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METHOD OF ITERATIVE EXTENSIONS FOR ANALYSIS OF SCREENED POISSON EQUATION IN THREE-DIMENSIONAL SPHERICAL DOMAIN

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Abstract. Screened Poisson equation is a type of elliptic equation that describes stationary systems. Numerical modelling solves it accurately and enables computational analysis. Despite the growth of computing power, numerical analysis of complex mathematical problems still requires development of optimal methods. Aim of this study is to develop asymptotically optimal method for analysis of mixed boundary value problem for screened Poisson equation in three-dimensional domains with complex geometry. The problem in domain with complex geometry is reduced to problem in simpler domain with fictitious domain method, and then discretized by system of linear algebraic equations. Method of iterative extensions is used to solve the system with optimal asymptotic in terms of number of operations. Algorithm of this method is presented in the paper, programmed with C++, and tested. Developed method has linear complexity in terms of number of operations and can be used for effective numerical analysis.

1. Introduction

The study of boundary value problems for elliptic equations remains an essential area of mathematical physics and numerical analysis as these equations emerge during analysis of stationary physical systems in mechanics, hydrodynamics, and electrical engineering. Numerical analysis of such systems requires development of efficient numerical methods while maintaining numerical stability and accuracy. Complexity of analysing boundary value problems depends on type of equation, domain of analysis itself, and its boundary conditions. A number of researchers, such as Aubin, Bank, have studied approximate solutions for elliptic equations [2, 3]. Aubin has done a fundamental theoretical work regarding approximations of various functional spaces. Afterwards, Bank has developed optimal marching algorithms for solving linear systems, which approximate elliptic equations.

The particular interest of this study lies in mixed boundary value problem for screened Poisson equation (SPE) in three-dimensional domains with complex geometry. SPE is a type of elliptic equation, which generalizes Poisson equation by introducing a screening parameter. In addition, we introduce Dirichlet and

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Neumann boundary conditions to describe unknown function on boundaries of domain. Dirichlet condition defines function values on boundary while Neumann condition defines normal derivatives on boundary.

Well-known methods, such as marching algorithms, are not applicable in domains with complex geometry. Numerous approaches have been studied to reduce complexity of the problem, such as domain decomposition [6], finite element method [4], fictitious domain method [7], etc. This study focuses on fictitious domain method (FDM), which consists of extending a domain with complex geometry to a simpler domain. This method simplifies discretization and allows employing of fast computational algorithms.

Existing research in this area includes the works of Astrakhantsev, Kaporin, and Matsokin who have studied and optimized the FDM for elliptic problems with Neumann or Dirichlet boundary condition [1, 9, 8, 10, 11]. This methodology improved further by adding a penalty term for correction of boundary values [5, 12]. In recent works of Ushakov [13, 14, 15], method of iterative extensions has been proposed, which generalizes fictitious domain method of Astrakhantsev for Poisson equations and achieves asymptotically optimal results in terms of number of required arithmetic operations.

Nonetheless, existing studies focus only on planar domains or consider boundary value problems with only one type of boundary condition, whereas practical problems can involve three-dimensional domains and mixed boundary constraints. In addition, implementation of FDM for SPE is understudied, as researchers usually consider its implementation only for Poisson equation.

The present research extends method of iterative extensions to asymptotically optimal analysis of mixed boundary value problem for SPE in three-dimensional domain with complex geometry. The proposed method includes formulation of continuation of the problem, its discretization into a linear system, and construction of an iterative process with asymptotically optimal convergence rate.

Thus, the study advances numerical analysis of boundary value problems for SPE. It extends method of iterative extensions to three-dimensional case, establishes asymptotic optimality for discrete extended system, and provides a computationally efficient algorithm for mixed boundary conditions. Achieved results contribute to general theory of elliptic boundary value problems and to practical development of high-performance methods in mathematical modelling of physical systems.

2. Boundary value problem for screened Poisson equation in three-dimensional spherical domain

The aim of this study is to develop asymptotically optimal method for analysis of mixed boundary value problem for SPEs in three-dimensional domain with complex geometry. In this section, we formulate the problem, domains, boundaries, and boundary conditions. Then, the initial problem is continued and discretized into a system of linear algebraic equations. Finally, we introduce iterative process for solving the system and consider convergence of this process.

2.1. Continuation of boundary value problem for screened Poisson equation. We consider following three-dimensional domains

$$\Omega_\omega, \Pi \subset \mathbb{R}^3,\ \omega \in \{1, \mathrm{II}\},$$
$$\Omega_1 \cap \Omega_{\mathrm{II}} = \emptyset,\ \overline{\Omega}_1 \cup \overline{\Omega}_{\mathrm{II}} = \overline{\Pi}.$$

Boundaries of each of these domains consist of closure of union of open disjoint parts

$$\partial\Pi = \overline{s},\ s = \Gamma_1 \cup \Gamma_2,\ \Gamma_1 \cap \Gamma_2 = \emptyset,$$
$$\partial\Omega_1 = \overline{s}_1, \partial\Omega_{\mathrm{II}} = \overline{s}_{\mathrm{II}}, s_1 = \Gamma_{1,1}, s_{\mathrm{II}} = \Gamma_{\mathrm{II},1} \cup \Gamma_{\mathrm{II},2}, \Gamma_{\mathrm{II},1} \cap \Gamma_{\mathrm{II},2} = \emptyset,$$
$$\partial\Omega_1 \cap \partial\Omega_{\mathrm{II}} = \overline{S}, S = \Gamma_{1,1} \cap \Gamma_{1,2} = \emptyset.$$

In this study we consider Ω_1 as the open ball and Π as the open cube in Cartesian coordinate system

$$\Omega_1 = \left\{(x; y; z) : x^2 + y^2 + z^2 < r^2\right\},\ r > 0,$$
$$\Pi = (-2r; 2r) \times (-2r; 2r) \times (-2r; 2r),$$
$$\Omega_{\mathrm{II}} = \Pi \backslash \overline{\Omega}_1.$$

Parts of boundaries of these domains are specified

$$\Gamma_{1,1} = \left\{(x; y; z) : x^2 + y^2 + z^2 = r^2\right\},$$
$$\Gamma_{\mathrm{II},1} = \{2r\} \times (-2r; 2r) \times (-2r; 2r) \cup (-2r; 2r) \times \{2r\} \times (-2r; 2r) \cup$$
$$\cup (-2r; 2r) \times (-2r; 2r) \times \{2r\},$$
$$\Gamma_{\mathrm{II},2} = \{-2r\} \times (-2r; 2r) \times (-2r; 2r) \cup (-2r; 2r) \times \{-2r\} \times (-2r; 2r) \cup$$
$$\cup (-2r; 2r) \times (-2r; 2r) \times \{-2r\} \cup \left\{(x; y; z) : x^2 + y^2 + z^2 = r^2\right\},$$
$$\Gamma_1 = \{2r\} \times (-2r; 2r) \times (-2r; 2r) \cup (-2r; 2r) \times \{2r\} \times (-2r; 2r) \cup$$
$$\cup (-2r; 2r) \times (-2r; 2r) \times \{2r\},$$
$$\Gamma_2 = \{-2r\} \times (-2r; 2r) \times (-2r; 2r) \cup (-2r; 2r) \times \{-2r\} \times (-2r; 2r) \cup$$
$$\cup (-2r; 2r) \times (-2r; 2r) \times \{-2r\}.$$

In Ω_1 boundary value problem for SPE with homogenous Dirichlet condition is considered

$$\begin{cases} -\Delta \check{u}_1 + \kappa_1 \check{u}_1 = \check{f}_1\ in\ \Omega_1, \\ \check{u}_1\big|_{\Gamma_{1,1}} = 0,\ \kappa_1 > 0, f_1 \in L_2(\Omega_1). \end{cases} \tag{2.1}$$

In Ω_{II} mixed boundary value problem for SPE with homogenous Dirichlet and Neumann conditions is considered

$$\begin{cases} -\Delta \check{u}_{\mathrm{II}} + \kappa_{\mathrm{II}} \check{u}_{\mathrm{II}} = \check{f}_{\mathrm{II}}\ in\ \Omega_{\mathrm{II}}, \\ \check{u}_{\mathrm{II}}\big|_{\Gamma_{\mathrm{II},1}} = 0,\ \frac{\partial \check{u}_{\mathrm{II}}}{\partial \vec{n}}\big|_{\Gamma_{\mathrm{II},2}} = 0,\ \kappa_{\mathrm{II}} > 0, f_{\mathrm{II}} \in L_2(\Omega_{\mathrm{II}}). \end{cases} \tag{2.2}$$

The problem (2.1) is initial problem and the problem (2.2) is fictitious problem with zero solution and zero function in right-hand side of equation.

Initial and fictitious problems can be formulated in variational form

$$\check{u}_\omega \in \check{H}_\omega \colon A_\omega(\check{u}_\omega, \check{v}_\omega) = F_\omega(\check{v}_\omega)\ \forall \check{v}_\omega \in \check{H}_\omega,\ \omega \in \{1, \mathrm{II}\}, \tag{2.3}$$

where $\check{u}_\omega$ is a function from Sobolev space

$$\check{H}_\omega = \check{H}_\omega(\Omega_\omega) = \left\{\check{v}_\omega \in W_2^1(\Omega_\omega) \colon \check{v}_\omega\big|_{\Gamma_{\omega,1}} = 0\right\}.$$

In the problem (2.3) left-hand side is the bilinear form and right-hand side is the linear form

$$A_\omega(\check{u}_\omega, \check{v}_\omega) = \int\limits_{\Omega_\omega} (\check{u}_{\omega_x}\check{v}_{\omega_x} + \check{u}_{\omega_y}\check{v}_{\omega_y} + \check{u}_{\omega_z}\check{v}_{\omega_z} + k_\omega \check{u}_\omega \check{v}_\omega) d\Omega_\omega,\ \omega \in \{1, \mathrm{II}\},$$

$$F_\omega(\check{v}_\omega) = (\check{f}_\omega, \check{v}_\omega) = \int\limits_{\Omega_\omega} \check{f}_\omega \check{v}_\omega d\Omega_\omega.$$

According to Lions–Lax–Milgram theorem, unique solution of the problem (2.3) exists, and these inequalities take place, see [2]

$$\exists c_1, c_2 > 0\colon c_1 \|\check{v}_\omega\|^2_{W_2^1(\Omega_\omega)} \le A_\omega(\check{v}_\omega, \check{v}_\omega) \le c_2 \|\check{v}_\omega\|^2_{W_2^1(\Omega_\omega)}\ \forall \check{v}_\omega \in \check{H}_\omega.$$

This guarantees existence and uniqueness of the solution with any positive κ_ω and any combination of boundary conditions.

Next, we continue the initial problem to simpler domain with FDM

$$\check{u} \in \check{V}\colon A(\check{u}, \check{v}) = A_1(\check{u}, I_1\check{v}) + A_{\mathrm{II}}(\check{u}, \check{v}) = F_1(I_1\check{v})\ \forall \check{v} \in \check{V}, \tag{2.4}$$

where $\check{V}$ is extended space of solutions of the problem (2.4)

$$\check{V} = \check{V}(\Pi) = \big\{\check{v} \in W_2^1(\Pi)\colon \check{v}\big|_{\Gamma_1} = 0\big\},$$

and I_1 is a projection operator from extended space of solutions onto subspace of solutions of initial problem (2.1), which nullifies function values outside Ω_1

$$I_1\colon \check{V} \mapsto \check{V}_1,\ \check{V}_1 = \mathrm{im} I_1,\ I_1 = I_1^2,$$

$$\check{V}_1 = \check{V}_1(\Pi) = \big\{\check{v}_1 \in \check{V}\colon \check{v}_1\big|_{\Pi\setminus\Omega_1} = 0\big\}.$$

Similarly with the problem (2.3), unique solution of the problem (2.4) exists, and these inequalities take place

$$\exists c_1, c_2 > 0\colon c_1 \|\check{v}\|^2_{W_2^1(\Pi)} \le A(\check{v}, \check{v}) \le c_2 \|\check{v}\|^2_{W_2^1(\Pi)}\ \forall \check{v} \in \check{V}.$$

The problem (2.4) is equivalent to next mixed boundary value problem for SPE

$$\begin{cases} -\Delta\check{u} + \kappa\check{u} = \check{f}\ in\ \Pi, \\ \check{u}\big|_{\Gamma_1} = 0,\ \frac{\partial\check{u}}{\partial\vec{n}}\big|_{\Gamma_2} = 0,\ \kappa > 0, f \in L_2(\Pi). \end{cases} \tag{2.5}$$

The problem (2.5) is continued problem, and its solution is solution of the initial problem (2.1) in Ω_1 continued by zero in Ω_{II}. Thus, we replace initial problem in domain Ω_1 with complex geometry by continued problem in domain Π with simpler geometry.

2.2. Approximation of continued boundary value problem for screened Poisson equation. Continued problem (2.5) is approximated with finite difference method. Grid is introduced with following nodes

$$(x_i; y_j; z_p) = \big((i-2m-1)h; (j-2m-1)h; (p-2m-1)h\big),\ h = 2r/(2m+1), h \ll r,$$

$$i, j, p = 0, \ldots, 4m+2,\ m-3 \in \mathbb{N}.$$

Note that these equalities take place

$$x_0 = y_0 = z_0 = -2r,$$

$$x_1 = y_1 = z_1 = -2r + h, (x_0 + x_1)/2 = (y_0 + y_1)/2 = (z_0 + z_1)/2 = -2r + h/2,$$
$$x_m = y_m = z_m = -r - h/2, x_{m+1} = y_{m+1} = z_{m+1} = -r + h/2,$$
$$(x_m + x_{m+1})/2 = (y_m + y_{m+1})/2 = (z_m + z_{m+1})/2 = -r,$$
$$x_{2m+1} = y_{2m+1} = z_{2m+1} = 0,$$
$$x_{3m+1} = y_{3m+1} = z_{3m+1} = r - h/2, x_{3m+2} = y_{3m+2} = z_{3m+2} = r + h/2,$$
$$(x_{3m+1} + x_{3m+2})/2 = (y_{3m+1} + y_{3m+2})/2 = (z_{3m+1} + z_{3m+2})/2 = r,$$
$$x_{4m+2} = y_{4m+2} = z_{4m+2} = 2r.$$

The equation in continued problem (2.5) is replaced by a system of differences

$$\frac{6u_{i,j,p} - u_{i-1,j,p} - u_{i+1,j,p} - u_{i,j-1,p} - u_{i,j+1,p} - u_{i,j,p-1} - u_{i,j,p+1}}{h^3} + \kappa_1 u_{i,j,p} =$$
$$= f_{i,j,p} \ if \ u_{i,j,p} \in \Omega_1,$$
$$\frac{6u_{i,j,p} - u_{i-1,j,p} - u_{i+1,j,p} - u_{i,j-1,p} - u_{i,j+1,p} - u_{i,j,p-1} - u_{i,j,p+1}}{h^3} + \kappa_{\mathrm{II}} u_{i,j,p} =$$
$$= f_{i,j,p} \ if \ u_{i,j,p} \in \Omega_{\mathrm{II}}.$$

Grid functions are considered as vectors on set of nodes of the grid

$$v_{i,j,p} = \overline{v}(x_i; y_j; z_p) \in \mathbb{R}, \ i, j, p = 0, \ldots, 4m+2, \ m-3 \in \mathbb{N}.$$

Components $v_{i,j,p}$ of vector $\overline{v}$ are distinguished in three blocks. Components corresponding to nodes of the grid are numerated first, when

$$[x_{i-1}; x_{i+1}] \times [y_{j-1}; y_{j+1}] \times [z_{p-1}; z_{p+1}] \subset \overline{\Omega}_1.$$

Components corresponding to nodes of the grid are numerated second, when

$$[x_{i-1}; x_{i+1}] \times [y_{j-1}; y_{j+1}] \times [z_{p-1}; z_{p+1}] \cap \Omega_1 = \emptyset \wedge$$
$$\wedge [x_{i-1}; x_{i+1}] \times [y_{j-1}; y_{j+1}] \times [z_{p-1}; z_{p+1}] \cap \Omega_{\mathrm{II}} = \emptyset.$$

Components corresponding to nodes of the grid are numerated third, when

$$[x_{i-1}; x_{i+1}] \times [y_{j-1}; y_{j+1}] \times [z_{p-1}; z_{p+1}] \subset [-2r; 2r] \times [-2r; 2r] \times [-2r; 2r] \backslash \Omega_1.$$

Then, the result of approximation is the system of linear algebraic equations in matrix form

$$\overline{u} \in \mathbb{R}^N \colon B\overline{u} = \overline{f}, \ \overline{f} \in \mathbb{R}^N, N = (4m+1)(4m+1)(4m+1), \tag{2.6}$$
$$B = \begin{bmatrix} A_{11} & A_{12} & 0 \\ 0 & A_{02} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix}, \overline{u} = \begin{bmatrix} \overline{u}_1 \\ \overline{0} \\ \overline{0} \end{bmatrix}, \overline{f} = \begin{bmatrix} \overline{f}_1 \\ \overline{0} \\ \overline{0} \end{bmatrix}.$$

Here, initial problem is approximated by system

$$\overline{u}_1 \in \mathbb{R}^M \colon A_{11}\overline{u}_1 = \overline{f}_1, \ \overline{f}_1 \in \mathbb{R}^M, 1 \le M \le (2m-1)(2m-1)(2m-1), m-3 \in \mathbb{N}.$$

And fictitious problem is approximated by system

$$\begin{bmatrix} A_{02} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \overline{u}_2 \\ \overline{u}_3 \end{bmatrix} = \begin{bmatrix} \overline{f}_2 \\ \overline{f}_3 \end{bmatrix}, \ \begin{bmatrix} \overline{f}_2 \\ \overline{f}_3 \end{bmatrix} = \begin{bmatrix} \overline{0} \\ \overline{0} \end{bmatrix}, \ \begin{bmatrix} \overline{u}_2 \\ \overline{u}_3 \end{bmatrix} = \begin{bmatrix} \overline{0} \\ \overline{0} \end{bmatrix}.$$

2.3. Asymptotically optimal analysis of continued boundary value problem for screened Poisson equation. Iterative process for solving system (2.6) uses method of iterative extensions

$$\overline{u}^k \in \mathbb{R}^N\colon C(\overline{u}^k - \overline{u}^{k-1}) = -\tau_{k-1}(B\overline{u}^{k-1} - \overline{f}), \tag{2.7}$$

$$\tau_{k-1} = \langle \overline{r}^{k-1}, \overline{\eta}^{k-1}\rangle / \langle \overline{\eta}^{k-1}, \overline{\eta}^{k-1}\rangle,\ \tau_0 = 1,\ \overline{u}^0 = \overline{0} \in \mathbb{R}^N$$

$$\overline{r}^{k-1} = B\overline{u}^{k-1} - \overline{f},\ \overline{w}^{k-1} = C^{-1}\overline{r}^{k-1},\ \overline{\eta}^{k-1} = B\overline{w}^{k-1},\ k \in \mathbb{N}.$$

Here, matrix C is extended matrix defined as a sum of matrices A_{I} and A_{II}, multiplied by additional positive parameter γ

$$C = A_{\mathrm{I}} + \gamma A_{\mathrm{II}},\ A_{\mathrm{I}} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{20} & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A_{\mathrm{II}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{02} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix},\ \gamma > 0,$$

and norm is

$$\|\overline{v}\|_{C^2} = \sqrt{\langle C^2\overline{v}, \overline{v}\rangle} \quad \forall \overline{v} \in \mathbb{R}^N.$$

We interpret method of iterative extensions as a generalization of FDM, when additional positive parameter is introduced in the definition of extended matrices. In this method, iterative parameters are selected using error minimization in a stronger norm than energy norm of emerging problem, i.e. minimum residual method is used instead of method of steepest descent. In the method of iterative extensions, sequence of relative errors converge with a geometric progression rate in a stronger norm than the energy norm, see [14]

$$\|\overline{u}^k - \overline{u}\|_{C^2} \le \varepsilon \|\overline{u}^0 - \overline{u}\|_{C^2},\ \varepsilon = 2(\gamma_2/\gamma_1)(\alpha/\gamma)^{k-1},\ k \in \mathbb{N}.$$

We conduct computational experiment to demonstrate efficiency of the method. Let $\check{u}_1 = \check{u}_1(x;y;z)$ be temperature field distribution in ball of radius $r > 0$ with thermal conductivity $T_1 > 0$, cooling rate performance $K_1 > 0$, constant zero temperature on boundary, and heat-flow density $P_1(x;y;z)$. Unknown function $\check{u}_1$ is derived by solving boundary value problem for SPE

$$\begin{cases} -T_1\Delta\check{u}_1 + K_1\check{u}_1 = P_1 \ in\ \Omega_1, \\ \check{u}_1\big|_{\Gamma_{1,1}} = 0, \end{cases}$$

where T_1 and K_1 are known values, and P_1 is known function

$$P_1(x;y;z) = 6T_1 + K_1(1 - x^2 - y^2 - z^2).$$

Also, we assume that

$$f_1 = P_1/T_1, \kappa_1 = K_1/T_1,$$

then the above boundary value problem is exactly problem (2.1). Note that analytical solution of this problem is

$$\check{u}_1(x;y;z) = 1 - x^2 - y^2 - z^2.$$

3. Results and Discussion

To achieve the aim of this study, initial problem (2.1) was continued to a simpler domain. Then, the continued problem (2.5) was discretized into linear system. Iterative process for solving the system was introduced as well as the numerical experiment to demonstrate effectiveness of this process.

The method of iterative extensions was implemented as an algorithm for an approximate solution of emerged problem after fictitious continuation and approximation applied to it. Choice of iteration parameters is automated, and stop criterion of this process is given parameter. Algorithm consists of 9 steps:

(1) Calculate norm value ε_0 of initial absolute error

$$\varepsilon_0 = \langle \overline{f}, \overline{f} \rangle.$$

(2) Calculate initial approximation $\overline{u}_1$

$$\overline{u}^1 \in \mathbb{R}^N \colon C\overline{u}^1 = \overline{f}.$$

(3) Calculate residuals $\overline{r}^{k-1}$

$$\overline{r}^{k-1} = B\overline{u}^{k-1} - \overline{f} = A_{\mathrm{II}}\overline{u}^{k-1},\ k \in \mathbb{N}\backslash\{1\}.$$

(4) Calculate norm value of current absolute error

$$\varepsilon_{k-1} = \langle \overline{r}^{k-1}, \overline{r}^{k-1} \rangle,\ k \in \mathbb{N}\backslash\{1\}.$$

(5) Calculate corrections $\overline{w}^{k-1}$

$$\overline{w}^{k-1} \in \mathbb{R}^N \colon C\overline{w}^{k-1} = \overline{r}^{k-1},\ k \in \mathbb{N}\backslash\{1\}.$$

(6) Calculate equivalent residuals $\overline{\eta}^{k-1}$

$$\overline{\eta}^{k-1} = B\overline{w}^{k-1} = A_{\mathrm{II}}\overline{w}^{k-1},\ k \in \mathbb{N}\backslash\{1\}.$$

(7) Calculate iterative parameter τ_{k-1}

$$\tau_{k-1} = \langle \overline{r}^{k-1}, \overline{\eta}^{k-1} \rangle / \langle \overline{\eta}^{k-1}, \overline{\eta}^{k-1} \rangle,\ k \in \mathbb{N}\backslash\{1\}.$$

(8) Calculate next approximation $\overline{u}^k$

$$\overline{u}^k = \overline{u}^{k-1} - \tau_{k-1}\overline{w}^{k-1},\ k \in \mathbb{N}\backslash\{1\}.$$

(9) Check stop criterion of iterative process.

$$\varepsilon_{k-1} \le \varepsilon^2 \varepsilon_0,\ k \in \mathbb{N}\backslash\{1\},\ \varepsilon \in (0;1).$$

If condition is not satisfied, repeat iterative process starting with step 3.

This algorithm was implemented with C++ and tested on numerical problem. In series of experiments, number of iterations of algorithm, number of iterations for process in steps 2 and 5, and maximum relative error were measured with variable parameters:

- thermal conductivity T_1;
- cooling rate performance K_1;
- screening parameter κ_{II} on fictitious domain;
- size of approximation grid m.

Results of experiments were presented in graphs. Fig. 1a shows the number of required iterations for solving systems emerged in steps 2 and 5 as the function of screening parameter κ_{II} in three different scenarios: $T_1 \sim K_1$, $T_1 \ll K_1$, and $T_1 \gg K_1$. Fig. 1b displays the number of iterations of developed algorithm as the function of screening parameter κ_{II}. Increasing of screening parameter reduces required number of iterations to two. Fig. 1c illustrates the number of iterations of algorithm as the function of size of approximation grid with a high absolute value of parameter κ_{II}. Fig. 1d demonstrates maximum relative error between analytical and approximate solution as the function of approximation grid size m with a high absolute value of parameter κ_{II}. In this experiment, increasing the grid size by three times reduced the error by two times.

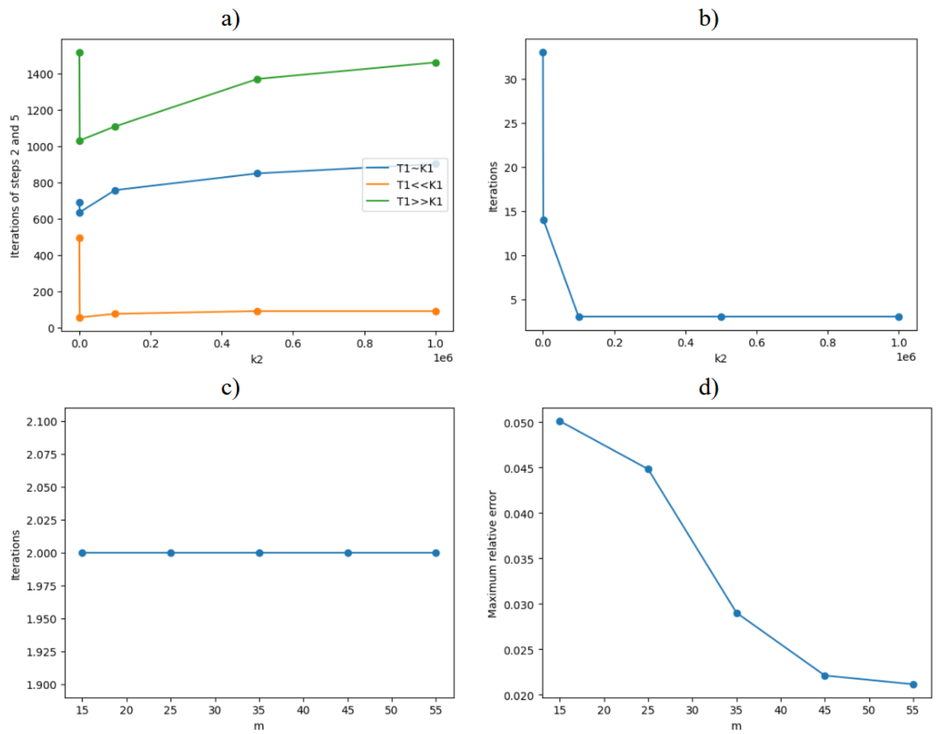

FIGURE 1. Graphs of experimantal results.

Obtained results indicate the significance of value of fictitious parameter κ_{II} in reducing the amount of required iterations of developed algorithm. This parameter is interpreted as penalty term that restricts values of vector $\overline{u}^k$. Similar results have been obtained in [12, 13, 14, 15] where penalty term was introduced as parameter of fictitious domain or additional term in error minimization problem. But in the case of SPE, penalty term comes naturally from the equation. However, increasing parameter κ_{II} also increases the amount of required iterations for solving systems emerged in steps 2 and 5. This fact indicates that increasing fictitious parameter also increases the condition number of the system.

As can be seen for figures 1c and 1d, grid size only affected the maximum relative error between analytical and approximate solution, and the number of required iterations of algorithm remained constant. This finding is consistent with a theory of method of iterative extensions [13, 14, 15] and demonstrates that the developed algorithm is asymptotically optimal in terms of number of required operations.

Proposed method extends method of iterative extensions to three-dimensional case and contributes to general theory of elliptic boundary value problems. Developed asymptotically optimal algorithm reduces computational cost of modelling of stationary physical systems and can be used as effective tool for analysis of mixed boundary value problem for SPE in domains with complex geometry.

4. Conclusion

The primary objective of this study was to develop asymptotically optimal method for effective analysis of mixed boundary value problems for SPE in three-dimensional domains with complex geometry. Algorithm of this method was constructed as an iterative process with automated parameter selection, and its asymptotic optimality was demonstrated through computational experiments.

The results are applicable to numerical modelling of stationary physical systems, including heat transfer, electrostatics, and diffusion models in domains with complex geometry. They offer an efficient computational tool for engineering and scientific applications requiring robust solvers for mixed boundary conditions.

As this study focused only on reducing computational cost of analysis, it does not consider preserving accuracy during computations, especially with curvilinear boundaries. In addition, proposed method is limited to homogeneous boundary conditions. An important next step will be to extend the method to inhomogeneous boundary conditions.

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