$

in which $\mathcal{E}(t) = E_{q,1}(-At^q)$, so that $X(t) = \mathcal{E}(t)X_0$ is a weak solution of (1.1) for $\sigma(u) = 0$.

Throughout the remainder of the work we shall assume that the following assumptions are satisfied.

Assumption 2.20. The measurable function $\sigma : \Omega \times H \to L_2^0$ satisfies the growth and Lipschitz conditions

$$\|\sigma(u)\|_{L_2^0} \le C\|u\|, \qquad \|\sigma(u) - \sigma(v)\|_{L_2^0} \le C\|u - v\|$$

for all $u, v \in H$.

Assumption 2.21. Assume that the initial value $X_0 : \Omega \to \dot{H}^\rho$ is an $\mathcal{F}_0$-measurable random variable for which the condition $\mathbb{E}\|X_0\|_\rho < \infty$ is satisfied for any $\rho \ge 0$.

3. Regularity properties of the fractional Itô integral

Here and in what follows we shall assume that the following condition holds.

(H) H is a separable Hilbert space and $W(t)$, $t \geq 0$, is an H-cylindrical Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$ relative to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

Remark 3.1. $W(t)$, $t \geq 0$, is an H-cylindrical Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$ relative to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if and only if it is a family of linear bounded operators acting from X into $L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that:

(i) for all $t \geq 0$ and $h_1, h_2 \in H$, $\mathbb{E}\big[W(t)h_1\, W(t)h_2\big] = t\langle h_1, h_2\rangle_H$;
(ii) for each $h \in H$, $W(t)h$, $t \geq 0$, is a real-valued Wiener process, adapted to the family $(\mathcal{F}_t)$.

Assume that E is a real separable Banach space such that H is densely and continuously embedded in E. We denote by $j : H \hookrightarrow E$ the natural embedding. If $j : H \hookrightarrow E$ is an abstract Wiener space and $w(t)$, $t \geq 0$, is a canonical E-valued Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then one can define a family of linear operators from E^* into $L^2(\Omega, \mathcal{F}, \mathbb{P})$:

$$W(t)h = \langle h, w(t)\rangle, \qquad h \in E^*,\ t \geq 0. \tag{3.1}$$

Since $W(t)$ is a continuous linear mapping from E^* with the topology induced by H, $W(t)$ will be uniquely extended to a linear bounded mapping from H into $L^2(\Omega, \mathcal{F}, \mathbb{P})$, so that the above conditions (i) and (ii) will be satisfied.

Conversely, if $W(t)$, $t \geq 0$, is an H-cylindrical Wiener process relative to the filtration $(F_t)_{t\geq 0}$ and $j : H \to E$ is an abstract Wiener space, then there exists an E-valued Wiener process $W(t)$, $t \geq 0$, such that condition (3.1) is satisfied. In fact, it will be sufficient to take a continuous version of the process

$$\tilde{w} = \sum W(t)(e_k)e_k, \qquad t \geq 0 \tag{3.2}$$

for some orthonormal basis $\{e_k\}_k$ of H.

Regarding the motivation for cylindrical Wiener processes and Itô integrals, we make several remarks. First, this will be more canonical and we shall not use many of the original objects. Second, as was noted earlier, any cylindrical Wiener process generates a true Wiener process. Finally, this is connected with the Itô integral, on which we shall dwell in more detail.

Definition 3.2. For a Hilbert space H and a Banach space X we set

$$\mathcal{M}(H, X) = \big\{L : H \to X \ :\ L \text{ linear bounded and } \gamma\text{-radonifying}\big\}.$$

Thus, a bounded linear operator belongs to $\mathcal{M}(H, X)$ if and only if the image $L(\gamma_H)$ of the canonical finitely additive Gaussian function γ_H on H through h is σ-additive on the algebra of cylindrical sets in X. By ν_L we denote the unique extension of $h(\gamma_H)$ to the Borel σ-algebra $\mathcal{B}(X)$.

For $h \in \mathcal{M}(H, X)$ we set

$$\|L\|_{\mathcal{M}(H,X)} = \left\{\int_X |x|^2\, d\nu_h(x)\right\}^{1/2}. \tag{3.3}$$

In [20] it is established that $\mathcal{M}(H,X)$ is a separable Banach space with norm $\|L\|_{\mathcal{M}(H,X)}$.

Let us specify a Banach space S. Usually S is one of the spaces X, $\mathcal{M}(H,X)$ or $\mathcal{L}(E,X)$. By $\mathcal{N}(a,b;S)$ we denote the space of functions $\xi:[a,b]\times\Omega\to S$ which are progressively measurable. For $q_1\in(0,q)$ we set

$$\mathcal{N}^{q_1}(a,b;S)=\left\{\xi\in\mathcal{N}(a,b;S):\ \int_a^b\|\xi(s)\|^{q_1}ds<\infty\text{ a.s.}\right\},\tag{3.4}$$

$$\mathcal{M}^{q_1}(a,b;S)=\left\{\xi\in\mathcal{N}(a,b;S):\ \mathbb{E}\int_a^b\|\xi(s)\|^{q}ds<\infty\right\}.\tag{3.5}$$

Further we denote by $\mathcal{N}_{\text{step}}(a,b;S)$ the space of all $\xi\in\mathcal{N}(a,b;S)$ for which there exists a partition $a=t_0<t_1<\cdots<t_n=b$ such that $\xi(t)=\xi(t_k)$ for $t\in[t_k,t_{k+1})$. We set $\mathcal{M}^{q_1}_{\text{step}}=\mathcal{M}^{q_1}\cap\mathcal{N}_{\text{step}}$.

Note that $\mathcal{M}^{q_1}_{\text{step}}(a,b;S)$ is a closed subspace of $L^{q_1}([a,b]\times\Omega;S)$. Assume that X satisfies assumption (H). Then for any $\xi\in\mathcal{M}\big(0,T;\mathcal{M}(H,X)\big)$ there exists a continuous X-valued process, denoted by $X(t)=\int_0^t\xi(s)\,dW(s)$, $0\le t\le T$, such that the Burkholder-type inequality is satisfied:

$$\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\xi(s)\,dW(s)\right|^r\le\left(\frac{r}{r-1}\right)^r C_r(X)\,\mathbb{E}\left(\int_0^T\|\xi(s)\|^r_{\mathcal{M}(H,X)}\,ds\right)^{r/2}.\tag{3.6}$$

By $(X,D(A))_{1/2,2}$ we denote the interpolation space with parameters $(1/2,2)$ between $D(A)$ and X. We also set $\sigma(x(t))\in\mathcal{M}^2\big(0,T;\mathcal{M}(H,X)\big)$ and $x_0\in L^2(\Omega,\mathcal{F}_0,X)$.

A process $x\in\mathcal{M}^2(0,T;X)$ is called a *mild solution* of the equation

$$\begin{cases}D^\alpha_{0t}x(t)+Ax(t)=I_{0t}\big(\sigma(x(t))\cdot W(t)\big), & t>0\\ x(0)=x_0\end{cases}\tag{3.7}$$

if and only if for all $t\in[0,T]$ a.s. the equation holds

$$x(t)=\mathcal{E}(t)x_0+\int_0^t\mathcal{E}(t-s)\sigma\big(x(s)\big)\,dW(s).\tag{3.8}$$

Now we indicate some sufficient conditions for the existence of the Itô integral in (3.8). The most general such condition will be

$$\mathcal{E}(t-s)\sigma(\cdot)\in\mathcal{M}^2\big(0,t;\mathcal{M}(H,X)\big),\qquad t\in[0,T].\tag{3.9}$$

Another condition is presented in the following theorem.

Theorem 3.3. *Denote $V=(X,D(A))_{1/2,2}$ and assume that $\sigma(x)$ is an operator-valued process such that $A^{-1}\sigma\in\mathcal{M}^2\big(0,T;\mathcal{M}(H,V)\big)$. Then for almost all $t\in[0,T]$, the Itô integral*

$$\vartheta(t)=\int_0^t\mathcal{E}(t-s)\sigma\big(x(s)\big)\,dW(s)$$

exists and the following inequality is satisfied:

$$\mathbb{E}\int_0^T |\vartheta(t)|^2\,dt \le C_2(X)\int_0^T \|A^{-1}\sigma(x(s))\|^2_{\mathcal{M}(H,V)}\,ds. \tag{3.10}$$

Proof. □

4. Martingale solutions (existence)

Consider the following stochastic equation:

$$\begin{cases} D_t^q u(t) + Au(t)\,dt^q = F\big(t,u(t)\big)dt + G\big(t,u(t)\big)dW(t), \\ \qquad\qquad u(0) = u_0. \end{cases} \tag{4.1}$$

Assume that X is a separable space and $-A$ is the generator of an analytic semigroup $S(t)_{t\ge 0}$ on X. We shall also need another Banach space B. Let assumptions (A3)–(A5) be satisfied.

Assumption ((A3))**.** B is a Banach space with norm $\|\cdot\|_B$ and

$$D(A^\delta) \hookrightarrow B \hookrightarrow X \tag{4.2}$$

for some $\delta \in [0, 1/2-\sigma)$. The embedding $D(A^\delta) \hookrightarrow B$ is compact. The semigroup $S(t)$, $t \ge 0$, on B is a strongly continuous semigroup on B.

Assumption ((A4))**.** The mapping $A^{-\sigma}G : [0,\infty)\times B \to \mathcal{M}(H,X)$ is bounded, continuous relative to the second variable and strongly measurable relative to the first variable.

Assumption ((A5))**.** The mapping $F : [0,\infty)\times B \to B$ is strongly measurable relative to the first variable and continuous relative to the second. Moreover, there exist $K \in \mathbb{R}$ and an increasing function $a : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{t\to\infty} a(t) = \infty$ such that for all $x \in D(A)$, $y \in B$ and $t \ge 0$:

$$\big(-Ax + F(t,x+y), z\big) \le a\big(|y|_B\big) - k|x|_B \tag{4.3}$$

for any $z \in \partial|x|_B$.

From assumption (A5) there follows the statement below.

Lemma 4.1. *Let B be a Banach space, $-A$ the generator of a strongly continuous semigroup of linear bounded operators on B, and let the mapping $F : [0,\infty)\times B \to B$ satisfy (A5). Let, for some $T > 0$, two continuous functions $z, \vartheta : [0,\tau] \to B$ satisfy the equation*

$$z(t) = \int_0^t e^{-(t-s)A} F\big(s, z(s) + \vartheta(s)\big)ds, \qquad t \le \tau.$$

Then

$$|z(t)|_B \le \int_0^t e^{-k(t-s)} a\big(|\vartheta(s)|_B\big)\,ds.$$

Proof. □

Definition 4.2. A *weak solution* of problem (4.1) is a B-valued admissible process $u(t)$, $0 \le t < T$, satisfying the equation

$$
\begin{aligned}
u(t) = \mathcal{T}(t)u_0 &+ \int_0^t (t-s)^{q-1}\mathcal{I}(t-s)F\big(s,u(s)\big)ds \\
&+ \int_0^t (t-s)^{q-1}\mathcal{I}(t-s)G\big(s,u(s)\big)dW(s), \qquad t \in [0,T],
\end{aligned}
$$

where

$$
\mathcal{T}(t) = \int_0^\infty \xi_q(\theta)S(t^q\theta)\,d\theta, \qquad \xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\omega_q\big(\theta^{-\frac{1}{q}}\big) \ge 0,
$$

$$
\omega_q(\theta) = \frac{1}{\pi}\sum_{n=1}^\infty \frac{(-1)^{n-1}\theta^{-qn-1}\Gamma(nq+1)}{n!}\sin(n\pi q), \qquad \theta > 0,
$$

ξ_q being the probability density function defined on $(0,\infty)$, so that

$$
\xi_q(\theta) \ge 0 \quad \text{for } \theta > 0, \qquad \int_0^\infty \xi_q(\theta)\,d\theta = 1,
$$

and

$$
\mathcal{I}(t) = q\int_0^\infty \theta\,\xi_q(\theta)S(t^q\theta)\,d\theta.
$$

Lemma 4.3. *For fixed $t > 0$ and any $x \in X$,*

$$
\|\mathcal{T}(t)x\|_X \le M\|x\|_X, \qquad \|\mathcal{I}(t)x\|_X \le \frac{M}{\Gamma(q)}\|x\|_X.
$$

Lemma 4.4. *The mappings $\mathcal{T}, \mathcal{I} : (0,\infty) \to L(X)$ are continuous.*

Assumption ((A6))**.** There exists a function $\bar{m} \in L^{1/q_2}\big([0,T],\mathbb{R}_+\big)$, $q_2 \in (0,q_1)$, and a nondecreasing continuous function $\omega : \mathbb{R}^+ \to \mathbb{R}_+$ such that

$$
\|F(t,u)\|_X \le \bar{m}(t)\,\omega\big(\|u\|_X\big)
$$

for all $u \in X$ and a.e. $t \in [0,T)$.

Assumption ((A7))**.** The inequality

$$
M_r + \frac{M\omega(r)}{\Gamma(q)}\int_0^T (t-s)^{q-1}\bar{m}(s)\,ds \le r
$$

holds for some $r > 0$.

Further we define the operator $F_1 : C([0,T],X) \to C([0,T],X)$ by the equality

$$
\begin{aligned}
(F_1u)(t) = \mathcal{T}(t)u_0 &+ \int_0^t (t-s)^{q-1}\mathcal{I}(t-s)F\big(s,u(s)\big)ds \\
&+ \int_0^t (t-s)^{q-1}\mathcal{I}(t-s)G\big(s,u(s)\big)dW(s), \qquad t \in [0,T].
\end{aligned}
$$

Lemma 4.5. *The operator $F_1 : C([0,T],X) \to C([0,T],X)$ is continuous.*

Definition 4.6. Assume that H is a separable Hilbert space and that all conditions concerning the operator A and the space B are satisfied. Let $F : [0, \infty) \times B \to B$ and let $u_0 \in B$ be fixed. A *martingale solution* of problem (4.1) is the system

$$\big(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0}, \{W(t)\}_{t\geq 0}, \{u(t)\}_{t\geq 0}\big) \tag{4.4}$$

such that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration on it, $\{W(t)\}_{t\geq 0}$ is an H-cylindrical Wiener process and $\{u(t)\}_{t\geq 0}$ is a B-valued admissible process such that for any $t \in [0, T]$

$$\begin{aligned} u(t) = \mathcal{T}(t)u_0 &+ \int_0^t (t-s)^{q-1}\mathcal{I}(t-s)F\big(s, u(s)\big)ds \\ &+ \int_0^t (t-s)^{q-1}\mathcal{I}(t-s)G\big(s, u(s)\big)dW(s) \qquad \text{a.s.} \end{aligned} \tag{4.5}$$

We shall say that the martingale solution (4.4)–(4.5) of (4.1) is unique if and only if any other martingale solution for (4.1),

$$\big(\Omega', \mathcal{F}', \mathbb{P}', \{\mathcal{F}'_t\}_{t\geq 0}, \{W'(t)\}_{t\geq 0}, \{u'(t)\}_{t\geq 0}\big),$$

has the same probability law as $u(t)$ on the space $C([0, T]; B)$.

We present the statement on the existence of a martingale solution.

Theorem 4.7. *Assume that a Banach space X, a Hilbert space H, a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and operators $A \to I$ (with some $\nu \geq 0$) are given, and that assumptions (H), (A1)–(A6) are satisfied. Assume also that for some $\beta < 1$ the function $A^{-\beta}F : [0, \infty) \times B \to X$ is strongly measurable relative to the first variable and bounded relative to the second variable. Then for any $x \in B$ there exists a martingale solution of* (4.1).

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