Global and Stochastic Analysis
Vol. 11 No. 4 (September, 2024)

Received: 24th June 2024 Revised: 21st July 2024 Accepted: 23rd August 2024

A GENERALIZATION OF THE RESULT OF V.V. SENATOV ON CHARACTERISTIC FUNCTIONS OF CONVOLUTIONS OF PROBABILITY DISTRIBUTIONS

V. N. SOBOLEV AND A. E. CONDRATENKO

Abstract. This paper gives new expansions of characteristic functions of convolution of symmetric probability distributions with an explicit estimate of the remainder.

1. Introduction

Let $\xi, \xi_1, \xi_2, \dots$ be independent identically distributed random variables with zero mean and unit variance. Let ξ be symmetric around zero with the real characteristic function $f(t)$. The normalized sums

$$\frac{\xi_1 + \ldots + \xi_n}{\sqrt{n}}$$

have distribution function $F_n(x)$ and characteristic function $f^n\left(\frac{t}{\sqrt{n}}\right)$.

The present paper is concerned with expansions of characteristic functions of convolution of distributions $f^n\left(\frac{t}{\sqrt{n}}\right)$ and estimating the remainders. Note that while constructing asymptotic expansions in the Central limit theorem (CLT), the expansions of characteristic functions of convolution of distributions are often used.

We use expansions containing the last known moment of the random variable ξ in their main part. It's idea was proposed by H. Prawitz in [5]

$$f(t) = \sum_{k=0}^{m-1} \alpha_k (it)^k + \frac{m}{2(m+1)} \alpha_m (it)^m + \frac{m+2}{2(m+1)} \beta_m (it)^m \gamma(t),$$

where for $k \le m+2$

$$\alpha_k = \alpha_k(\mathrm{P}) = \frac{\mathsf{M}\xi^k}{k!}, \qquad \beta_k = \beta_k(\mathrm{P}) = \frac{\mathsf{M}|\xi|^k}{k!},$$

$\gamma(t)$, here and futher, are different complex functions such that $|\gamma| \le 1$.

2000 *Mathematics Subject Classification.* Primary 60F05; Secondary 60F99.

Key words and phrases. Central limit theorem; charachteristic functions; asymptotic expansions; convolution; accuracy of approximation; explicit estimates; Chebyshev — Hermite polynomials; Senatov moments.

This result of Prawitz has been generalized by I. G. Shevtsova [12]. In the notation of this paper, one of her results can be represented by the following theorem.

Theorem 1.1. *For any r.v. ξ with the characteristic function $f(t)$ and $\mathsf{M}|\xi|^m < \infty$ for some $m \in \mathbb{N}$, for all $t \in \mathbb{R}$ the following estimates hold:*

$$\left| f(t) - \sum_{k=0}^{m-1} \alpha_k (it)^k \right| \le \inf_{\lambda \ge 0} \Big(\lambda |\alpha_m| + q_m(\lambda) \beta_m \Big) \cdot |t|^m, \tag{1.1}$$

where

$$q_m(\lambda) = \sup_{x>0} \frac{m!}{x^m} \left| e^{ix} - \sum_{k=0}^{m-1} \frac{(ix)^k}{k!} - \lambda \frac{(ix)^m}{m!} \right|, \qquad \lambda \ge 0. \tag{1.2}$$

V. V. Senatov obtained in [11, p. 189] two expansions of the real characteristic function, wich are generalized in this paper. It is easily seen that expansions in [11, p. 189] are valid for any even number $m+2$ and

$$f(t) = \sum_{k=0}^{m} \alpha_k (it)^k + \lambda \alpha_{m+2} (it)^{m+2} + \overline{\lambda} \alpha_{m+2} (it)^{m+2} \gamma(t), \tag{1.3}$$

where

$$\overline{\lambda} = \max\{\lambda, 1-\lambda\}, \qquad 0 \le \lambda < 1.$$

For the comparison of (1.2) and (1.3) it is worth noting (see [12]) that $q_m(\lambda) \ge \max\{\lambda, |1-\lambda|\}$ and

$$q_m(\lambda) = 1 - \lambda \qquad \text{for} \qquad 0 \le \lambda \le \frac{m}{2(m+1)}.$$

While constructing in the CLT the expansions with and explicit estimation of the remainder, it is convenient to apply [15] the decomposition for the function $f(t)e^{t^2/2}$ in terms of the Senatov moments [7, 8]

$$\theta_k = \theta_k(\mathrm{P}) = \frac{1}{k!} \int_{-\infty}^{\infty} H_k(x)\, \mathrm{d}F(x),$$

where

$$H_k(x) = (-1)^k \cdot \frac{\varphi^{(k)}(x)}{\varphi(x)}$$

are the Chebyshev — Hermite polynomials [1, p. 21] of degree k, $\varphi(x)$ is the probability density function of the standard normal (cumulative) distribution function $\Phi(x)$.

As well known [4]

$$\frac{H_k(x)}{k!} = \sum_{j=0}^{[k/2]} (-1)^j \alpha_{2j}(\varphi) \frac{x^{k-2j}}{(k-2j)!}\,.$$

The latter formula for $H_k(x)$ allows us to express moments θ_k in terms of moments α_k and $\alpha_{2j}(\varphi)$, so [3]

$$\theta_k = \sum_{j=0}^{[k/2]} (-1)^j \alpha_{2j}(\varphi)\alpha_{k-2j}(\mathrm{P}). \tag{1.4}$$

For the before introduced random variable ξ it follows that

$$\alpha_0 = 1, \qquad \alpha_1 = 0, \qquad \alpha_2 = \frac{1}{2}. \tag{1.5}$$

The respective quantities for the standard normal law denote by $\alpha_k(\varphi)$, $\beta_k(\varphi)$, for example,

$$\alpha_{2j}(\varphi) = \frac{1}{2^j \cdot j!}, \qquad \alpha_{2j+1}(\varphi) = 0, \qquad j = 0, 1, 2, \dots .$$

At the same time, it is clear that always $\theta_0 = 1$. From (1.5) and (1.4) it is seen that

$$\theta_1 = \alpha_1(\mathrm{P})\alpha_2(\varphi) = 0,$$
$$\theta_2 = \alpha_2(\mathrm{P})\alpha_0(\varphi) - \alpha_0(\mathrm{P})\alpha_2(\varphi) = \frac{1}{2} - \frac{1}{2} = 0,$$
$$\theta_3 = \alpha_3(\mathrm{P})\alpha_0(\varphi) - \alpha_1(\mathrm{P})\alpha_2(\varphi) = \alpha_3 = 0.$$

Due to the symmetry of the distribution P of ξ, it follows that the odd Senatov moments are equal to zero,

$$\theta_{2k+1} = \sum_{j=0}^{k} (-1)^j \alpha_{2j}(\varphi)\alpha_{2k+1-2j}(\mathrm{P}) = 0. \tag{1.6}$$

The incomplete Senatov moments

$$\theta_k^{(k-2)} = \sum_{j=1}^{[k/2]} (-1)^j \alpha_{2j}(\varphi) \cdot \alpha_{k-2j}(\mathrm{P}),$$

are included in the Senatov moments with the parameter λ (see [11])

$$\theta_{m+2}^{(\lambda)} = \lambda\alpha_{m+2} + \theta_{m+2}^{(m)}.$$

The Senatov moments $\theta_l(\mathrm{P}_n)$ of probability distribution P_n of normalized sum for $l \geq 3$ can be expressed in terms of the moments $\theta_3, \dots, \theta_l$ of the probability distribution P as (see [7, 8, 3])

$$\theta_l(\mathrm{P}_n) = \sum \frac{n!}{j_0! j_3! \cdots j_l!} \left(\frac{\theta_3}{n^{3/2}}\right)^{j_3} \cdots \left(\frac{\theta_l}{n^{l/2}}\right)^{j_l}, \tag{1.7}$$

where summation is performed for all sets of integer nonnegative numbers $j_0, j_3, \dots, j_l$ such that

$$3j_3 + \ldots + lj_l = l, \qquad j_0 + j_3 + \ldots + j_l = n.$$

While constructing asymptotic expansions, the Senatov quasi-moments $\theta_l^{(s)}(\mathrm{P}_n)$ are used for $s = 1, 2, \dots, l-1$. To calculate them, we can apply (1.7) for $\theta_l(\mathrm{P}_n)$, the summand with moments $\theta_l^{(s)}(\mathrm{P}_n)$, $l > s$, not to be taken into account. That quasi-moments were first used by V. V. Senatov and more information about them can be found in [9, ch. 4, §4].

In the statement of main result below, the Senatov moments will appear as the sum of its product:

$$\Theta_{s,l} = \sum_{k_1+\dots+k_s=l} \theta_{k_1}\dots\theta_{k_s}, \qquad k_j \geq 4, \qquad j = 1,\dots,m-1. \tag{1.8}$$

The asymptotic expansions will be considered under the following assumptions: the distribution P with zero mean and variance one has a finite even moment of order $m+2$, some positive number $\nu > 0$ exists so that the function $|f(t)|^\nu$ is integrable on the whole real line, e.g.

$$\int_{-\infty}^{\infty} |f(t)|^\nu dt < \infty\,. \tag{1.9}$$

The convergence of the last integral guarantees the existence of a continuous density $p_n(x)$ of distribution P_n for all $n \geq \nu$. In this case, the inverse Fourier transform gives density

$$p_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx} f^n\left(\frac{t}{\sqrt{n}}\right) dt\,. \tag{1.10}$$

In addition, this assumption guarantees that the value

$$\alpha(T) = \max\Big\{|f(t)|:\ t \geq T\Big\}$$

is strictly less than one for all $T > 0$.

Applying the function

$$\mu(t) = \max\left\{|f(t)|;\ e^{-t^2/2}\right\}$$

proposed by V.Yu. Korolev, define the integrals

$$B_{l,n-k} = \frac{1}{2\pi}\int_{-T\sqrt{n}}^{T\sqrt{n}} |t|^l\,\mu^{n-k}\left(\frac{t}{\sqrt{n}}\right) dt\,, \tag{1.11}$$

for any non-negative l, k, n with $n \geq k$.

These integrals turn out to be connected with moments of the standard normal distribution. So, for any distribution P with the finite fourth moment, the parameter $T > 0$ can be chosen (see, for example, [11], [9, p. 154]) so that for all l there is a limit

$$\lim_{n\to+\infty} B_{l,n-k} = B_l\,, \tag{1.12}$$

where

$$B_l = \frac{\beta_l(\varphi)}{\sqrt{2\pi}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |t|^l\,e^{-t^2/2}\,dt\,.$$

It is assumed further that the property (1.12) is valid. The value

$$A_n(T) = \frac{\sqrt{n}}{\pi}\alpha^{n-\nu}(T)\int\limits_T^\infty \left|f(t)\right|^\nu dt\,,$$

$$L_l(u) = \frac{1}{2\pi}\int\limits_{|t|\geq u} |t|^l e^{-t^2/2}\,dt\,. \tag{1.13}$$

will be used through the paper.

In this paper, as in [10], Zolotarev's ideal metric $\zeta_3 = \zeta_3(\mathrm{P}, \Phi)$ (see, for example, [18, 6]) is

$$\sup\left\{\left|\int_{-\infty}^{\infty} u(x)\big(\mathrm{P}(dx) - \Phi(dx)\big)\right| :\ u \in \tilde{\mathfrak{F}}_3\right\}\,,$$

where $\tilde{\mathfrak{F}}_3$ is the class of the bounded real-valued function $u(x)$, $-\infty < x < \infty$, three times differentiable, and $\left|u^{(3)}(x)\right| \leq 1$ [6, p. 101]. For all real $t \in \mathbb{R}$ the inequalities

$$\left|f(t) - e^{-t^2/2}\right| \leq |t|^3\,\zeta_3 \tag{1.14}$$

and $\frac{|\theta_3|}{3!} \leq \zeta_3$ are valid. Last inequality can not be improved in the sense that for any number $0 < c < 1$ there is a distribution P with zero mean and unit variance such $\frac{|\theta_3|}{3!} > c\zeta_3$.

The function $\mu(t)$ is used as estimates for functions $f(t)$ and $g(t) = e^{-\frac{t^2}{2}}$. In case $k < n$ for the sum

$$S_k = \sum_{0\leq j_1+\ldots+j_k\leq n-k} f^{n-j_1-\ldots-j_k-k} g^{j_1+\ldots+j_k+k}$$

the estimate

$$|S_k| \leq C_n^k \mu^n\,. \tag{1.15}$$

holds. Indeed,

$$|S_k| \leq \left|\sum_{0\leq j_1+\ldots+j_k\leq n-k} \mu^{n-j_1-\ldots-j_k-k}\mu^{j_1+\ldots+j_k+k}\right| = \mu^n \sum_{0\leq j_1+\ldots+j_k\leq n-k} 1 = C_n^k\mu^n\,.$$

The last equality is true becouse if $k < n$ then the number of non-negative integer solutions $j_1, ..., j_k$ of the inequality $0 \leq j_1 + ... + j_k \leq n - k$ is C_n^k.

We use (1.14) only in following reasoning. From (1.14) for

$$\psi = \psi(t) = f(t)\,e^{\frac{t^2}{2}} - 1 \tag{1.16}$$

follows the next inequality

$$|\psi(t)| \leq |t|^3\,\zeta_3\mu^{-1}\,, \tag{1.17}$$

and application of (1.15) gives

$$|S_k\psi(t)| \leq C_n^k\zeta_3\,|t|^3\,\mu^{n-1}\,. \tag{1.18}$$

2. Expansions of the characteristic function of symmetric distributions

The following two expansions of characteristic functions will be used in constructing expansions of characteristic functions of convolution distributions. The first of them is quite standard in line with the article [14], the parameter λ is not included. The second expansion contains the parameter λ, and its proof relies on equality (1.3). It is worth noting that in the proof of the theorem, the expansion of characteristic functions with parameter λ will be used only once.

To estimate the remainder of the expansions of characteristic functions, we need the following values

$$\|\theta_s\| = \sum_{j=0}^{[s/2]} |\alpha_{s-2j}| \, \alpha_{2j}(\varphi) \tag{2.1}$$

and

$$\left\|\theta_{s+2}^0\right\| = \beta_{s+2} + \|\theta_s\| \, \alpha_2(\varphi) \,. \tag{2.2}$$

Proposition 2.1. *Let symmetric probability distribution* P *has a finite moment of even order* $(m+2) \geq 2$. *Then for all even* $2 \leq s \leq m$ *the characteristic function* $f(t)$ *of distribution* P *has the expansions*

$$f(t) = e^{-\frac{t^2}{2}} \sum_{k=0}^{s} \theta_k \, (it)^k + R_{f,s}(t) \,. \tag{2.3}$$

Moreover

$$|R_{f,s}(t)| \leq \left\|\theta_{s+2}^0\right\| |t|^{s+2} \,. \tag{2.4}$$

Proof. Let $t \in \mathbb{R}$ and $\omega = it$. It is known that

$$f(t) = \sum_{k=0}^{s+1} \alpha_k \omega^k + \rho_{s+1}(t) \,, \quad |\rho_{s+1}(t)| \leq \beta_{s+2} t^{s+2}$$

and

$$e^{\frac{t^2}{2}} = e^{-\omega^2/2} = \sum_{k=0}^{\infty} b_{2j} \omega^{2j} \,,$$

where

$$b_{2j} = \frac{(-1)^j}{j! 2^j} = (-1)^j \alpha_{2j}(\varphi) \,.$$

Observe that the product

$$f(t) e^{\frac{t^2}{2}} = \sum_{j=0}^{\infty} b_{2j} \omega^{2j} \sum_{k=0}^{s+1} \alpha_k \omega^k + \rho_s(t) e^{\frac{t^2}{2}}$$

can be transformed to

$$f(t) e^{\frac{t^2}{2}} = \sum_{k=0}^{s+1} \theta_k \omega^k + \sum_{k=s+2}^{\infty} \theta_k^{(s+1)} \omega^k + \rho_s(t) e^{\frac{t^2}{2}} \,.$$

Let's break the third component from the right part of the last equality into two parts

$$\sum_{k=0}^{\infty}\theta_{s+2+k}^{(s+1)}\omega^{s+2+k}=\sum_{k=0}^{\infty}\theta_{s+2+k}^{(s)}\omega^{s+2+k}=$$

$$=\omega^{s+2}\sum_{k=0}^{\infty}\theta_{s+2+2k}^{(s)}\omega^{2k}+\omega^{s+3}\sum_{k=0}^{\infty}\theta_{s+3+2k}^{(s)}\omega^{2k}.$$

The symmetring of distribution P for even s leads to $\theta_{2k+1}=0$ and $\theta_{s+3+2k}^{(s)}=0$ for all natural $k\in\mathbb{N}$.

For this reason,

$$f(t)e^{\frac{t^2}{2}}=\sum_{k=0}^{s}\theta_k\omega^k+\omega^{s+2}\sum_{k=0}^{\infty}\theta_{s+2+2k}^{(s)}\omega^{2k}+\rho_{s+1}(t)\,e^{\frac{t^2}{2}}.$$

We can represent Senatov quasi-moments from last formula as follows

$$\theta_{s+2+2k}^{(s)}=\sum_{j=0;\,2j\geq 2k+2}^{[(s+2k+2)/2]}\alpha_{s+2k+2-2j}b_{2j}=$$

$$=\sum_{j=k+1}^{[s/2]+k+1}\alpha_{s-2(j-k-1)}b_{2j}=\sum_{j=0}^{[s/2]}\alpha_{s-2j}b_{2(k+1+j)}.$$

Then

$$\left|\theta_{s+2+2k}^{(s)}\right|\leq\alpha_{2k}(\varphi)\sum_{j=0}^{[s/2]}\left|\alpha_{s-2j}\right|\alpha_{2j+2}(\varphi)=\alpha_2(\varphi)\,\alpha_{2k}(\varphi)\,\|\theta_s\|$$

because

$$\left|b_{2(k+1+j)}\right|\leq\left|b_{2k}\right|\left|b_{2j+2}\right|.$$

Hence

$$\left|\omega^{s+1}\sum_{k=0}^{\infty}\theta_{s+2+2k}^{(s)}\omega^{2k+1}\right|\leq$$

$$\leq\alpha_2(\varphi)\,|t|^{s+2}\sum_{k=0}^{\infty}\alpha_{2k}(\varphi)\,|t|^{2k}\,\|\theta_s\|=$$

$$=t^{s+2}e^{t^2/2}\alpha_2(\varphi)\,\|\theta_s\|.$$

So the proof is completed. □

Corollary. *From* (2.3) *for* (1.16) *it follows that*

$$\psi\left(\frac{t}{\sqrt{n}}\right)=e^{\frac{t^2}{2n}}f\left(\frac{t}{\sqrt{n}}\right)-1=$$

$$=\sum_{k=4}^{s}\theta_k\left(\frac{t}{\sqrt{n}}\right)^k+e^{\frac{t^2}{2}}R_{f,s}\left(\frac{t}{\sqrt{n}}\right)\tag{2.5}$$

for all even $2\leq s\leq m$. *In case* $s=2$ *the sum by* k *in* (2.5) *is not included.*

Now let us obtain the expression for characteristic function $f(t)$ with the parameter λ in its expansion folloing the main ideas of [15].

Proposition 2.2. *Let symmetric probability distribution* P *with the characteristic function* $f(t)$ *has a finite moment of even order* $(m+2) \geq 2$. *Then*

$$f(t) = \left(\sum_{k=0}^{m} \theta_k (it)^k + \theta_{m+2}^{(\lambda)} (it)^{m+2} \right) e^{-\frac{t^2}{2}} +$$

$$+\gamma \bar{\lambda} \alpha_{m+2} |t|^{m+2} + \gamma \left\| \theta_{m+4}^{(m+2,\lambda)} \right\| |t|^{m+4} , \tag{2.6}$$

where

$$\theta_{m+2}^{(\lambda)} = \lambda \alpha_{m+2} + \theta_{m+2}^{(m)} \tag{2.7}$$

and

$$\left\| \theta_{m+4}^{(m+2,\lambda)} \right\| = \lambda \alpha_{m+2} \alpha_2 (\varphi) + \| \theta_m \| \alpha_4 (\varphi) . \tag{2.8}$$

Proof. The generalization of the above mentioned result of the article [11, p. 189] gives (see more about in [15])

$$f(t) = \sum_{k=0}^{m+1} \alpha_k \omega^k + \lambda \alpha_{m+2} \omega^{m+2} + \rho_{m+2}(\lambda, t) , \quad |\rho_{m+2}(\lambda, t)| \leq \bar{\lambda} \alpha_{m+2} |t|^{m+2} .$$

As at the initial steps of the proof in the previous proposition, we get

$$f(t) e^{\frac{t^2}{2}} = \sum_{j=0}^{\infty} b_{2j} \omega^{2j} \sum_{k=0}^{m} \alpha_k \omega^k + \lambda \alpha_{m+2} \omega^{m+2} e^{\frac{t^2}{2}} + \rho_{m+2}(\lambda, t) e^{\frac{t^2}{2}} =$$

$$= \sum_{k=0}^{m} \theta_k \omega^k + \omega^{m+2} \sum_{k=0}^{\infty} \theta_{m+2+2k}^{(m)} \omega^{2k} + \lambda \alpha_{m+2} \omega^{m+2} e^{\frac{t^2}{2}} +$$

$$+\rho_{m+2}(\lambda, t) e^{\frac{t^2}{2}} = \sum_{k=0}^{m} \theta_k \omega^k + \omega^{m+2} \sum_{k=1}^{\infty} \theta_{m+2+2k}^{(m)} \omega^{2k} + \theta_{m+2}^{(m)} \omega^{m+2} +$$

$$+\lambda \alpha_{m+2} \omega^{m+2} e^{\frac{t^2}{2}} + \rho_{m+2}(\lambda, t) e^{\frac{t^2}{2}} = \sum_{k=0}^{m} \theta_k \omega^k + \omega^{m+4} \sum_{k=0}^{\infty} \theta_{m+4+2k}^{(m)} \omega^{2k} +$$

$$+\theta_{m+2}^{(m)} \omega^{m+2} + \lambda \alpha_{m+2} \omega^{m+2} e^{\frac{t^2}{2}} + \rho_{m+2}(\lambda, t) e^{\frac{t^2}{2}} .$$

Senatov quasi-moments can be represented as

$$\theta_{s+4+2k}^{(s)} = \sum_{j=0;\, 2j \geq 2k+4}^{[(s+2k+4)/2]} \alpha_{s+2k+4-2j} b_{2j} =$$

$$= \sum_{j=k+2}^{[s/2]+k+2} \alpha_{s-2(j-k-2)} b_{2j} = \sum_{j=0}^{[s/2]} \alpha_{s-2j} b_{2(k+j+2)} .$$

It is not hard to verify that

$$\left| b_{2(k+2+j)} \right| \leq |b_{2k}| \, |b_4| \, |b_{2j}|$$

and

$$\left|\theta_{s+4+2k}^{(s)}\right| \le \alpha_4(\varphi)\,\alpha_{2k}(\varphi)\sum_{j=0}^{[s/2]}|\alpha_{s-2j}|\,\alpha_{2j}(\varphi) = \\ = \alpha_4(\varphi)\,\alpha_{2k}(\varphi)\,\|\theta_s\|\ .$$

Hense

$$\left|\omega^{s+4}\sum_{k=0}^{\infty}\theta_{s+4+2k}^{(s)}\omega^{2k}\right| \le \alpha_4(\varphi)\,|t|^{s+4}\sum_{k=0}^{\infty}\alpha_{2k}(\varphi)\,|t|^{2k}\,\|\theta_s\| = t^{s+4}e^{t^2/2}\alpha_4(\varphi)\,\|\theta_s\|\ .$$

Thus, it remains in the asymptotic expansion

$$f(t)e^{\frac{t^2}{2}} = \sum_{k=0}^{m}\theta_k\omega^k + \omega^{m+4}\sum_{k=0}^{\infty}\theta_{m+4+2k}^{(m)}\omega^{2k} + \\ + \theta_{m+2}^{(m)}\omega^{m+2} + \lambda\alpha_{m+2}\omega^{m+2}e^{\frac{t^2}{2}} + \rho_{m+2}(\lambda,t)\,e^{\frac{t^2}{2}}$$

to consider only the term $\lambda\alpha_{m+2}\omega^{m+2}e^{\frac{t^2}{2}}$, since the estimate for $\rho_{m+2}(\lambda,t)\,e^{\frac{t^2}{2}}$ follows immediately from the estimate $|\rho_{m+2}(\lambda,t)| \le \overline{\lambda}\alpha_{m+2}\,|t|^{m+2}$.

Write

$$\lambda\alpha_{m+2}\omega^{m+2}e^{\frac{t^2}{2}} = \lambda\alpha_{m+2}\omega^{m+2} + \lambda\alpha_{m+2}\omega^{m+2}\sum_{j=1}^{\infty}b_{2j}\omega^{2j}$$

where

$$\left|\lambda\alpha_{m+2}\omega^{m+2}\sum_{j=1}^{\infty}b_{2j}\omega^{2j}\right| \le \left|\lambda\alpha_{m+2}\omega^{m+4}\sum_{j=0}^{\infty}b_{2j+2}\omega^{2j}\right| \le \\ \le \left|\lambda\alpha_{m+2}\omega^{m+4}\alpha_2(\varphi)\sum_{j=0}^{\infty}b_{2j}\omega^{2j}\right| = \lambda\alpha_{m+2}\alpha_2(\varphi)\,e^{\frac{t^2}{2}}\,|t|^{m+4}\ .$$

Then (2.3) follows and the proof is completed.

□

3. Main result: Expansion of characteristic functions of convolutions of distributions

The following theorem presents the main result about the expansion of the characteristic function of the convolution of distributions with an explicit estimate of the remainder. Note, that the last known moment of the initial distribution P is included in the main part of the expansion. This allows us to obtain the best explicit estimate of the remainder.

Theorem 3.1. *Let $f(t)$ is the characteristic function of symmetric distributions* P *with finite moments up to even $m+2 \ge 4$ included. If the condition* (1.9) *is fulfilled, then for $n \ge m+2$*

$$f^n\left(\frac{t}{\sqrt{n}}\right) = e^{-t^2/2} + e^{-t^2/2}\sum_{s=1}^{m/2}C_n^s\sum_{l=4s}^{m-4+4s}\Theta_{s,l}\left(\frac{it}{\sqrt{n}}\right)^l +$$

$$+\theta_{m+2}^{(\lambda)}\frac{(it)^{m+2}}{(\sqrt{n})^{m}}e^{-\frac{t^2}{2}}+\sum_{s=1}^{m/2}r_s+R_0+R^{\psi}_{\frac{m+2}{2},m}\,, \tag{3.1}$$

where

$$|r_1|\le\left(\bar{\lambda}\alpha_{m+2}\frac{|t|^{m+2}}{n^{m/2}}+\left\|\theta_{m+4}^{(m+2,\lambda)}\right\|\frac{|t|^{m+4}}{n^{m/2+1}}\right)\mu^{n-1}\left(\frac{t}{\sqrt{n}}\right)\,,$$

$$|R_0|\le\frac{1}{2}\zeta_3\left|\theta_{m+2}^{(\lambda)}\right|\frac{|t|^{m+5}}{n^{m/2+1/2}}\mu^{n-1}\left(\frac{t}{\sqrt{n}}\right)\,,$$

$$|r_s|\le\|\Theta_{s,m+4s-2}\|\frac{|t|^{m+4s-2}}{(\sqrt{n})^{m+2s-2}}\mu^{n-1}\left(\frac{t}{\sqrt{n}}\right)\,,\ 2\le s\le\frac{m}{2}\,,$$

$$\left|R^{\psi}_{\frac{m+2}{2},m}\right|\le C_n^{\frac{m+2}{2}}\zeta_3\sum_{k=2m}^{3m-4}|\Theta_{s-1,k}|\left|\frac{t}{\sqrt{n}}\right|^{k+3}\mu^{n-1}\left(\frac{t}{\sqrt{n}}\right)\,.$$

So the last theorem can be used to build a new asymptotic expansions [10, 13] in the CLT with an explicit estimate of the remainder.

Remark 3.2. The estimate of the remainder can be written as

$$\overline{\lambda}\cdot\frac{\alpha_{m+2}}{n^{m/2}}|t|^{m+2}\mu^{n-1}\left(\frac{t}{\sqrt{n}}\right)+O\left(\frac{|t|^{m+5}}{n^{m/2+1/2}}\right)\,. \tag{3.2}$$

4. Proof of theorem

We break our proof up into 4 steps.

1. Here we consider the difference $f^n\left(\frac{t}{\sqrt{n}}\right)-e^{-t^2/2}$, which can be written as $f^n\left(\frac{t}{\sqrt{n}}\right)-g^n\left(\frac{t}{\sqrt{n}}\right)$, where $g(t)=e^{-t^2/2}$. (Further, the arguments of some functions will often be suppressed.)

The equality

$$a^n-b^n=(a-b)\sum_{j=0}^{n-1}a^{n-j-1}b^j$$

for any complex numbers a and b is well known. It follows that

$$f^n\left(\frac{t}{\sqrt{n}}\right)-g^n\left(\frac{t}{\sqrt{n}}\right)=f^n-g^n=$$

$$=(f-g)\sum_{j_1=0}^{n-1}f^{n-j_1-1}g^{j_1}=\psi\sum_{j_1=0}^{n-1}f^{n-j_1-1}g^{j_1+1}=\psi S_1\,, \tag{4.1}$$

where ψ given by the formula (1.16).

From (2.6) that for $m \geq 4$ we have

$$\psi\left(\frac{t}{\sqrt{n}}\right) = \sum_{k=4}^{m}\theta_k\left(\frac{it}{\sqrt{n}}\right)^k + \theta_{m+2}^{(\lambda)}\left(\frac{it}{\sqrt{n}}\right)^{m+2} + e^{\frac{t^2}{2}}R_{f,m+2}\left(\frac{t}{\sqrt{n}}\right).$$

Therefore, the right part of (4.1) is equal to

$$\psi S_1 = \left(\sum_{k=4}^{m}\theta_k\left(\frac{it}{\sqrt{n}}\right)^k + \theta_{m+2}^{(\lambda)}\left(\frac{it}{\sqrt{n}}\right)^{m+2}\right)S_1 + S_1 e^{\frac{t^2}{2}}R_{f,m+2}\left(\frac{t}{\sqrt{n}}\right) =$$

$$= \left(\sum_{k=4}^{m}\theta_k\left(\frac{it}{\sqrt{n}}\right)^k + \theta_{m+2}^{(\lambda)}\left(\frac{it}{\sqrt{n}}\right)^{m+2}\right)S_1 + r_1\,,$$

where

$$|r_1| \leq \left|R_{f,m+2}\left(\frac{t}{\sqrt{n}}\right)\right|\mu^{n-1}\left(\frac{t}{\sqrt{n}}\right)$$

or more explicitly

$$|r_1| \leq \left(\bar{\lambda}\alpha_{m+2}\frac{|t|^{m+2}}{n^{m/2}} + \left\|\theta_{m+4}^{(m+2,\lambda)}\right\|\frac{|t|^{m+4}}{n^{m/2+1}}\right)\mu^{n-1}\left(\frac{t}{\sqrt{n}}\right).$$

Recall that the quantities $\left\|\theta_{m+4}^{(m+2,\lambda)}\right\|$ are determined in (2.8). Next conversions

$$S_1 = \sum_{j_1=0}^{n-1}f^{n-j_1-1}g^{j_1+1} = \sum_{j_1=0}^{n-2}\left(f^{n-j_1-1} - g^{n-j_1-1}\right)g^{j_1+1} + \sum_{j_1=0}^{n-1}g^n =$$

$$= \sum_{j_1=0}^{n-1}g^n + \sum_{j_1=0}^{n-2}\left((f-g)\sum_{j_2=0}^{n-j_1-2}f^{n-j_1-j_2-2}g^{j_2}\right)g^{j_1+1} =$$

$$= \sum_{j_1=0}^{n-1}g^n + (f-g)\sum_{j_1=0}^{n-2}\sum_{j_2=0}^{n-j_1-2}f^{n-j_1-j_2-2}g^{j_1+j_2+1} =$$

$$= ng^n + \psi\sum_{0\leq j_1+j_2\leq n-2}f^{n-2-j_1-j_2}g^{j_1+j_2+2} =$$

$$= ng^n + \psi S_2 = C_n^1 g^n + \psi S_2$$

allow us to write out the equality

$$S_1 = C_n^1 e^{-\frac{t^2}{2}} + \psi S_2 \tag{4.2}$$

and get the representation

$$f^n - g^n = \left(\sum_{k=4}^{m}\theta_k\left(\frac{it}{\sqrt{n}}\right)^k + \theta_{m+2}^{(\lambda)}\left(\frac{it}{\sqrt{n}}\right)^{m+2}\right)S_1 + r_0 =$$

$$= \left(\sum_{k=4}^{m}\theta_k\left(\frac{it}{\sqrt{n}}\right)^k + \theta_{m+2}^{(\lambda)}\left(\frac{t}{\sqrt{n}}\right)^{m+2}\right)\left(ne^{-\frac{t^2}{2}} + \psi S_2\right) + r_0 =$$

$$= C_n^1 e^{-\frac{t^2}{2}} \left(\sum_{k=4}^{m} \theta_k \left(\frac{it}{\sqrt{n}} \right)^k + \theta_{m+2}^{(\lambda)} \left(\frac{it}{\sqrt{n}} \right)^{m+2} \right) + \psi S_2 \sum_{k=4}^{m} \theta_k \left(\frac{it}{\sqrt{n}} \right)^k + R_0 + r_1 , \tag{4.3}$$

where

$$|R_0| \le \left| \theta_{m+2}^{(\lambda)} \right| \left| \frac{t}{\sqrt{n}} \right|^{m+2} |\psi S_2| \le \left| \theta_{m+2}^{(\lambda)} \right| \left| \frac{t}{\sqrt{n}} \right|^{m+5} \zeta_3 C_n^2 \mu^{n-1} \left(\frac{t}{\sqrt{n}} \right) \le$$
$$\le \frac{\zeta_3}{2} \left| \theta_{m+2}^{(\lambda)} \right| \frac{|t|^{m+5}}{(\sqrt{n})^{m+1}} \mu^{n-1} \left(\frac{t}{\sqrt{n}} \right) .$$

by (1.18).

2. Obviously, if in (4.3) $m = 2$ the summand

$$R_{2,m}^{\psi} = \psi S_2 \sum_{k=4}^{m} \theta_k \left(\frac{it}{\sqrt{n}} \right)^k$$

will be missing (it will completely enter into R_0). In this case, (4.3) takes the form

$$f^n \left(\frac{t}{\sqrt{n}} \right) - e^{-t^2/2} = \lambda \alpha_4 \frac{t^4}{n} e^{-t^2/2} + R_0 + r_1 , \tag{4.4}$$

where

$$|r_1| \le \left(\bar{\lambda} \alpha_4 + \left\| \theta_6^{(4,\lambda)} \right\| \frac{t^2}{n} \right) \cdot \frac{t^4}{n} \cdot \mu^{n-1} \left(\frac{t}{\sqrt{n}} \right) ,$$
$$|R_0| \le \frac{1}{2} \zeta_3 \left| \theta_4^{(\lambda)} \right| \cdot \frac{|t|^7}{n^{3/2}} \cdot \mu^{n-1} \left(\frac{t}{\sqrt{n}} \right) .$$

It coincides with the statement of the teorem when $m = 2$. The expression (4.4) was obtained in [11, p. 191] with a slightly different estimate of the remainder.

3. For $m = 4$ the rate of decrease of the estimate for the remainder equal to $\frac{1}{n^2}$, and the relation (4.3) turns into

$$f^n - g^n = C_n^1 e^{-\frac{t^2}{2}} \left(\theta_4 \left(\frac{it}{\sqrt{n}} \right)^4 + \theta_6^{(\lambda)} \left(\frac{it}{\sqrt{n}} \right)^6 \right) + \psi S_2 \theta_4 \left(\frac{it}{\sqrt{n}} \right)^4 + R_0 + r_1 , \tag{4.5}$$

where

$$|r_1| \le \left(\bar{\lambda} \alpha_6 \frac{t^6}{n^2} + \left\| \theta_8^{(6,\lambda)} \right\| \frac{t^8}{n^3} \right) \mu^{n-1} \left(\frac{t}{\sqrt{n}} \right) ,$$
$$|R_0| \le \frac{\zeta_3 \left| \theta_6^{(6,\lambda)} \right|}{2} \frac{|t|^9}{n^{5/2}} \mu^{n-1} \left(\frac{t}{\sqrt{n}} \right) .$$

The second sum from the right side (4.5)

$$\psi \cdot S_2 \theta_4 \left(\frac{it}{\sqrt{n}} \right)^4$$

by the inequality (1.18) can be estimated as

$$\left| \psi S_2 \theta_4 \left(\frac{it}{\sqrt{n}} \right)^4 \right| \le C_n^2 \zeta_3 |\mu|^{n-1} \frac{|t|^3}{n^{3/2}} |\theta_4| \left| \frac{t}{\sqrt{n}} \right|^4 \le \frac{\zeta_3 |\theta_4|}{2} \frac{|t|^7}{n^{3/2}} |\mu|^{n-1} .$$

The rate of convergence of the last estimate is $\frac{1}{n^{3/2}}$ and not $\frac{1}{n^2}$. We found a lower convergence rate in the estimate than it is needed. Therefore, we split this summand into two parts using the representation (2.5). Thus,

$$R^{\psi}_{2,4} = \psi S_2 \sum_{k=4}^{m} \theta_k \left(\frac{it}{\sqrt{n}}\right)^k =$$
$$= S_2 \sum_{k_1=4}^{m} \theta_{k_1} \left(\frac{it}{\sqrt{n}}\right)^{k_1} \cdot \left(\sum_{k_2=4}^{m+4-k_1} \theta_{k_2} \left(\frac{it}{\sqrt{n}}\right)^{k_2} + e^{\frac{t^2}{2}} R_{f,m+4-k_1}\right) =$$
$$= S_2 \sum_{k_1=4}^{m} \sum_{k_2=4}^{m+4-k_1} \theta_{k_1}\theta_{k_2} \left(\frac{it}{\sqrt{n}}\right)^{k_1+k_2} +$$
$$+ \sum_{k_1=4}^{m} \theta_{k_1} \left(\frac{it}{\sqrt{n}}\right)^{k_1} e^{\frac{t^2}{2}} R_{f,m+4-k_1}\left(\frac{t}{\sqrt{n}}\right) S_2 =$$
$$= S_2 \left(\sum_{8\le k_1+k_2\le m+4} \theta_{k_1}\theta_{k_2} \left(\frac{it}{\sqrt{n}}\right)^{k_1+k_2}\right) + r_2\,,$$

where $k_1, k_2 \ge 4$.

It follows from (1.15) and (2.4) that

$$|r_2| \le C_n^2 \sum_{k_1=4}^{m} |\theta_{k_1}| \left|\frac{t}{\sqrt{n}}\right|^{k_1} \left|R_{f,m+4-k_1}\left(\frac{t}{\sqrt{n}}\right)\right| \mu^{n-1}\left(\frac{t}{\sqrt{n}}\right) \le$$
$$\le \frac{1}{2} \sum_{k_1=4}^{m} |\theta_{k_1}| \left|\frac{t}{\sqrt{n}}\right|^{k_1} \left\|\theta^0_{m+6-k_1}\right\| \frac{|t|^{m+6-k_1}}{(\sqrt{n})^{m+2-k_1}} \mu^{n-1}\left(\frac{t}{\sqrt{n}}\right) =$$
$$= \frac{1}{2} \sum_{k_1=4}^{m} |\theta_{k_1}| \left\|\theta^0_{m+6-k_1}\right\| \frac{|t|^{m+6}}{(\sqrt{n})^{m+2}} \mu^{n-1}\left(\frac{t}{\sqrt{n}}\right) =$$
$$= \left\|\Theta_{2,m+6}\right\| \frac{|t|^{m+6}}{(\sqrt{n})^{m+2}} \mu^{n-1}\left(\frac{t}{\sqrt{n}}\right),$$

where

$$\left\|\Theta_{2,m+6}\right\| = \frac{1}{2} \sum_{k_1=4}^{m} |\theta_{k_1}| \left\|\theta^0_{m+6-k_1}\right\|.$$

Applying the equality

$$\sum_{k_1=4}^{m} \sum_{k_2=4}^{m+4-k_1} \theta_{k_1}\theta_{k_2} \left(\frac{it}{\sqrt{n}}\right)^{k_1+k_2} =$$
$$= \sum_{8\le k_1+k_2\le m+4} \theta_{k_1}\theta_{k_2} \left(\frac{it}{\sqrt{n}}\right)^{k_1+k_2} = \sum_{k=8}^{m+4} \Theta_{2,k} \left(\frac{it}{\sqrt{n}}\right)^k,$$

we get that

$$f^n - g^n = C_n^1 e^{-\frac{t^2}{2}} \left(\sum_{k=4}^{m} \theta_k \left(\frac{it}{\sqrt{n}}\right)^k + \theta_{m+2}^{(\lambda)} \left(\frac{it}{\sqrt{n}}\right)^{m+2} \right) +$$

$$+ S_2 \sum_{k=8}^{m+4} \Theta_{2,k} \left(\frac{it}{\sqrt{n}}\right)^k + R_0 + r_1 + r_2 \,. \tag{4.6}$$

The reasoning that leads to (4.2) allows us to write the equality

$$S_2 = C_n^2 g^n + \psi \cdot S_3 \,.$$

The last formula allows to divide the summand containing S_2 on the right part of (4.6) into two parts

$$S_2 \sum_{k=8}^{m+4} \Theta_{2,k} \left(\frac{it}{\sqrt{n}}\right)^k = C_n^2 g^n \sum_{k=8}^{m+4} \Theta_{2,k} \left(\frac{it}{\sqrt{n}}\right)^k + \psi \cdot S_3 \sum_{k=8}^{m+4} \Theta_{2,k} \left(\frac{it}{\sqrt{n}}\right)^k =$$

$$= C_n^2 g^n \sum_{k=8}^{m+4} \Theta_{2,k} \left(\frac{it}{\sqrt{n}}\right)^k + R_{3,m}^{\psi} \,.$$

Thus, (4.6) can be represented in the form

$$f^n - g^n = C_n^1 e^{-\frac{t^2}{2}} \left(\sum_{k=4}^{m} \theta_k \left(\frac{it}{\sqrt{n}}\right)^k + \theta_{m+2}^{(\lambda)} \left(\frac{it}{\sqrt{n}}\right)^{m+2} \right) +$$

$$+ C_n^2 e^{-\frac{t^2}{2}} \sum_{k=8}^{m+4} \Theta_{2,k} \left(\frac{it}{\sqrt{n}}\right)^k + R_0 + r_1 + r_2 + R_{3,m}^{\psi} \,. \tag{4.7}$$

From the inequality (1.18) for $\psi \cdot S_3$ it follows that

$$\left| R_{3,m}^{\psi} \right| = \left| \psi \cdot S_3 \sum_{k=8}^{m+4} \Theta_{2,k} \left(\frac{it}{\sqrt{n}}\right)^k \right| \le C_n^3 \zeta_3 \left|\mu\right|^{n-1} \frac{|t|^3}{n^{3/2}} \sum_{k=8}^{m+4} \left|\Theta_{2,k}\right| \left| \frac{t}{\sqrt{n}} \right|^k ,$$

and, putting $m = 4$, we have

$$\left| R_{3,4}^{\psi} \right| = \left| \psi \cdot S_3 \Theta_{2,8} \left(\frac{it}{\sqrt{n}}\right)^8 \right| \le \frac{\zeta_3 \left|\Theta_{2,8}\right|}{3!} \frac{|t|^{11}}{n^{5/2}} \left|\mu\right|^{n-1} \,.$$

Thus we get the estimate for the convergence rate of order $\frac{1}{n^{5/2}}$ and it is better than $\frac{1}{n^2}$. Clearly order of this estimate is the same for any $m \ge 4$.

So, at this step we have obtained the following asymptotic expansion

$$f^n = e^{-\frac{t^2}{2}} + C_n^1 e^{-\frac{t^2}{2}} \left(\sum_{k=4}^{m} \theta_k \left(\frac{it}{\sqrt{n}}\right)^k + \theta_{m+2}^{(\lambda)} \left(\frac{it}{\sqrt{n}}\right)^{m+2} \right) +$$

$$+ C_n^2 e^{-\frac{t^2}{2}} \sum_{k=8}^{m+4} \Theta_{2,k} \left(\frac{it}{\sqrt{n}}\right)^k + R_0 + r_1 + r_2 + R_3^{\psi} =$$

$$= e^{-\frac{t^2}{2}} + C_n^1 \theta_{m+2}^{(\lambda)} \left(\frac{it}{\sqrt{n}}\right)^{m+2} e^{-\frac{t^2}{2}} + e^{-\frac{t^2}{2}} \sum_{s=1}^{2} C_n^s \sum_{k=4s}^{m+4s-4} \Theta_{s,k} \left(\frac{it}{\sqrt{n}}\right)^k +$$

$$+R_0 + r_1 + r_2 + R_{3,m}^{\psi} \tag{4.8}$$

with the order of approximation $\frac{1}{n^2}$ for $m \geq 4$.

In the case $m = 4$, the expansion (4.8) takes the form

$$f^n = e^{-\frac{t^2}{2}} + C_n^1 e^{-\frac{t^2}{2}} \left(\theta_4 \left(\frac{it}{\sqrt{n}}\right)^4 + \theta_6^{(6,\lambda)} \left(\frac{it}{\sqrt{n}}\right)^6\right) +$$

$$+C_n^2 \theta_4^2 \left(\frac{it}{\sqrt{n}}\right)^8 e^{-\frac{t^2}{2}} + R_0 + r_1 + r_2 + R_{3,4}^{\psi}\,, \tag{4.9}$$

where

$$|r_1| \leq \left(\bar{\lambda}\alpha_6 \frac{t^6}{n^2} + \left\|\theta_8^{(6,\lambda)}\right\| \frac{t^8}{n^3}\right) \mu^{n-1}\left(\frac{t}{\sqrt{n}}\right),$$

$$|R_0| \leq \frac{\zeta_3 \left|\theta_6^{(6,\lambda)}\right|}{2} \frac{|t|^9}{n^{5/2}} \mu^{n-1}\left(\frac{t}{\sqrt{n}}\right),$$

$$|r_2| \leq \|\Theta_{2,10}\| \frac{|t|^{10}}{n^4} |t|^{10} \mu^{n-2}\left(\frac{t}{\sqrt{n}}\right),$$

$$\left|R_{3,4}^{\psi}\right| \leq \frac{\zeta_3 |\Theta_{2,8}|}{3!} \frac{|t|^{11}}{n^{5/2}} |\mu|^{n-1}.$$

The estimate of this asymptotic expansion has only one term of maximum order:

$$\bar{\lambda}\alpha_6 \frac{t^6}{n^2} \mu^{n-1}\left(\frac{t}{\sqrt{n}}\right),$$

while other's has higher order as n grows.

Hence we have verified the validity of the desired representation (3.1) for $m = 4$. The expression (4.9) (as well as (4.4)) was obtained in [11, p. 196].

The asymptotic expansion obtained below is given for the first time. To get it split $R_{3,m}^{\psi}$ into two parts. The remaining main part of asymptotic expansion remains unchanged.

4. To complete the proof use induction. It can be seen that it is enough to conduct an induction for $R_{s,m}^{\psi}$.

We will divide

$$R_{s,m}^{\psi} = \psi \cdot S_s \sum_{k=4(s-1)}^{m+4(s-2)} \Theta_{s-1,k} \left(\frac{it}{\sqrt{n}}\right)^k. \tag{4.10}$$

into two parts using (2.5). So,

$$R^{\psi}_{s,m} = S_s \sum_{k=4(s-1)}^{m+4(s-2)} \Theta_{s-1,k} \left(\frac{it}{\sqrt{n}}\right)^k \left(\sum_{k_s=4}^{m+4s-4-k} \theta_{k_s} \left(\frac{it}{\sqrt{n}}\right)^{k_s} + e^{\frac{t^2}{2}} R_{f,m+4s-4-k} \right) =$$

$$= S_s \sum_{k=4(s-1)}^{m+4(s-2)} \sum_{k_s=4}^{m+4s-4-k} \Theta_{s-1,k} \theta_{k_s} \left(\frac{it}{\sqrt{n}}\right)^{k+k_s} +$$

$$+ e^{\frac{t^2}{2}} S_s \sum_{k=4(s-1)}^{m+4(s-2)} \Theta_{s-1,k} \left(\frac{it}{\sqrt{n}}\right)^k R_{f,m+4s-4-k} =$$

$$= S_s \sum_{l=4s}^{m+4s-4} \Theta_{s,l} \left(\frac{it}{\sqrt{n}}\right)^l + r_s \,.$$

Note briefly that we used (1.8) and the equality

$$\sum_{k=4(s-1)}^{m+4(s-2)} \sum_{k_s=4}^{m+4s-4-k} \Theta_{s-1,k} \theta_{k_s} = \sum_{l=4s}^{m+4s-4} \Theta_{s,l} \,.$$

Applying (1.15) and (2.4), we have that

$$|r_s| \le C_n^s \sum_{k=4(s-1)}^{m+4(s-2)} |\Theta_{s-1,k}| \left|\frac{t}{\sqrt{n}}\right|^k \left| R_{f,m+4s-4-k}\left(\frac{t}{\sqrt{n}}\right)\right| \mu^{n-1}\left(\frac{t}{\sqrt{n}}\right) \le$$

$$\le \frac{n^s}{s!} \sum_{k=8}^{m+4} |\Theta_{s-1,k}| \left|\frac{t}{\sqrt{n}}\right|^k \left\|\theta^0_{m+4s-2-k}\right\| \frac{|t|^{m+4s-2-k}}{(\sqrt{n})^{m+4s-2-k}} \mu^{n-1}\left(\frac{t}{\sqrt{n}}\right) =$$

$$= \frac{1}{s!} \sum_{k=4(s-1)}^{m+4(s-2)} |\Theta_{s-1,k}| \left\|\theta^0_{m+4s-2-k}\right\| \frac{|t|^{m+4s-2}}{(\sqrt{n})^{m+2s-2}} \mu^{n-1}\left(\frac{t}{\sqrt{n}}\right) \le$$

$$\le \|\Theta_{s,m+4s-2}\| \frac{|t|^{m+4s-2}}{(\sqrt{n})^{m+2s-2}} \mu^{n-1}\left(\frac{t}{\sqrt{n}}\right) ,$$

where

$$\|\Theta_{s,m+4s-2}\| = \frac{1}{s!} \sum_{k=4(s-1)}^{m+4(s-2)} |\Theta_{s-1,k}| \left\|\theta^0_{m+4s-2-k}\right\| .$$

Inserting

$$S_s = C_n^s g^n + \psi \cdot S_{s+1}$$

into

$$R^{\psi}_{s,m} = S_s \sum_{l=4s}^{m+4s-4} \Theta_{s,l} \left(\frac{it}{\sqrt{n}}\right)^l + r_s$$

in light (4.10), we get

$$R^{\psi}_{s,m} = (C^s_n g^n + \psi \cdot S_{s+1}) \sum_{l=4s}^{m+4s-4} \Theta_{s,l} \left(\frac{it}{\sqrt{n}}\right)^l + r_s =$$

$$= C^s_n g^n \sum_{l=4s}^{m+4s-4} \Theta_{s,l} \left(\frac{it}{\sqrt{n}}\right)^l + r_s + R^{\psi}_{s+1,m} .$$

This completes the inductive step. It can be seen that the first summand in the right-hand side of the last inequality forms expansions of characteristic functions of convolution of distributions, the second summand is contained in the estimate of the remainder term of the expansion, and the induction step can be applied to the third summand again.

Then from (1.18) it follows that

$$\left|R^{\psi}_{\frac{m+2}{2},m}\right| \le \left|\psi \cdot S_{\frac{m+2}{2}}\right| \sum_{k=4\left(\frac{m+2}{2}-1\right)}^{m+4\left(\frac{m+2}{2}-2\right)} |\Theta_{s-1,k}| \left|\frac{t}{\sqrt{n}}\right|^k \le$$

$$\le C_n^{\frac{m+2}{2}} \zeta_3 \sum_{k=2m}^{3m-4} |\Theta_{s-1,k}| \left|\frac{t}{\sqrt{n}}\right|^{k+3} \mu^{n-1} .$$

The last quantity has order $n^{-m/2-1/2}$. Therefore, to form the main part of our expansion, it is enough to perform summation up s to $\frac{m}{2}+1$. The theorem is proved.

5. Acknowledgment

We are grateful to V.V. Senatov for the attention and support. V.V. Senatov (10.03.1951 — 22.06.2021), who made an exceptional contribution to solving several central probability problems [19]. For instance the accaracy of asymptotic expansions in CLT with explicit evaluation of accuracy that can be aproved computationally. Senatovs moment characteristics based on Chebyshev — Hermite polynomials institutes for the construction of this expansions, can be rightfully named as Senatov moments in his memory [20, 16]. We would also like to thank M.V. Kozlov who read the first version and made his comments on the translation into english.

The main results of this article were reported on XXXVI International Seminar on Stability Problems for Stochastic Models (Petrozavodsk, Karelia, Russia, June 21-25, 2021). They form the basis of the proof of the main result in the work [**?**].

It was supposed that this paper would be published in the journal Mathematical Sciences, dedicated to the XXXVI International Seminar on the Stability of Stochastic Models, but for reasons unknown to the authors, its release is postponed. In this regard, the authors thank I.G. Shevtsova for the attention to their work, despite the fact that the article publishes it in this journal.

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V. N. Sobolev: MSU, Moscow, Russia
Email address: `sobolev_vn@mail.ru`

A. E. Condratenko: MSU, Moscow, Russia
Email address: `ae_cond@mech.math.msu.su`