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THE STABILIZATION OF SOLUTIONS FOR THE WENTZELL STOCHASTIC EVOLUTION SYSTEM IN A CIRCLE AND ON ITS BOUNDARY

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Abstract. The paper considers the problem of stabilizing the solutions of the deterministic and stochastic Wentzell equations, which describe the evolution of a liquid in a circle and on its boundary. The authors address the issue of exponential stability and instability of the deterministic Wentzell equations solutions. They consider different signs of the parameters that describe the medium and the properties of the liquid. The instability gives rise to solving the problem of stabilization using a feedback loop. The obtained results are used in the stochastic Wentzell equations. The Nelson–Gliklikh derivative is considered, and a stochastic process is a solution.

Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a manifold with edge Γ. The system of two Dzekzer equations [1], which describing the evolution of the free surface of a filtered liquid is defined on the compact $\Omega \cup \Gamma$

$$(\lambda - \Delta)u_t = \alpha_0 \Delta u - \beta_0 \Delta^2 u - \gamma_0 u,\ u = u(t,x),\ (t,x) \in \mathbb{R} \times \Omega, \tag{0.1}$$

$$(\lambda - \Delta)v_t = \alpha_1 \Delta v - \beta_1 \Delta^2 u + \frac{\partial u}{\partial \nu} - \gamma_1 v,\ v = v(t,x),\ (t,x) \in \mathbb{R} \times \Gamma, \tag{0.2}$$

$$\text{tr } u = v,\ \text{ on } \mathbb{R} \times \Gamma. \tag{0.3}$$

Here the symbol Δ in (0.1) denotes the Laplace – Beltrami operator on the smooth Riemannian manifold Ω, and in (0.2) the same symbol denotes the Laplace – Beltrami operator on the smooth Riemannian manifold Γ. The symbol $\nu = \nu(t,x),\ (t,x) \in \mathbb{R} \times \Gamma$, denotes the external normal to $\mathbb{R} \times \Omega$. The parameters $\alpha_0, \alpha_1, \lambda, \beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{R}$ characterize the medium.

We will study the stabilizing the solutions of the system (0.1), (0.2) in the case: $\Omega = \{(r,\theta) : r \in [0,R), \varphi \in [0,2\pi)\}$ is a circle in $\mathbb{R}^2$, and $\Gamma = \{\theta : \theta \in [0,2\pi)\}$ is the edge of the circle. In this case (0.1), (0.2) is transformed to the form

$$(\lambda - \Delta_{r,\theta})u_t = \alpha_0 \Delta_{r,\theta} u - \beta_0 \Delta^2_{r,\theta} u - \gamma_0 u,\ u = u(t,r,\theta),\ (t,r,\theta) \in \mathbb{R} \times \Omega, \tag{0.4}$$

$$(\lambda - \Delta_\theta)v_t = \alpha_1 \Delta_\theta v - \beta_1 \Delta^2_\theta u + \partial_\theta u - \gamma_1 v,\ v = v(t,\theta),\ (t,\theta) \in \mathbb{R} \times \Gamma, \tag{0.5}$$

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where the Laplace – Beltrami operator $\Delta_{r,\theta}$ on the circle and the Laplace – Beltrami operator Δ_θ on the edge of the circle have the following form

$$\begin{aligned} \Delta_{\theta,\varphi} &= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}, \\ \Delta_\varphi &= \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}, \quad \partial_\theta = \frac{\partial}{\partial\theta}\bigg|_{\theta=\frac{\pi}{2}}. \end{aligned} \tag{0.6}$$

To this system we add the matching condition (0.3) and equip it with initial conditions

$$u(0,r,\theta) = u_0(r,\theta), \quad v(0,\theta) = v_0(\theta). \tag{0.7}$$

Let us call the solution of the problem (0.3) – (0.7) the deterministic solution of the Wentzell system. If we replace the functions u and v, defined by Ω and Γ respectively, on $\eta = \eta(t)$ and $\kappa = \kappa(t)$ are a stochastic processes on the interval $(0,\tau)$, we obtain stochastic Wentzell system, where the derivative of stochastic processes is understand by the Nelson – Gliklikh derivative [2] of the process. This paper is devoted to studying the stability of solutions of a stochastic evolutionary system of Wentzel equations in a circle and on its boundary, as well as solving the stabilization problem for unstable solutions of this system.

In addition to the introduction and the list of references, the work consists of four parts. The first part examines the existence and uniqueness of a deterministic evolutionary system. Wentzell in the circle and on its border. The second part contains the stability of the obtained solution for the deterministic evolutionary Wentzell system. The third part contains the stabilization of the solution for the deterministic evolutionary Wentzell system. The fourth part contains the stabilization of the solution for the stochastic Wentzell evolutionary system.

1. Analytical solution of a deterministic Wentzell system

We give an analytical study for the corresponding system (0.4)–(0.7) following the results in [3]. In this case, let us consider the following series

$$\begin{aligned} u = \sum_{k=2}^{\infty} \exp\left(t\frac{-\beta_0 k^4 - \alpha_0 k^2 - \gamma_0}{\lambda + k^2}\right)\frac{(R-r)^k}{2R^k}\left(a_k\cos k\theta + b_k \sin k\theta\right) + \\ + \sum_{k=1}^{\infty} \exp\left(t\frac{-\beta_0 k^4 - \alpha_0 k^2 - \gamma_0}{\lambda + k^2}\right)\left(c_k\cos k\theta + d_k\sin k\theta\right), \end{aligned} \tag{1.1}$$

where

$$a_k = \int_0^R \int_0^{2\pi} u_0(r,\theta) \frac{(R-r)^k}{2R^k} \cos k\varphi d\theta r dr,$$

$$b_k = \int_0^R \int_0^{2\pi} u_0(r,\theta) \frac{(R-r)^k}{2R^k} \sin k\varphi d\theta r dr,$$

$$c_k = \int_0^{2\pi} v_0(\theta) \cos k\varphi d\theta, \;\; d_k = \int_0^{2\pi} v_0(\theta) \sin k\varphi d\theta.$$

It is easy to see that the above series is a formal solution to the equation (0.4). Moreover, if the rows in (1.1) converge evenly, then we have a solution to the problem (0.4), (0.7), where $\partial_R u = 0$. Taking this into account, it is possible to construct a solution to the problem (0.6), (0.7) on the edge

$$v = \sum_{k=1}^{\infty} \exp\left(t \frac{-\beta_1 k^4 - \alpha_1 k^2 - \gamma_1}{\lambda + k^2}\right)\left(c_k \cos k\theta + d_k \sin k\theta\right), \tag{1.2}$$

where in the case of $\alpha_0 = \alpha_1$, $\beta_0 = \beta_1$, $\gamma_0 = \gamma_1$ solutions to the (0.4) – (0.7) problem will satisfy the (0.3) matching condition.

Closure of span$\{(R^k)^{-1}(R-r)^k \cos k\varphi, (R^k)^{-1}(R-r)^k \sin k\varphi : k \in \mathbb{N}\backslash\{1\}, r \in (0,R), \theta \in [0,2\pi)\}$ generated by the dot product

$$\langle \varphi, \psi \rangle_{A(\Omega)} = \int_0^{2\pi} \int_0^R \varphi(r,\theta)\psi(r,\theta) r dr d\theta,$$

is denoted by the symbol $A(\Omega)$. Next, the closure of span$\{\cos k\varphi,\ \sin k\varphi, \theta \in [0,2\pi)\}$ by the norm generated by the dot product

$$\langle \varphi, \psi \rangle_{A(\Gamma)} = \int_0^{2\pi} \varphi(\theta)\psi(\theta) d\theta,$$

is denoted by $A(\Gamma)$. Thus, the following theorem holds.

Theorem 1.1. *For any $u_0 \in A(\Omega)$ and $v_0 \in A(\Gamma)$ such that the matching condition is met, and for coefficients $\alpha_0, \alpha_1 \beta_0, \beta_1, \gamma_0, \gamma_1, \lambda \in \mathbb{R}$, such that the following condition is met $\alpha_0 = \alpha_1$, $\beta_0 = \beta_1$, $\gamma_0 = \gamma_1$, and $\lambda \neq k^2$, where $k \in \mathbb{N}$, there is a unique solution to $(u,v) \in C^\infty(\mathbb{R}; A(\Omega) \oplus A(\Gamma))$ a Wentzell evoluation system consisting of the equations* (0.4)–(0.7).

2. The stability of solutions of a deterministic Wentzell's system

Let $\mathfrak{U} = A(\Omega) \oplus A(\Gamma)$. Norm $\|\cdot\|_{\mathfrak{U}} = \|\cdot\|_{A(\Omega)} + \|\cdot\|_{A(\Gamma)}$, where $\|\cdot\|_{A(\Omega)}$ and $\|\cdot\|_{A(\Gamma)}$ are generated by scalar products in $A(\Omega)$ and $A(\Gamma)$.

Definition 2.1. Solution $u = u(t)$ of the (0.4)–(0.7) system *is exponentially stable* if there are constants $N > 0$ and $\nu > 0$ such that for all $t \in \mathbb{R}_+$ and any $u_0 \in A(\Omega)$,

$v_0 \in A(\Gamma)$ The solution $u = u(t)$ of the (0.4)–(0.7) system satisfies exponential estimation

$$\|u(t)\|_{\mathfrak{U}} \leq Ne^{-\nu t}(|u_0\|_{A(\Omega)} + \|v_0\|_{A(\Gamma)}). \tag{2.1}$$

Definition 2.2. Solution $u = u(t)$ of the (0.4)–(0.7) system is called *unstable* if it exists $\varepsilon > 0$ for some $u_0 \in A(\Omega)$, $v_0 \in A(\Gamma)$ and $t > 0$ fulfilled

$$\|u(t) - u_0\|_{A(\Omega)} + \|u(t) - v_0\|_{A(\Gamma)} \geq \varepsilon.$$

Theorem 2.3. *Let $\alpha_0,\ \beta_0,\ \gamma_0 > 0$, $\beta_0 > -\dfrac{\alpha_0}{k^2} - \dfrac{\gamma_0}{k^4}$ and $\lambda > -1$. Then for any $u_0 \in A(\Omega)$, $v_0 \in A(\Gamma)$ the solution of the (0.4)–(0.7) system is exponentially stable.*

Indeed, $\dfrac{-\beta_0 k^4 - \alpha_0 k^2 - \gamma_0}{\lambda + k^2} < 0$ for $\beta_0 > -\dfrac{\alpha_0}{k^2} - \dfrac{\gamma_0}{k^4}$, $\lambda > -1$. Let's denote

$$-\nu_1 = \max\left\{\frac{-\beta_0 k^4 - \alpha_0 k^2 - \gamma_0}{\lambda + k^2},\ k \in \mathbb{N}\right\}.$$

Then the assessment is fair $e^{\frac{-\beta_0 k^4 - \alpha_0 k^2 - \gamma_0}{\lambda + k^2}t} \leq e^{-\nu_1 t}$, $\nu_1 > 0$. For the solution $u = u(t)$ of the (0.4)–(2.14) system, has been completed

$$u = \left\|\sum_{k=2}^{\infty} \exp\left(t\frac{-\beta_0 k^4 - \alpha_0 k^2 - \gamma_0}{\lambda + k^2}\right)\frac{(R-r)^k}{2R^k}\Big(a_k \cos k\theta + b_k \sin k\theta\Big)\right\|_{A(\Omega)} +$$

$$+\left\|\sum_{k=1}^{\infty} \exp\left(t\frac{-\beta_0 k^4 - \alpha_0 k^2 - \gamma_0}{\lambda + k^2}\right)\Big(c_k \cos k\theta + d_k \sin k\theta\Big)\right\|_{A(\Gamma)} \leq$$

$$\leq C_1 e^{-\nu_1 t}\|u_0\|_{A(\Omega)} + C_2 e^{-\nu_1 t}\|v_0\|_{A(\Gamma)}.$$

Theorem 2.4. *Let $\alpha_0,\ \beta_0,\ \gamma_0 > 0$ and $\beta_0 > -\dfrac{\alpha_0}{k^2} - \dfrac{\gamma_0}{k^4}$. If $\lambda < -1$, then for any $u_0 \in A(\Omega)$, $v_0 \in A(\Gamma)$ the solution of the (0.4)–(0.7) system is unstable.*

Let $\alpha_0,\ \beta_0,\ \gamma_0 > 0$ and $\beta_0 > \dfrac{\alpha_0}{k^2} + \dfrac{\gamma_0}{k^4}$, $\lambda < -1$. Then $\dfrac{-\beta_0 k^4 - \alpha_0 k^2 - \gamma_0}{\lambda + k^2} < 0$ for $\lambda > -k^2$ and $\dfrac{-\beta_0 k^4 - \alpha_0 k^2 - \gamma_0}{\lambda + k^2} > 0$ for $\lambda < -k^2$. Let's denote $-\nu_1 = \max\left\{\dfrac{-\beta_0 k^4 - \alpha_0 k^2 - \gamma_0}{\lambda + k^2},\ \lambda > -k^2\right\}$, $\nu_2 = \min\left\{\dfrac{-\beta_0 k^4 - \alpha_0 k^2 - \gamma_0}{\lambda + k^2}, \lambda < -k^2\right\}$, $\nu_1, \nu_2 > 0$, and $m = \max\{k :\ \lambda < -k^2\}$. Then the solution $u = u(t)$ of the (0.4)–(0.7) system can be represented as $u = u_1 + u_2$, where

$$u_1(t) = \sum_{k=m+1}^{\infty} \exp\left(t\frac{-\beta_0 k^4 - \alpha_0 k^2 - \gamma_0}{\lambda + k^2}\right)\frac{(R-r)^k}{2R^k}\Big(a_k \cos k\theta + b_k \sin k\theta\Big) +$$
$$+\sum_{k=m}^{\infty} \exp\left(t\frac{-\beta_0 k^4 - \alpha_0 k^2 - \gamma_0}{\lambda + k^2}\right)\Big(c_k \cos k\theta + d_k \sin k\theta\Big), \tag{2.2}$$

$$
\begin{aligned}
u_2(t) = \sum_{k=2}^{m} \exp\left(t\frac{-\beta_0 k^4 - \alpha_0 k^2 - \gamma_0}{\lambda + k^2}\right)\frac{(R-r)^k}{2R^k}\Big(a_k \cos k\theta + b_k \sin k\theta\Big)+ \\
+ \sum_{k=1}^{m-1} \exp\left(t\frac{-\beta_0 k^4 - \alpha_0 k^2 - \gamma_0}{\lambda + k^2}\right)\Big(c_k \cos k\theta + d_k \sin k\theta\Big).
\end{aligned}
\tag{2.3}
$$

The following estimates are valid:

$$
\|u_1(t)\|_{\mathfrak{U}} \le C_3 e^{-\nu_1 t}\|u_0\|_{A(\Omega)} + C_4 e^{-\nu_1 t}\|v_0\|_{A(\Gamma)}, \tag{2.4}
$$

$$
\|u_2(t)\|_{\mathfrak{U}} \le C_5 e^{\nu_2 t}\|u_0\|_{A(\Omega)} + C_6 e^{\nu_2 t}\|v_0\|_{A(\Gamma)}. \tag{2.5}
$$

Obviously, the solution $u = u(t)$ of the (2.13)–(0.7) system is unstable.

Let the coefficients be $\alpha_0 = \alpha_1,\ \beta_0 = \beta_1,\ \gamma_0 = \gamma_1,\ -\lambda \ne k^2,\ \lambda < -1,\ \beta_0 > -\dfrac{\alpha_0}{k^2} - \dfrac{\gamma_0}{k^4}$. Then, by virtue of Theorem 2.4, the solution $u = u(t)$ of the (2.13)–(0.7) system is unstable. Let's set the following stabilization task. It is required to find a control in the domain f_u and on the boundary of the domain f_v such that the solutions of the system

$$
(\lambda - \Delta_{r,\theta})u_t = \alpha_0\Delta_{r,\theta}u - \beta_0\Delta^2_{r,\theta}u - \gamma_0 u + f_u,\ u = u(t,r,\theta),\ (t,r,\theta) \in \mathbb{R}\times\Omega, \tag{2.6}
$$

$$
(\lambda - \Delta_{\theta})v_t = \alpha_1\Delta_{\theta}v - \beta_1\Delta^2_{r,\theta}u + \partial_R u - \gamma_1 v + f_v,\ v = v(t,\theta),\ (t,\theta) \in \mathbb{R}\times\Gamma, \tag{2.7}
$$

$$
tr\ u = v, \tag{2.8}
$$

will be exponentially stable. We will look for the control of f_u and f_v using the feedback loop

$$
f_u = Bu,\ f_v = Bv, \tag{2.9}
$$

Denote by n the number of the maximum value of

$$
\left\{\frac{-\beta_0 k^4 - \alpha_0 k^2 - \gamma_0}{\lambda + k^2}, \lambda < -k^2\right\}. The
$$

operator B can be represented as

$$
B = \begin{cases} \mathbb{O},\ \lambda > -k^2; \\ \frac{\varepsilon + \beta_0 n^4 + \alpha_0 n^2 + \gamma_0}{\lambda + k^2}\mathbb{I},\ \lambda < -k^2, \end{cases} \tag{2.10}
$$

where ε can be chosen small enough. Then the solution of the (2.2)–(2.9) closed feedback loop (2.9) has the form

$$
\begin{aligned}
u(t) = u_1(t) + \sum_{k=2}^{m} \exp\left(t\frac{\varepsilon + \beta_0(n^4 - k^4) + \alpha_0(n^2 - k^2)}{\lambda + k^2}\right)\frac{(R-r)^k}{2R^k}\Big(a_k \cos k\theta + \\
+ b_k \sin k\theta\Big) + \sum_{k=1}^{m-1} \exp\left(t\frac{\varepsilon + \beta_0(n^4 - k^4) + \alpha_0(n^2 - k^2)}{\lambda + k^2}\right)\Big(c_k \cos k\theta + d_k \sin k\theta\Big)
\end{aligned}
\tag{2.11}
$$

and by virtue of (2.4), it is exponentially stable. Fair enough

Theorem 2.5. *Let* $\alpha_0 = \alpha_1,\ \beta_0 = \beta_1,\ \gamma_0 = \gamma_1,\ \beta_0 > -\dfrac{\alpha_0}{k^2} - \dfrac{\gamma_0}{k^4},\ \lambda < -1,$ *then for any* $u_0 \in A(\Omega),\ v_0 \in A(\Gamma)$ *the solution of the* (2.2)–(2.9) *system, closed-loop* (2.9)*, where the operator* B *has the form* (2.10)*, is exponentially stable.*

The stabilization of the stochastic Wentzell system's solution

Let $\mathfrak{U} = \{u \in W_2^2(\Omega) \oplus W_2^2(\Gamma) : \partial_R u = 0\}$, $\mathfrak{F} = L_2(\Omega) \oplus L_2(\Gamma)$. Following [?] let us construct spaces *of random* $\mathbf{K}$ *- values. The random* $\mathbf{K}$ *- values* ξ, $\chi \in \mathbf{U_K L_2}$ have the form

$$\xi = \sum_{k=1}^{\infty} \lambda_k \xi_k \phi_k, \quad \chi = \sum_{k=1}^{\infty} \lambda_k \chi_k \psi_k, \tag{2.12}$$

where $\{\phi_k\}$ – the set of eigenfunctions of the modified Laplace – Beltrami operator $\Delta_{r,\theta} \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ orthonormal in the sense of the scalar product $\langle \cdot, \cdot \rangle$ of $L_2(\Omega)$; $\{\psi_k\}$ – the set of eigenfunctions of the modified Laplace – Beltrami operator $\Delta_\theta \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ orthonormal in the sense of the scalar product $\langle \cdot, \cdot \rangle$ of $L_2(\Gamma)$. The norm in $\mathbf{U_K L_2}$ is given by the formula

$$\|\xi\|_{\mathbf{U_K L_2}} + \|\chi\|_{\mathbf{U_K L_2}} = \sum_{k=1}^{\infty} \lambda_k^2 \mathbf{D} \xi_k + \sum_{k=1}^{\infty} \lambda_k^2 \mathbf{D} \chi_K.$$

Let us consider the stochastic Wentzel system

$$(\lambda - \Delta_{r,\theta}) \overset{\circ}{\xi} (t) = \alpha_0 \Delta_{r,\theta} \xi(t) - \beta_0 \Delta_{r,\theta}^2 \xi(t) - \gamma_0 \xi(t),\ t \in (0, \tau),\ \xi \in \Omega, \tag{2.13}$$

$$(\lambda - \Delta_\theta) \overset{\circ}{\chi} (t) = \alpha_1 \Delta_\theta \chi(t) - \beta_1 \Delta_\theta^2 \chi(t) + \partial_\theta \xi(t) - \gamma_1 \chi(t),\ t \in (0, \tau),\ \chi \in \Gamma. \tag{2.14}$$

We will add a matching condition to this system, which guarantees the uniqueness of the solution obtained.,

$$\text{tr}\ \xi(t) = \chi(t) \tag{2.15}$$

and provide it with the initial conditions

$$\xi(0) = \xi_0, \quad \chi(0) = \chi_0. \tag{2.16}$$

Definition 2.6. The solution $\xi = \xi(t)$ of the system (2.13)–(2.16) *is exponentially stable* if there are constants $N > 0$ and $\nu > 0$ such that for all $t \in \mathbb{R}_+$ and any $\xi_0 \in \mathbf{U_K L}_2(\Omega)$ and $\chi_0 \in \mathbf{U_K L}_2(\Gamma)$ the solution $\xi = \xi(t)$ of the system (2.13)–(2.16) satisfies exponential estimation

$$\|\xi(t)\|_{\mathbf{U_K L_2}} \le N e^{-\nu t} (\|\xi_0\|_{\mathbf{U_K L_2}} + \|\chi_0\|_{\mathbf{U_K L_2}}). \tag{2.17}$$

Definition 2.7. The solution $\xi = \xi(t)$ of the (2.13)–(2.16) system is called *unstable* if it exists $\varepsilon > 0$ for some $\xi_0 \in \mathbf{U_K L}_2(\Omega)$ and $\chi_0 \in \mathbf{U_K L}_2(\Gamma)$ and $t > 0$ completed

$$\|\xi(t) - \xi_0\|_{\mathbf{U_K L_2}} + \|\chi(t) - \chi_0\|_{\mathbf{U_K L_2}} \ge \varepsilon.$$

By analogy with the deterministic case, it is easy to see that the following theorems hold.

Theorem 2.8. *Let*$\alpha_0 = \alpha_1$, $\beta_0 = \beta_1$, $\gamma_0 = \gamma_1$, $\beta_0 > -\frac{\alpha_0}{k^2} - \frac{\gamma_0}{k^4}$, $\lambda > -1$, *then for any* $\xi_0 \in \mathbf{U_K L}_2(\Omega)$ *and* $\chi_0 \in \mathbf{U_K L}_2(\Gamma)$ *the solution of the* (2.13)–(2.16) *system is exponentially stable.*

Theorem 2.9. *Let* $\alpha_0 = \alpha_1$, $\beta_0 = \beta_1$, $\gamma_0 = \gamma_1$, $\beta_0 > -\frac{\alpha_0}{k^2} - \frac{\gamma_0}{k^4}$, $\lambda < -1$, *then for any* $\xi_0 \in \mathbf{U_K L}_2(\Omega)$ *and* $\chi_0 \in \mathbf{U_K L}_2(\Gamma)$ *the solution of the* (2.13)–(2.16) *system is unstable.*

Let the coefficients be $\alpha_0 = \alpha_1$, $\beta_0 = \beta_1$, $\gamma_0 = \gamma_1$, $\beta_0 > -\frac{\alpha_0}{k^2} - \frac{\gamma_0}{k^4}$, $\lambda < -1$. Then, by virtue of Theorem 2.9, the solution $u = u(t)$ of the (2.13)–(2.16) system is unstable. Let's set the following stabilization task. It is required to find such a control in the area of η_ξ and on the border of the area of η_χ, what are the solutions of the system

$$(\lambda - \Delta_{r,\theta}) \overset{\circ}{\xi}(t) = \alpha_0 \Delta_{r,\theta}\xi(t) - \beta_0 \Delta^2_{r,\theta}\xi(t) - \gamma_0\xi(t) + \eta_\xi,\ t \in (0,\tau),\ \xi \in \Omega, \tag{2.18}$$

$$(\lambda - \Delta_\theta) \overset{\circ}{\chi}(t) = \alpha_1 \Delta_\theta\chi(t) - \beta_1 \Delta^2_\theta\chi(t) + \partial_\theta\xi(t) - \gamma_1\chi(t) + \eta_\chi,\ t \in (0,\tau),\ \chi \in \Gamma. \tag{2.19}$$

$$\mathrm{tr}\ \xi(t) = \chi(t), \tag{2.20}$$

will be exponentially stable. Management of η_ξ and η_χ we will search using the feedback loop

$$\eta_\xi = B\xi,\ \eta_\chi = B\chi, \tag{2.21}$$

where B is a linear operator.

Theorem 2.10. *Let $\alpha = \gamma$, $\beta = \delta$, $-\lambda \neq k^2$, $\frac{\beta}{\alpha} \neq k^2$, $\alpha, \beta \in \mathbb{R}_+$. If $\alpha k^2 > \beta$, $\lambda < -1$, then for any $\xi_0 \in \mathbf{U_K L}_2(\Omega)$ and $\chi_0 \in \mathbf{U_K L}_2(\Gamma)$ the solution of the* (2.18)–(2.16) *system, closed-loop* (2.21)*, where the operator B has the form* (2.10)*, is exponentially stable.*

All arguments and estimates in proving theorems in the stochastic case are similar to the deterministic case and therefore are not given.

Conclusion

In this paper, the problems of exponential stability and instability of solutions to deterministic and stochastic Wentzell equations describing the evolution of a liquid in a circular domain and on its boundary were investigated. Conditions ensuring stability and instability were obtained for various values of the model parameters characterizing the medium and liquid properties. For unstable regimes, a feedback-based stabilization approach was proposed and justified. The results obtained for deterministic equations were extended to the stochastic case using the Nelson–Gliklikh derivative framework. The conducted analysis demonstrates the effectiveness of the proposed stabilization method and provides a theoretical basis for controlling distributed systems with dynamic boundary conditions under deterministic and stochastic perturbations.

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References

1. Dzektser E.S. Generalization of the equation of motion of groundwater with a free surface, *Doklady Akademii Nauk SSSR*, **202** N5 (1972) 1031–1033.
2. Gliklikh Yu.E.: *Global and Stochastic Analysis with Applications to Mathematical Physics*, London; Dordrecht; Heidelberg; N.-Y., Springer, 2011.
3. Goncharov N.S., Sviridyuk G.A. Analysis of the stochastic Wentzell system of fluid filtration equations in a circle and on its boundary *Bulletin of the South Ural State University. Series: Mathematics. Mechanics. Physics*, **15** (2023) N3 15–22.

4. Goncharov N.S., Kitaeva O.G., Sviridyuk G.A. Stabilization of solutions for a stochastic dynamical Wentzell system in a circle and on its boundary *Bulletin of the South Ural State University. Series: Mathematics. Mechanics. Physics*, **17** N3 (2025) 5–12.
5. Goncharov N.S., Zagrebina S.A., Sviridyuk G.A. Non-Uniqueness of Solutions to Boundary Value Problems with Wentzell Condition Bulletin of the South Ural State University. Series: Mathematical Modeling, Programming and Computer Software, **14** N4 (2021) 102–105.
6. Favini A., Zagrebina S.A., Sviridyuk G.A. Multipoint Initial-Final Value Problem for Dynamical Sobolev-Type Equation in the Space of Noises Electron. J. Evol. Equ, **2018** N128 (2018) 1–10.
7. Favini A., Zagrebina S.A., Sviridyuk G.A. The multipoint initial-final value condition for the Hoff equations in geometrical graph in spaces of K-"noises" Mediterr. J. Math., **19** N2 (2022) 53.
8. Goncharov N.S. Stochastic Barenblatt – Zheltov – Kochina Model on the Interval with Wentzell Boundary Conditions Global and Stochastic Analysis, **7** N1 (2020) 11–23.
9. Giuseppe M.Coclite., Favini A., Gal Ciprian G., Goldstein G.R.: The Role of Wentzell Boundary Conditions in Linear and Nonlinear Analysis, *Tubinger Berichte*, **132**, (2008) 279–292.
10. Wentzell A.D.: Semigroups of operators corresponding to a generalized differential operator of second order, *Doklady Academii Nauk SSSR*, **111**, (1956) 269–272. (in Russian)
11. Wentzell A.D.: On boundary conditions for multidimensional diffusion processes, *Theory of Probability and its Applications*, **4**, (1959) 164–177.

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