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ENUMERATION AND CLASSIFICATION OF CERTAIN CLASSES OF GRACEFUL DIGRAPHS USING AN ANALYTICAL AND ALGORITHMIC APPROACH

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Abstract. A graceful directed graph is said to be graceful if there exists an one to one labeling of its vertices by integers such that each edge receives a label that is the modular difference between its terminal vertex and initial vertex, and all edge labels are distinct. A $(v, k, \lambda)-$ difference set is a $k-$subset D of Z_v so that each non-identity element appears exactly λ times as a difference of two distinct elements of D. A digraph can be labeled gracefully from the set of edge and vertex labels generated from the $(v, k, \lambda)-$ difference set. This study classifies certain classes of graceful directed graphs obtained from $(v, k, \lambda)-$ difference set for $v = k = \lambda$ as weakly connected, symmetric, and non-symmetric disconnected digraphs. In addition, we use analytical proofs and a depth-first search (DFS) algorithm to identify and enumerate certain classes of graceful digraphs.

1. Introduction

An assignment of real numbers or a subset of a set to the edges or vertices of a graph or both that is subjected to specific constraints is called labeling of graphs. A graph or a digraph could be labeled in numerous ways. However, when specific conditions are applied to vertice, edges, or both, these problems become highly fascinating. A large variety of labeling techniques are categorized, and graceful labeling is one among them.

Labeled graphs have become increasingly useful in the different fields of mathematics to solve some of the structural models as given by Bloom and Golomb in [2]. We referred to Gallian[4] for a detailed survey on labeling problems Bloom et al. [3] prolonged the idea of graph labelings from undirected graphs to directed graphs. They have also studied the connection between graceful directed graphs and various types of algebraic structures, such as difference sets and complete mappings, etc.

Definition 1.1. [6]A directed graph D containing p vertices and q edges is labeled by allotting a distinct integer $h(v)$ from the set $\{0, 1, \ldots, q + 1\}$ to every vertex, v. The values of vertices, again, find a value of $h(v, w)$ on each edge (v, w) where

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$h(v,w) = \{h(w) - h(v)\} \pmod{q} + 1$. If all the edge values are different, then the labeling is called a graceful labeling of a directed graph.

Definition 1.2. [12] A digraph D is said to be *weakly connected* if the underlying undirected graph(all edges replaced by undirected edges) of D is connected.

Definition 1.3. [12] A symmetric digraph is a digraph where, for every edge (a,b)(i.e., from vertex a to b), there is also an edge (b,a).

Definition 1.4. [8] A $(v,k,\lambda)-$difference set $D = \{d_1, d_2, \ldots, d_k\}$ is a set of k residues modulo v such that for each non-zero residue $\alpha \pmod{v}$ the congruence $d_i - d_j \equiv \alpha \pmod{v}$ has exactly λ solution pairs (d_i, d_j) with d_i and d_j in D.

The study of graceful directed trees is important because it extends the conventional graceful tree conjecture to directed structures, which allows for a richer theoretical understanding. These techniques assist in crafting effective routing and addressing strategies within directed networks. They achieve this by guaranteeing that each directed edge possesses a distinct difference value. Moreover, the graceful labeling of ditrees is useful in coding theory, particularly for creating error-resistant sequences and communication codes. In hierarchical data structures and information-flow models, graceful labels allow efficient indexing and decrease ambiguity. Bloom and Hsu[3] suggested the notion that all trees are disgraceful. This study makes some headway in solving the conjecture.

In combinatorial analysis, enumerative methods have often been seen as less methodical. As more robust methodologies and general frameworks are developed, this viewpoint is projected to evolve gradually. Arthur Cayley, a pioneer in graph theory, explored the problem of counting trees in 1857. This work was a key step in the development of graph enumeration. The count of distinct trees with n vertices facilitates calculating the number of isomers of a saturated hydrocarbon with n carbon atoms, that is C_nH_{2n+2} specifically. Since Cayley's discovery, a lot of study has been done on counting different types of graphs. The algorithms that have come from this research have proven useful in many real-world situations. Cayley, Redfield, and Pólya made important contributions to the basic ideas of graphical enumeration theory. Redfield's 1927 study [10] was a major advance, foreshadowing almost all of the graphical enumeration methods used today, as noted in [5]. Regrettably, this work languished in obscurity for a long time.

In modern graph theory, particularly for solving problems that are too huge or complex for manual computation, the use of computers and efficient algorithms is essential. This study uses the Depth-First Search (DFS) algorithm to find the number of specific classes of graceful digraphs. Moreover, we show that weakly connected graceful digraphs obtained from $(v,k,\lambda)-$ difference set when $v = k = \lambda$ are graceful directed trees. In these graceful directed trees, some structures are either directed paths or unidirectional paths. Using the Depth-First Search (DFS) algorithm, we enumerated several types of graceful digraphs that came from the previous construction.

2. Graceful digraphs obtained from the CDR for a $(v,k,\lambda)-$ difference set with $v=k=\lambda$

In [7], the construction of graceful digraphs using cyclic difference sets are given as below.

Definition 2.1. Let $B=\{B_1,B_2,\ldots,B_n\}$ be a subset of nonempty X, A system of distinct representatives (SDR) for B is a set $\{b_1,b_2,\ldots,b_n\}$ such that, for any i, $i=1,2,\ldots,n$, where $b_i\in B_i$ and $b_i\neq b_j$ whenever $i\neq j$.

For a $(v,k,\lambda)-$ difference set $D=\{a_1,a_2,a_3,\ldots a_k\}$,

Let $U=\{b_{(a_i,a_j)}|a_j-a_i\equiv y \pmod v : \forall i,j \text{ where } i,j=1,2.,\ldots,k,\ i\neq j\}$ be the set of $\lambda(v-1)$ differences of the components of the difference set B.

Let $B=\{C_1,C_2,\ldots,C_{(v-1)}\}$ represent the family of $v-1$ sub-multisets of S, with $C_y=\{y_{(a_i,a_j)}|a_j-a_i\equiv y \pmod v$, for some i,j where $i,j=1,2.,\ldots,k,i\neq j\}$. Then each $C_y, 1\leq y\leq (v-1)$ is a set of one of the elements of Z_v-0 appearing exactly λ times for some $a_i,a_j\in D, i\neq j$.

From the above, it can be seen that,

(1) $C_i\cap C_j=\emptyset\ \forall i,j, \quad i,j=1,2.,\ldots,k,i\neq j$,
(2) $\bigcup_{y=1}^{v-1}C_y=S$, that is the λ copies of components of $Z_v-\{0\}$.

Definition 2.2. [7] Let $B=\{C_1,C_2,\ldots,C_{(v-1)}\}$be the family of $v-1$ sub-multisets of U, where U is the set of $\lambda(v-1)$ differences of the entries of the difference set $D=\{a_1,a_2,a_3,\ldots,a_k\}$ with $C_y=\{y_{(a_i,a_j)}|a_j-a_i\equiv y \pmod v$ for some i,j where $i,j=1,2.,\ldots,k,i\neq j\}$. Subsequently, a collection of a different representative (CDR) for B is a method of choosing a member $z_{(a_i,a_j)}$ for some $a_i,a_j\in D$, from each $C_z, 1\leq z\leq v-1$ such that they are all distinct in nature.

The following result is proved in [7].

Theorem 2.3. *For a (v,k,λ) of a difference set $D=\{a_1,a_2,a_3,\ldots,a_k\}$ there exist a graceful digraph.*

Theorem 2.3 gives an illustration of any graceful digraph derived from the difference set, but some results for graceful ditrees using $v=k=\lambda$ are obtained in [11] as follows.

Proposition 2.4. *Proposition 2.2. For a $(v,k,\lambda)-$difference set with $v=k=\lambda$, there exists a graceful ditree.*

Proposition 2.5. *Proposition 2.3. For a $(v,k,\lambda)-$ difference set with $v=k=\lambda$, there exists a unidirectional path graph $\overrightarrow{P}_\lambda$ when λ is even.*

Also, the existence of an oriented path when λ is odd is established in [11].

With the reference of theorem 2.3, this study explores all the possible graceful digraphs obtained from the $(v,k,\lambda)-$ difference set with $v=k=\lambda=4$, which listed below in the table 1.

It is observed that the graceful digraphs obtained from the $(v,k,\lambda)-$difference set with $v=k=\lambda$ include weakly connected digraphs and disconnected digraphs which are graceful. Additionally, within these classes of graceful digraphs, some are ditrees, oriented paths, unidirectional path graphs, symmetric digraphs or

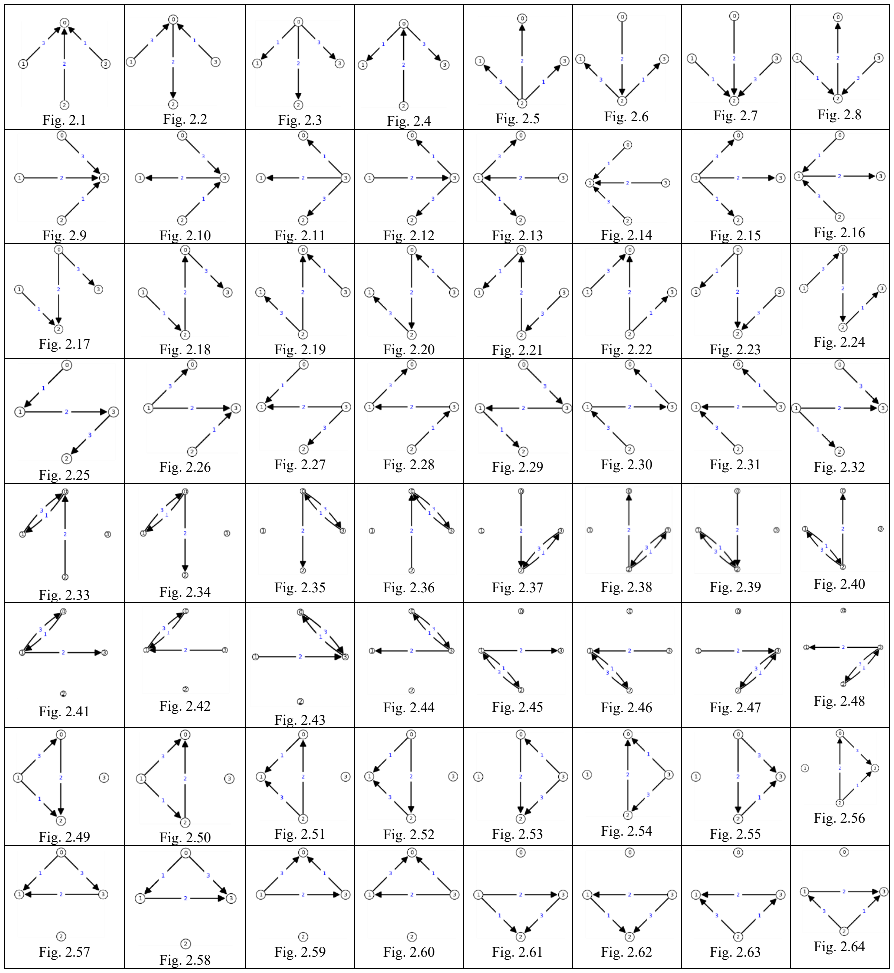

TABLE 1. All graphs when $\lambda = 4$

symmetric digraphs with one asymmetric edge. In this section, weakly connected graceful digraphs are obtained from the $(v, k, \lambda)-$difference set for $v = k = \lambda$ and are proven to be ditrees, and also established the existence of certain classes of graceful digraphs with imposed constraints. The following results are proven using the notion of CDR.

Proposition 2.6. *A graceful digraph obtained from the* $(v, k, \lambda)-$ *difference set with* $v = k = \lambda$ *has* λ *vertices and* $\lambda - 1$ *edges.*

Proof. To construct a graceful digraph from the $(v, k, \lambda)-$ difference set with $v = k = \lambda$ using the construction method given in definition 2.2, the CDR for B, denoted by $z_{(a_{i_z},a_{j_z})}$, for each z, $1 \leq z \leq \lambda - 1$, must be selected from the sub-multisets $C_1, C_2, \ldots, C_{\lambda-1}$. These CDRs correspond to the edge labels of the graceful digraph with the vertex labels $\{0, 1, 2, \ldots, \lambda-1\}$. Hence the corresponding graceful digraph has λ vertices and $\lambda - 1$ edges.

Theorem 2.7. [1] *A connected graph on n vertices and $n - 1$ edges is a tree.*

Theorem 2.8. *A graceful digraph obtained from the $(v, k, \lambda)-$difference set with $v = k = \lambda$ is a ditree if and only if it is weakly connected.*

Proof. Suppose that the graceful digraph obtained from the $(v, k, \lambda)-$difference set with $v = k = \lambda$ is a weakly connected digraph. It has λ vertices and $\lambda - 1$ edges (proposition 2.6) and its underlying graph is a connected graph having $\lambda-1$ edges and λ vertices. Hence from theorem 2.7, we say that the underlying graph is a tree. Therefore, the corresponding weakly connected graceful digraph is a graceful directed tree.
Conversely, assume the graceful digraph obtained from the $(v, k, \lambda)-$difference set with $v = k = \lambda$ is a ditree with λ vertices and $\lambda - 1$ edges, then the underlying graph is a tree and hence it is connected. Therefore, the corresponding digraph is weakly connected. □

Example 2.9. In table 1, figures from 2.1 to 2.32 represents weakly connected graceful digraphs and figures from 2.33 to 2.64 represents disconnected graceful digraphs when $v = k = \lambda = 4$

Theorem 2.10. *For a (v, k, λ)difference set with $v = k = \lambda$, there exists a directed graceful inward star graph and a directed graceful outward star graph.*

Proof. Using the construction method given in definition 2.2, for a fixed i,$1 \leq i \leq \lambda - 1$, if CDR $y_{(i,y(i))}$ for B from C_y for all $y = 1, 2, 3, \ldots, \lambda - 1$ is selected, so that $y(i) - i \equiv y \mod$ with $y(i) \neq i$, $y(i) = 0, 1, .., \lambda - 1$. Here the fixed i value represents the central vertex and will have out-degree as $\lambda - 1$ and in-degree 0. Then from the CDR obtained out of it gives a graceful ditree, which represents a graceful outward-directed star graph. Similarly, for a fixed $j, 1 \leq j \leq v - 1$, if the C_y from $y = 1, 2, 3, \ldots, \lambda - 1$ is selected as $y_{(y(j),j)}$ such that $j - y(j) \equiv y \pmod{v}$ with $y(j) \neq j$, $y(j) = 0, 1, .., \lambda - 1$. Here the fixed j value represents the central vertex and will have in-degree as $\lambda - 1$ and an out-degree of 0. Then from the CDR obtained out of it gives a graceful ditree, which represents a graceful inward-directed star graph. □

Example 2.11. In table 1, figures 2.1, 2.7, 2.9 and 2.14 represent graceful inward star digraphs and figures 2.3, 2.5, 2.11 and 2.15 represent graceful outward star digraphs when $v = k = \lambda = 4$.

The following facts are observed from the above theorem.

Observation 2.1. There is no strongly connected digraph obtained from the $(v, k, \lambda)-$difference set for $v = k = \lambda$. Because, if it is strongly connected, then it should have a closed walk, but any connected digraph obtained from a $(v, k, \lambda)-$difference set with $v = k = \lambda$ is a ditree, hence it does not contain any closed walk, so it is not strongly connected.

Observation 2.2. Any graceful digraphs constructed from a $(v, k, \lambda)-$difference set with $v = k = \lambda$ is a graceful digraph with λ vertices and $\lambda - 1$ edges, so any such graceful digraph that is not a ditree is a digraph whose underlying graph is a disconnected graph having trees as subgraphs or a disconnected graph having cyclic subgraphs.

The following disconnected graceful digraphs are obtained from the $(v, k, \lambda)-$difference set with $v = k = \lambda$.

Theorem 2.12. *For a $(v, k, \lambda)-$difference set with $v = k = \lambda$, there exists a graceful symmetric directed graph, when λ is odd.*

Proof. Let a set $B = \{C_1, C_2, \ldots, C_{\lambda-1}\}$ of $\lambda - 1$ multisubsets of S, where S is the set of $\lambda(\lambda - 1)$ differences from the difference set D and for $y = 1, 2, .., \lambda - 1$, $C_y = \{b_{(a_i,a_j)} | a_j - a_i \equiv y \pmod{v} : \text{for some } i, j \text{ where } i, j = 1, 2., \ldots, \lambda, i \neq j\}$. If a CDR for B is chosen so that whenever $y_{(a_i,a_j)}$ is an component selected from C_y for $1 \leq y \leq (\lambda - 1)/2$, then $z_{(a_j,a_i)}$ must be an element selected from C_z, for $(\lambda - 1)/2 \leq z \leq (\lambda - 1)$. Since λ is odd, one can see that the values of y and z represent the inverse elements of each other with respect to *modulo* v.

Using this CDR for B assign the components of the difference set D as the label for vertex. Then the edges of the set comprise the elements from 1 to $(v - 1)$ and vertex set D has distinct elements of Z_v. Hence satisfies the conditions of graceful diagraph labeling and we observe that a graceful symmetric digraph is constructed. □

Example 2.13. When $v = k = \lambda = 5$ the difference set $D = \{0, 1, 2, 3, 4\}$ with $B = \{C_1, C_2, C_3, C_4\}$, where $C_1 = \{1_{(0,1)}, 1_{(1,2)}, 1_{(2,3)}, 1_{(3,4)}, 1_{(4,0)}\}, \ldots, C_4 = \{4_{(0,4)}, 4_{(1,0)}, 4_{(2,1)}, 4_{(3,2)}, 4_{(4,3)}\}$.

Suppose $\{1_{(0,1)}, 2_{(0,2)}, 3_{(2,0)}, 4_{(1,0)}\}$ is a CDR for B, then from this CDR one can construct a graceful symmetric graph as given below.

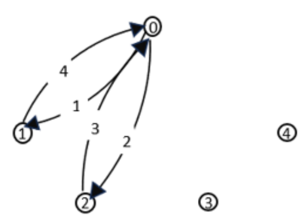

FIGURE 1. Symmetric graceful digraph

If λ is even, it is not possible to get symmetric graceful digraph. But there is a symmetric graceful digraph with exactly one asymmetric edge whose underlying graph is a tree.

Theorem 2.14. *For a $(v, k, \lambda)-$difference set with λ, there exists a graceful symmetric directed graph with one asymmetric edge, when λ is even.*

Proof. Let a set $B = \{C_1, C_2, \ldots, C_{\lambda-1}\}$ of $\lambda-1$ multisubsets of S, where S is the collection of $\lambda(\lambda-1)$ differences of the difference set D and $C_y = \{b_{(a_i,a_j)} | a_j - a_i \equiv y \pmod{v}$: for some i, j where $i, j = 1, 2., \ldots, \lambda, i \neq j\}$ for $y = 1, 2, .., \lambda-1$. If a CDR for A is chosen so that whenever $y_{(a_i,a_j)}$ is an component selected from C_y for $1 \leq y \leq (\lambda-2)/2$, $z_{(a_j,a_i)}$ must be an element selected from C_z, for $(\lambda+2)/2 \leq z \leq (\lambda-1)$ and any one $b_{(a_k,a_l)}$ element from the set $C_(\lambda/2)$. Since $\lambda-1$ is odd, it can be observed that the values of y and z represent the inverse elements of each other with respect to mod v and b is value without having any inverse element. Using this CDR for B assign the components of the difference set D as the label for the vertex. Then the edges of the set comprise the elements from 1 to $(\lambda-1)$ and vertex set D has distinct elements of Z_v. Hence, it satisfies the conditions of graceful diagraph labeling and it gives a graceful symmetric digraph with one asymmetric edge. □

Example 2.15. In table 1, figures from 2.33 to fig 2.48 represent graceful symmetric directed graphs with one asymmetric edge when $v = k = \lambda = 4$.

Remark 2.16. If all the orientation are ignored in the graceful symmetric digraph mentioned in theorem 2.12 and the graceful symmetric digraph with one asymmetric edge mentioned in theorem 2.14, their underlying undirected graphs contains disconnected trees.

Observation 2.3. It is also observed that, for $(v, k, \lambda)-$ difference set with λ in addition to the graceful digraphs mentioned in the above results, there exist digraphs whose underlying graphs contain cyclic subgraphs. Such classes of graceful digraphs are enumerated in the following section.

3. Enumeration of Graceful Digraphs

Although it is a tedious work to find the possible graceful labeling of a digraph, here we have made an attempt to find the possible number of graceful digraphs obtained from $(v, k, \lambda)-$ difference set with $v = k = \lambda$.

Theorem 3.1. *There exists exactly $\lambda^{(\lambda-1)}$ graceful directed graphs that can be constructed from the a $(v, k, \lambda)-$ difference set with $v = k = \lambda$.*

Proof. Since there are $\lambda-1$ sub-multisets and each sub-multiset has λ copies of all the differences in $D = 1, 2, \ldots, \lambda-1$, then from any selected combination of CDR for B from S(as defined above), one can construct graceful digraph. Therefore, there are exactly $\lambda^{(\lambda-1)}$ different choices for CDRs for B. Hence there are exactly $\lambda^{(\lambda-1)}$ graceful directed graph. □

Example 3.2. All possible graceful digraphs obtained when $\lambda = 4$ are listed in table 1, there are total 64 graceful digraphs.

Theorem 3.3. *There exists λ inward directed graceful star digraphs and λ outward directed graceful star digraphs for a $(v, k, \lambda)-$difference set with λ.*

Proof. From the theorem 2.10, to construct inward directed graceful digraph, we select a CDR from C_y for $y = 1, 2, 3, \ldots, \lambda-1$ as $y_{(i,y(i))}$ such that $y(i) - i \equiv y \pmod{v}$ with $y(i) \neq i$, $y(i) = 0, 1, .., \lambda-1$. Hence there are exactly λ choice for $y(i)$.

So there exists λ inward directed graceful star digraphs. Similarly, for outward directed graceful star graphs, we select CDR for $y_{(y(j),j)}$ such that $j - y(j) \equiv y \pmod{v}$ with $y(j) \neq j$, $y(j) = 0, 1, .., \lambda - 1$. Since there are λ choices for $y(j)$, there exists λ outward directed graceful star graphs. □

Theorem 3.4. *There exists $\lambda^{(\lambda-1)/2}$ symmetric graceful directed graphs, when λ is odd for a $(v, k, \lambda)-$ difference set with λ.*

Proof. From the theorem 2.12, to construct graceful symmetric digraph, one can choose any element $y_(a_i, a_j)$ from the setC_y for y in $1, 2, \ldots, (\lambda - 1)/2$ as CDR for B, all these elements are assigns edge labels of $(\lambda - 1)/2$ non-symmetric edges of the digraph. But the corresponding inverse element $z_{(a_j,a_i)}$ is always belongs to the set C_z with for $z \equiv \lambda - y \pmod{v}$ in $\{(\lambda - 1)/2, \ldots, \lambda - 1\}$. Since each C_y has λ elements, therefore there are $\lambda^{(\lambda-1)/2}$ different CDRs for B. Hence there are exactly $\lambda^{\lambda-1/2}$ symmetric graceful digraphs, when λ is odd.

Theorem 3.5. *There exist $\lambda^{\lambda/2}$ symmetric graceful directed graphs with one asymmetric edge, when λ is even for $(v, k, \lambda)-$ difference set with λ.*

Proof. From theorem 2.14, to construct graceful symmetric digraph, one can choose any element $y_{(a_i,a_j)}$ from the set C_y for y in $1, 2, \ldots, (\lambda - 2)/2$ as CDR for A, all these elements are assigns edge labels of $(\lambda - 2)/2$ non-symmetric edges of the digraph. But the corresponding inverse element $z_{(a_j,a_i)}$ is always belongs to the set C_z with for $z \equiv \lambda - y \pmod{v}$ in $(\lambda + 2)/2, \ldots, \lambda - 1$ and one chose any element form the set $C_(\lambda/2)$. Since each C_y has λ elements, therefore there are $\lambda^{(\lambda-1)/2}$ different CDRs for A. Hence there are exactly $\lambda^{(\lambda-1)/2}$ symmetric graceful digraphs, when λ is odd.

Since there are $\lambda^{\lambda-1}$ graceful digraphs—a very large class including several different classes of graceful digraphs—it is difficult to generalize all possible classes of graceful digraphs through an analytical approach. Therefore, an attempt is made to enumerate certain specific classes of digraphs by using algorithms.

4. Algorithm for Enumerating Some Classes of Graceful Digraphs

This section describes the algorithm developed to enumerate and categorize the certain classes of graceful directed graphs constructed from $(v, k, \lambda)-$difference set with $v = k = \lambda$. The algorithm systematically constructs all possible digraphs obtained by selecting exactly one directed edge from each sub-multiset $A_j = \{(i, (i + k) mod\ n)\}$ for $j = 1, 2, \ldots, n - 1$. Each generated digraph is made up of n vertices and $n - 1$ directed edges. The algorithm then analyzes each digraph to determine its structural properties, such as weak connectivity, presence of directed cycles, path structure, star configuration, and other related characteristics by using the Depth-First Search (DFS) algorithm [9]. This method involves systematically moving down the edges of a graph, ensuring each edge is used exactly once and each vertex is visited at least once, using the adjacency matrix. The approach combines exhaustive enumeration with efficient tests from graph theory. This allows us to classify certain types of graceful directed graphs obtained from

$(v, k, \lambda)-$ difference set with λ. The algorithm developed below receives a number of vertices of the digraph as an input value and counts the number of certain classes of graceful digraphs.

Algorithm for enumerating specific classes of graceful digraphs:

Input: The number of vertices n, an integer ¿ 2, is a key factor.

Output: This counts the number of graceful digraphs classified as:

- Weakly connected graceful digraphs
- Disconnected graceful digraphs
- Graceful digraphs containing directed cycles
- Graceful weakly connected digraphs whose underlying graph is a path
- Unidirectional graceful directed paths
- Graceful digraphs whose underlying graph contains a cycle Oriented graceful star graphs.

The flowchart of the algorithm is: Flowchart of the algorithm is given below.

The algorithm proceeds via the following steps:

Step 1: First, get a value n from the user.

Step 2: Building multisubsets

1. For each $k,\ 1 \leq k \leq n-1$, construct the set $A_k = \{(i, (i+k) mod\ n) | i = 0, 1, \ldots, n-1\}$.
2. Represent each A_k as a collection of n directed edges.

Each set represents all feasible directed edges that correspond to a specific difference.

Step 3: Initialization of the counters

Begin by setting the following counters to zero:

- connected - weakly connected graceful digraphs
- disconnected– weakly disconnected graceful digraphs
- cycle – graceful digraphs containing at least one directed cycle
- path – graceful digraphs whose underlying graph is an undirected path
- uni_path– unidirectional graceful directed paths
- underlying_cycle – graceful directed graphs whose underlying graph contains a cycle
- star – graceful directed graphs whose underlying graph is a star.

Step 4: Counting total number of graceful digraphs obtained

- From each A_k select precisely one edge.
- The Cartesian product of these selections creates all possible directed graphs, that are formed by collecting one edge from each A_k.
- Total combinations n^{n-1}.

Step 5: Generate graph for each combination

- Each digit selects single edge from the corresponding set A_k.
- Construct a directed graceful graph by combining the chosen edges, then the corresponding graceful digraph has
  - the vertex set $\{0, 1, \ldots, n-1\}$, and
  - exactly $n-1$ directed edges.

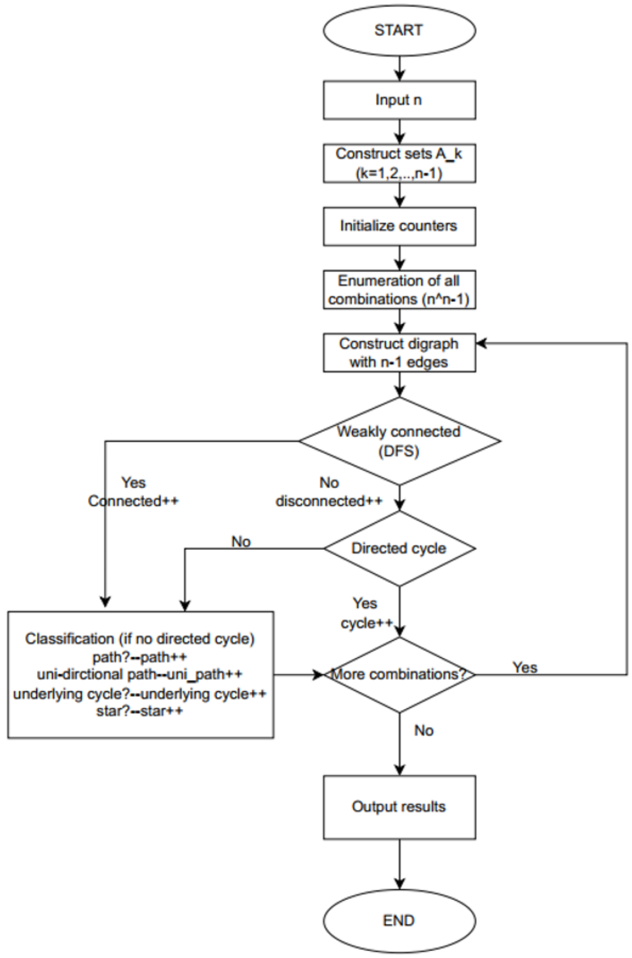

FIGURE 2. Flowchart

Step 6: Use the Depth- First Search algorithm to check if the digraph is weakly connected.

Transform the directed graph into its underlying undirected graph by disregarding the directionality of its edges. Perform Depth-First Search (DFS) starting from a randomly selected vertex.

- During DFS:
  - Each vertex that has been visited should be marked
  - Recursively, explore all neighbouring vertices.
- Once the DFS process wraps up:
  - A digraph is considered as, weakly connected if all its vertices have been visited;

- ○ otherwise, it is weakly disconnected digraph.

The corresponding counters are updated accordingly.
Step 7: Detecting and counting Directed Cycle Detection Using DFS
If a digraph is disconnected digraph then,

- Construct the directed adjacency structure(matrix) of the graceful digraph.
- Initialize two arrays:
  - ○ The visited vertices were recorded in the visited[] arrary,
  - ○ The recStack[] array is used to track the vertices currently being explored in the DFS recursion.
- Starting from each unvisited vertex, perform a depth-first search:
  - ○ mark the vertex as visited, then add it into the recursion stack;
  - ○ explore the outgoing edges recursively.
- When DFS finds a vertex that's already in recStack[], it means a directed cycle has been found.
- When backtracking, vertices should be removed from the recursion stack.

If a directed cycle is discovered, the directed cycle counter should be increased.
Step 8: Identifying the Underlying Path Graphs
If a directed graph is weakly connected;

- Consider the underlying undirected graph.
- Employ DFS to verify the connectivity.
- Calculate the degree of each vertex.
- A graph is considered as a path if:
  - ○ exactly two vertices have degree of 1,
  - ○ each of the remaining vertices has degree of 2,
  - ○ no vertex has degree larger than 2.

If these conditions hold, increment the path counter.
Step 9: Identifying of Unidirectional Directed Paths
To find directed paths that maintain a continuous orientation:

- Calculate the indegree and outdegree of every vertex.
- Check the following conditions:
  - ○ exactly one vertex has indegree 0 and outdegree 1; this is the starting vertex,
  - ○ exactly one vertex has indegree 1 and outdegree 0; this is the terminal vertex
  - ○ Each of the remaining vertices has indegree = outdegree = 1.
- Employ DFS to verify weak connectivity.

If all the above criteria are satisfied, classify the graph as a unidirectional directed path.
Step 10: Employ DFS to detect oriented cycles.

- Neglect edge directions and consider the graph as undirected.
- Perform DFS by keeping track of the parent nodes:
  - ○ if a visited vertex is reached that is not the parent, an undirected cycle exists.
- If such a cycle is found, increment the underlying cycle counter.

Step 11: Detection of Star Graphs

- Use DFS to confirm connectivity of the underlying undirected graph.
- Compute vertex degrees.
- The graph is classified as a star if:
  - exactly one vertex has degree n-1,
  - all other vertices have degree 1.

Increment the star counter if the condition holds.
Step 12: Termination and Output
After all, n^{n-1}graphs are processed:

- display execution time,
- output the counts of each graph class.

An illustration of the above algorithm for n=4 is given below. Step 1: Input : $n = 4, V = \{0, 1, 2, 3\}$ **Step 2:** Construct multisubsets A_k
$A_1 = \{(0, 1), (1, 2), (2, 3), (3, 0)\}$
$A_2 = \{(0, 2), (1, 3), (2, 0), (3, 1)\}$
$A_3 = \{(0, 3), (1, 0), (2, 1), (3, 2)\}$

Step 3: Initialize all counters as zero
connected = 0; disconnected = 0; cycle = 0; path = 0; uni_path = 0; underlying_cycle = 0; star =0;
Step 4: Enumerate all graphs
So total graphs: $4^{4-1} = 4^3 = 64$.
So the algorithm will generate 64 digraphs, each with 3 directed edges.
Step 5: One sample combination (graceful digraph)
Let us pick: From $A_1 : (0, 1)$, from $A_2 : (1, 3)$ and from $A_3 : (3, 2)$. So the digraph G has edge set: $E = (0, 1), (1, 3), (3, 2)$
Step 6: Detection of weak connectivity:
By ignore directions, we get the underlying undirected graph whose edges are: $\{0, 1\}, \{1, 3\}, \{3, 2\}$. This connects all vertices 0,1,2,and 3, so it is weakly connected. Therefore, weakly connected graphs count will be incremented as **connected=connected+1**.
We selected (for $n = 4$) $E = \{(0, 1), (1, 3), (3, 2)\}$,so the directed graph is $0 \to 1 \to 3 \to 2$ (fig. 2.25 in table 1).
Since the graph is weakly connected, it is not executing step 7. So, the cycle counter is not incremented.
Step 8: Underlying path graph detection
Since the digraph is weakly connected, check if the underlying graph is a path. Now compute the degrees of the vertices of the underlying graph: We get deg(0)=1; deg(1)=2; deg(3)=2; deg(2)=1;
There are exactly two vertices have degree 1, the remaining vertices have degree 2, and there is no vertex with $degree < 2$. So the underlying graph is a path. Hence, the path counter is increased by 1 unit as **path=path+1.**
Step 9: Unidirectional directed path detection Now check the indegree and outdegree of the graceful graph with the edges $(0, 1), (1, 3), (3, 2)$. The following table shows the indegree and outdegree of the vertices:

Vertex	Indegree	Outdegree
0	0	1
1	1	1
3	1	1
2	1	0

From the above table, vertex 0 has indegree 0 and outdegree 1 (that is the starting vertex), vertex 2 has indegree 1 and outdegree 0 (that is the end vertex), and the other vertices, 1 and 3 have indegree and outdegree as 1. So, this is a graceful unidirectional directed path. Therefore, **uni_path = uni_path +1.**
Step 10: Underlying cycle detection (undirected):
Check if the underlying graph contains an undirected cycle. Corresponding underlying edges are $\{0,1\},\{1,3\},\{3,2\}$.
This is a simple chain, so there is no closed loop. So there is no undirected cycle. Therefore, underlying_cycle counter is not incremented.
Step 11: Star graph detection: To detect the oriented star graph, find the degrees of vertices by ignoring the orientations. Degrees of vertices as follows for the graph with edges $\{0,1\},\{1,3\},\{3,2\}$,
deg(0)=1; deg(1)=2; deg(3)=2 and deg(2)=1;
Here, none of the vertices have degree 3. Therefore, it is not a star graph. Hence, star counter is not incremented.
For the above selected sample, following counters are incremented:
connected=connected + 1; path=path+1; uni_path = uni_path +1;
Other counters are not incremented.
For n=4, the graceful digraph with the edges $(0,1),(1,3),(3,2)$ is a weakly connected, cycle-free, path and unidirectional directed path digraph.

The following table exhibits the results of the previously mentioned algorithm's execution for graceful digraphs to 11 vertices in the MATLAB environment using a parallel computing method on a computer setup with 96 cores, 512 GM RAM, and a GPU card NVIDIA RTX A5000. The elapsed times taken for the for n=10 and n=11 were 2892.02 and 77874.05 seconds, respectively.

The following graphs as shown in the figure 3 illustrate various classes of graceful digraphs obtained for different values of n listed in table 2. From these graphs, it is observed that for $n \geq 7$, the number of possible certain classes of graceful digraphs discussed in the above algorithm increases exponentially, except for the number of graceful unidirectional paths. Moreover, when n is odd, there is no graceful unidirectional path, while for even n, the number of graceful unidirectional path graphs increases exponentially.

4.1. Computational time complexity analysis. For a fixed integer n, the algorithm constructs modular edge sets $A_k = \{(i,(i+k) mod\, n) : i = 0,1,\ldots,n-1\}, k = 1,2,\ldots,n-1$.

No. of vertices	Total no. of graceful digraphs	No. of weakly connected digraphs	No. of disconnected digraphs	No. of digraphs with directed cycles
3	9	6	3	3
4	64	32	32	16
5	625	220	405	235
6	7776	1872	5904	3000
7	117649	19544	98105	50827
8	2097152	249856	1847296	8999504
9	43046721	3510864	39535857	20695923
10	1000000000	59004800	940995200	472224500
11	25937424601	1063793632	24873630969	13115817737

No. of vertices	No. of digraphs whose underlying graph is path	No. of unidirectional paths	No. of digraphs whose underlying graphs with cycles	No. of directed star graphs
3	6	0	0	6
4	16	8	0	6
5	120	0	200	20
6	528	24	4176	48
7	2016	0	69762	56
8	14848	192	1498272	128
9	101088	0	32651640	144
10	970880	2880	8290236000	320
11	7244160	0	22207852392	352

Table 2

Each set A_k contains exactly n edges. The algorithm selects exactly one edge from each A_k. Hence, the total number of directed graphs enumerated is n^{n-1}. This exponential growth dominates the overall running time. For each generated graph, the algorithm performs several DFS-based and degree-based tests. Since each graph has n vertices, exactly $n-1$ edges, the complexity of DFS on any such graph is: $O(n+(n-1)) = O(n)$. Thus for each of the followings, Weak Connectivity Test, Directed Cycle Detection, Underlying Path Detection, Unidirectional Directed Path Detection, Undirected Cycle Detection, Star Graph Detection, the time complexity is O(n). Hence Total Cost Per Graph is $T(n) = O(n \cdot n^{n-1}) = O(n^n)$.

The proposed algorithm performs an exhaustive enumeration of modular directed graphs and classifies each graph using DFS-based structural tests. Although each graph is processed in linear time with respect to the number of vertices, the

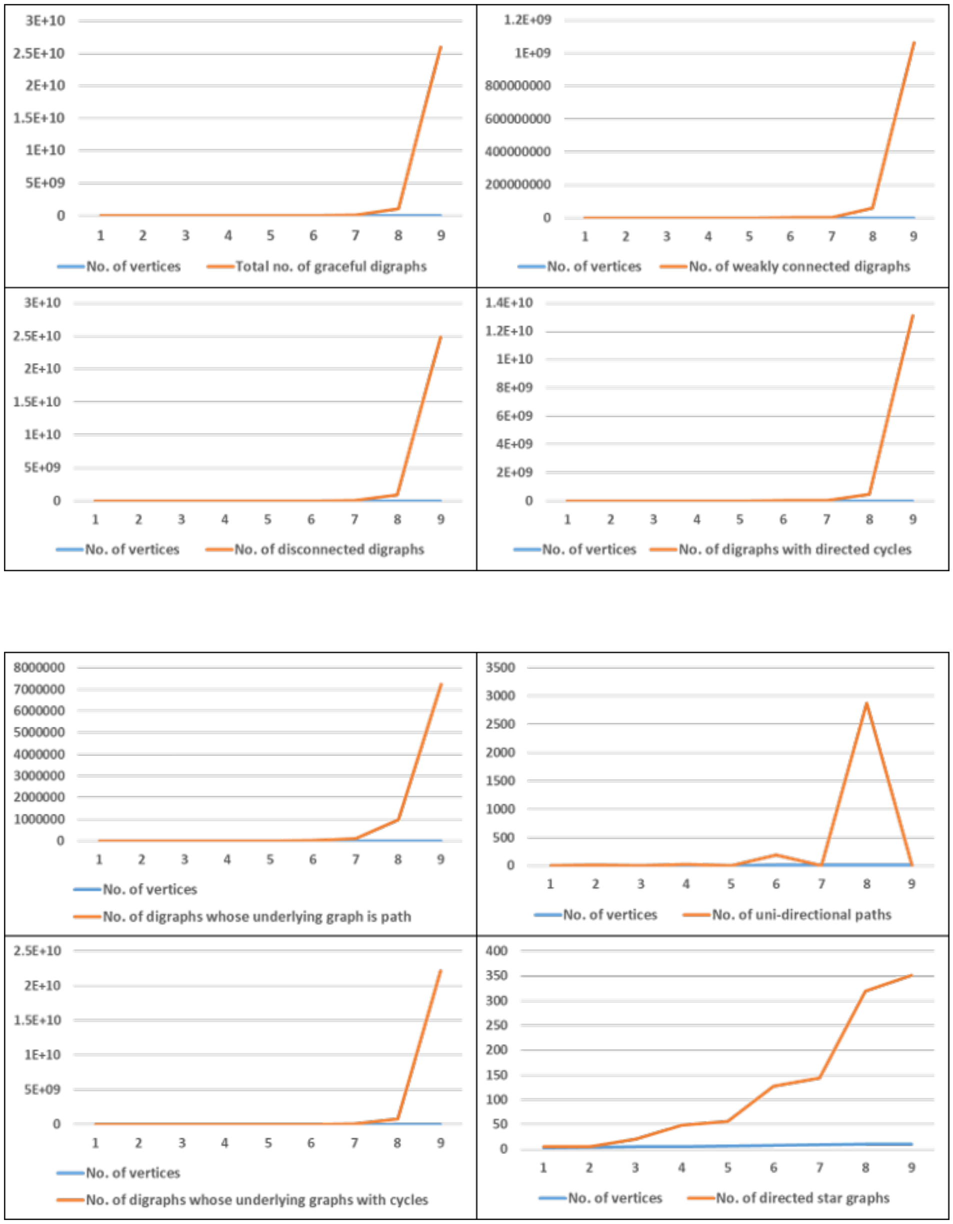

Figure 3

total running time grows as $O(n^n)$ due to the exponential size of the search space. Consequently, the algorithm is suitable for small to moderate values of n, while larger values require analytical reductions or combinatorial pruning techniques.

5. Conclusion

This paper investigates connected and disconnected classes of graceful digraphs obtained from the $(v, k, \lambda)-$difference set with $v = k = \lambda$. Several subclasses of graceful digraphs yielded from theses difference sets identified, and selected classes of digraphs were enumerated. While the present work is focusing on the enumeration of some specific subclasses, the method and the algorithm developed here can be extended further to investigate broader classes of graceful digraphs and to constructions derived from the other difference sets.

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