Global and Stochastic Analysis
Vol. 13 No. 3 (2026)

DOI: 10.64837/GSA.13.3.5

Received: 08th April 2026 *Revised: 07th June 2026* *Accepted: 16th June 2026*

SPECTRAL BOUNDS FOR FUZZY GRAPH LAPLACIANS UNDER INTERVAL UNCERTAINTY

ASOKAN VASUDEVAN, YOGEESH N*, KOYEL ROY, P. WILLIAM, ZETTY PAKIR MASTAN, AND THULASI MANOHARAN

Abstract. We study interval-valued fuzzy weighted graphs whose edge memberships are known only through closed intervals and investigate how this uncertainty propagates to the Laplacian spectrum. For every admissible realization we define the fuzzy weighted Laplacian and prove that the edge-to-Laplacian map is monotone in the Loewner order. Consequently the k-th Laplacian eigenvalue is confined to the exact interval $[\lambda_k(L^-), \lambda_k(L^+)]$ generated by the lower and upper endpoint graphs. We derive robust connectivity criteria, midpoint perturbation bounds of Weyl type, Gershgorin-type global enclosures, and explicit formulas for homogeneous interval uncertainty on paths, cycles, stars, and complete graphs. Numerical examples show that exact spectral widths can be substantially smaller than generic perturbation radii. The results place interval-valued fuzzy graph Laplacians into a rigorous spectral-uncertainty framework.

1. Introduction

Since Zadeh introduced fuzzy sets [36], graph-theoretic models with graded uncertainty have become a natural interface between combinatorics, matrix analysis, and decision theory. Rosenfeld's fuzzy graphs [32] initiated the area, after which structural operations, line graphs, domination and strength concepts, and several generalized fuzzy graph classes were developed in [8, 25, 26, 9, 10, 6, 18, 27]. Interval-valued fuzzy graph models are especially attractive when an edge membership cannot be specified by a single number and only lower and upper confidence levels are available. This direction was systematized by Akram and Dudek and further expanded in [3, 1, 2, 4, 29].

A parallel literature studies spectral and energy-type invariants of fuzzy graphs. Recent contributions include interval-valued fuzzy Laplacian energy [28], signless Laplacian energy [31], picture fuzzy graph energies [33], hesitant fuzzy Laplacian energy [30], connectivity of intuitionistic fuzzy graphs [7], degree-energy bounds in spectral fuzzy graphs [11], and Seidel Laplacian bounds in bipolar fuzzy graphs [19]. These works show that spectral data retain discriminatory power in uncertain graph models; however, most existing studies emphasize a single chosen fuzzy graph rather than a full uncertainty set of admissible realizations.

2000 *Mathematics Subject Classification.* Primary 05C50; Secondary 05C72, 15A18, 65G40.

Key words and phrases. interval-valued fuzzy graph, Laplacian spectrum, algebraic connectivity, interval uncertainty, Gershgorin bound, Weyl perturbation, Process innovation.

* Corresponding author: yogeesh.r@gmail.com.

The present paper addresses that gap for the Laplacian spectrum. On the classical side, the Laplacian matrix is a central object in algebraic graph theory, with foundational results due to Fiedler, Anderson-Morley, Grone-Merris-Sunder, Merris, Li-Pan, and Das [13, 5, 14, 15, 23, 24, 21, 22, 12]. On the interval-analysis side, spectral enclosures for interval matrices have been studied by Shih-Lur, Hladík-Daney-Tsigaridas, Leng-He, and Yuan-Yang [34, 16, 17, 20, 35]. Our aim is to combine these two lines of work in the context of interval-valued fuzzy graphs.

We consider a finite interval-valued fuzzy graph whose edge-membership values vary independently inside prescribed intervals. Every admissible realization determines a symmetric weighted graph and hence a Laplacian matrix. The main idea is that the Laplacian map is order-preserving: larger edge weights produce a larger Laplacian in the Loewner order. This simple observation has strong consequences. In particular, the entire uncertain Laplacian spectrum is bracketed by the lower and upper endpoint Laplacians, and the corresponding intervals are exact.

Our contributions are as follows:

- We formulate interval-valued fuzzy graph Laplacians as a family of admissible weighted Laplacians constrained by endpoint matrices.
- We prove exact eigenvalue enclosures
$$\lambda_k(L^-) \le \lambda_k(L(W)) \le \lambda_k(L^+), \quad k = 1, \dots, n,$$
for every admissible realization W.
- We obtain robust connectivity criteria, midpoint perturbation bounds of Weyl type, and width estimates controlled by the uncertainty degree profile.
- We derive Gershgorin-type global spectral boxes and explicit exact formulas for homogeneous interval uncertainty on paths, cycles, stars, and complete graphs.

2. Interval-valued fuzzy graphs and Laplacians

We work on a fixed finite vertex set $V = \{1, \dots, n\}$. Throughout the paper all matrices are real and symmetric unless noted otherwise, and the eigenvalues of a symmetric matrix A are arranged in nondecreasing order

$$\lambda_1(A) \le \lambda_2(A) \le \cdots \le \lambda_n(A).$$

Definition 2.1. An interval-valued fuzzy graph is a triple $\tilde{G} = (V, \sigma, \mu)$ where $\sigma = (\sigma_1, \dots, \sigma_n) \in (0, 1]^n$ is the vertex-membership vector and, for each $1 \le i < j \le n$,

$$\mu_{ij} = [\mu_{ij}^-, \mu_{ij}^+]$$

is a closed interval satisfying $0 \le \mu_{ij}^- \le \mu_{ij}^+ \le \min(\sigma_i, \sigma_j)$. We set $\mu_{ii} = [0, 0]$ and require symmetry, $\mu_{ij}^- = \mu_{ji}^-$, $\mu_{ij}^+ = \mu_{ji}^+$.

This definition is standard in interval-valued fuzzy graph theory; see [3, 1, 2, 4]. For spectral purposes, the vertex-membership vector acts as a feasibility constraint, whereas the uncertainty propagates through the edge-membership intervals.

Definition 2.2. An admissible realization of $\tilde{G}$ is a symmetric matrix $W = (w_{ij}) \in \mathbb{R}^{n\times n}$ such that $w_{ii} = 0$, $w_{ij} \in [\mu_{ij}^-, \mu_{ij}^+]$ for $i \neq j$. Its degree vector and Laplacian are defined by

$$d_i(W) = \sum_{j=1}^{n} w_{ij}, \quad D(W) = \operatorname{diag}(d_1(W), \dots, d_n(W)), \quad L(W) = D(W) - W.$$

The endpoint matrices $W^- = (\mu_{ij}^-)$ and $W^+ = (\mu_{ij}^+)$ are themselves admissible realizations. We also use the midpoint and radius matrices

$$M = \frac{W^- + W^+}{2}, \quad R = \frac{W^+ - W^-}{2},$$

so that every admissible realization can be written as $W = M + E$ with $|E_{ij}| \leq R_{ij}$ and $E_{ii} = 0$. We denote the lower and upper degree vectors by $d_i^- = \sum_{j=1}^n \mu_{ij}^-$ and $d_i^+ = \sum_{j=1}^n \mu_{ij}^+$, and write

$$\Delta^+ = \max_{1\leq i\leq n} d_i^+, \quad r_i = \sum_{j=1}^{n} R_{ij}, \quad r^* = \max_{1\leq i\leq n} r_i.$$

The Laplacian inherits the basic positivity structure of classical weighted graphs.

Proposition 2.3. *For every admissible realization W,*

$$L(W) = \sum_{1\leq i<j\leq n} w_{ij}(e_i - e_j)(e_i - e_j)^\top,$$

where $(e_1, \dots, e_n)$ is the standard basis of $\mathbb{R}^n$. Consequently,

$$x^\top L(W)x = \sum_{1\leq i<j\leq n} w_{ij}(x_i - x_j)^2 \quad \text{for all } x \in \mathbb{R}^n,$$

and therefore $L(W)$ is positive semidefinite with $L(W)\mathbf{1} = 0$.

Proof. The (i,j) contribution $w_{ij}(e_i - e_j)(e_i - e_j)^\top$ adds w_{ij} to the diagonal positions (i,i) and (j,j) and adds $-w_{ij}$ to the off-diagonal positions (i,j) and (j,i). Summing over $i < j$ produces exactly the matrix $D(W) - W$. The Rayleigh quotient identity follows by direct expansion: $x^\top(e_i - e_j)(e_i - e_j)^\top x = (x_i - x_j)^2$. Since every summand is positive semidefinite, so is $L(W)$. Finally, $(e_i - e_j)^\top \mathbf{1} = 0$ for every $i < j$, hence $L(W)\mathbf{1} = 0$. $\square$

Proposition 2.3 is the weighted fuzzy analogue of the standard graph Laplacian decomposition used in spectral graph theory [13, 23, 12]. It immediately yields a Rayleigh-quotient sandwich between the lower and upper endpoint graphs.

Proposition 2.4. *For every admissible realization W and every vector $x \in \mathbb{R}^n$,*

$$x^\top L(W^-)x \leq x^\top L(W)x \leq x^\top L(W^+)x.$$

Equivalently,

$$\sum_{i<j} \mu_{ij}^-(x_i - x_j)^2 \leq \sum_{i<j} w_{ij}(x_i - x_j)^2 \leq \sum_{i<j} \mu_{ij}^+(x_i - x_j)^2.$$

Proof. Because $\mu_{ij}^- \leq w_{ij} \leq \mu_{ij}^+$ for all $i < j$, the conclusion follows by termwise comparison in the Rayleigh quotient representation from Proposition 2.3. $\square$

The order structure is even stronger at the matrix level.

Lemma 2.5 (Loewner monotonicity). *If U and W are two admissible realizations satisfying $u_{ij} \le w_{ij}$ for all $i \ne j$, then $L(U) \preceq L(W)$; that is, $L(W) - L(U)$ is positive semidefinite.*

Proof. By Proposition 2.3,

$$L(W) - L(U) = \sum_{1 \le i < j \le n} (w_{ij} - u_{ij})(e_i - e_j)(e_i - e_j)^\top.$$

Every coefficient $w_{ij} - u_{ij}$ is nonnegative, so the right-hand side is a sum of positive semidefinite matrices. □

3. Exact spectral enclosure and robust connectivity

The first main theorem shows that the uncertain Laplacian spectrum is bracketed exactly by the lower and upper endpoint Laplacians.

Theorem 3.1 (Exact eigenvalue enclosure). *Let $\tilde{G}$ be an interval-valued fuzzy graph and let W be any admissible realization. Then, for every $k = 1, \dots, n$,*

$$\lambda_k(L(W^-)) \le \lambda_k(L(W)) \le \lambda_k(L(W^+)).$$

Moreover, these endpoint values are exact: for each k, the lower and upper bounds are attained by the admissible realizations W^- and W^+, respectively.

Proof. By definition, $W^- \le W \le W^+$ entrywise. Lemma 2.5 therefore gives $L(W^-) \preceq L(W) \preceq L(W^+)$. For Hermitian matrices, Loewner order implies eigenvalue monotonicity; this follows from the Courant-Fischer min-max principle. Hence

$$\lambda_k(L(W^-)) \le \lambda_k(L(W)) \le \lambda_k(L(W^+))$$

for each k. Exactness is immediate because W^- and W^+ are admissible realizations. □

Theorem 3.1 may be viewed as a fuzzy-graph version of an interval eigenvalue enclosure, but here the proof is particularly transparent because the admissible set is generated by edgewise monotonicity. The theorem yields immediate consequences for the algebraic connectivity and the spectral radius.

Corollary 3.2 (Robust connectivity). *Let $a(W) = \lambda_2(L(W))$ denote the algebraic connectivity of an admissible realization.*

(a) *If $\lambda_2(L(W^-)) > 0$, then every admissible realization is connected.*
(b) *If $\lambda_2(L(W^+)) = 0$, then every admissible realization is disconnected.*

Proof. Part (a) follows from Theorem 3.1: $\lambda_2(L(W)) \ge \lambda_2(L(W^-)) > 0$ for every admissible realization W. A weighted graph is connected if and only if its Laplacian has a simple zero eigenvalue, equivalently, if and only if $\lambda_2 > 0$ [13, 14, 24]. This proves robust connectivity.

For part (b), Theorem 3.1 gives $0 \le \lambda_2(L(W)) \le \lambda_2(L(W^+)) = 0$, hence $\lambda_2(L(W)) = 0$ for every admissible realization. □

Corollary 3.3 (Extremal spectral quantities)**.** *For every admissible realization W,*

$$\lambda_n(L(W^-)) \le \lambda_n(L(W)) \le \lambda_n(L(W^+)).$$

In particular, the interval for the largest Laplacian eigenvalue is exact. Furthermore,

$$2\sum_{i<j}\mu_{ij}^- \le \operatorname{tr} L(W) \le 2\sum_{i<j}\mu_{ij}^+$$

and this trace interval is also exact.

Proof. The first claim is Theorem 3.1 with $k = n$. For the trace, observe that $\operatorname{tr} L(W) = \sum_{i=1}^n d_i(W) = 2\sum_{i<j} w_{ij}$. Since every w_{ij} lies in $[\mu_{ij}^-, \mu_{ij}^+]$, the stated interval follows. Exactness again occurs at W^- and W^+. □

Remark 3.4. Theorem 3.1 is stronger than a Gershgorin-type box because it respects the ordered Laplacian spectrum. In particular, it separates the behavior of λ_2, interior eigenvalues, and λ_n without any extra optimization.

4. Computable bounds from midpoint perturbation and Gershgorin theory

The exact endpoint enclosure may still require computing the spectra of both $L(W^-)$ and $L(W^+)$. In many applications one also wants a quick a priori bound centered at the midpoint matrix M. This is the role of the next theorem.

Theorem 4.1 (Midpoint perturbation bound)**.** *Let W be any admissible realization and set $\Delta(W) = L(W) - L(M)$. Then*

$$\|\Delta(W)\|_\infty \le 2r^*, \quad \|\Delta(W)\|_2 \le 2r^*.$$

Consequently, for every $k = 1, \dots, n$,

$$|\lambda_k(L(W)) - \lambda_k(L(M))| \le 2r^*.$$

Proof. Write $W = M+E$ with $|E_{ij}| \le R_{ij}$ and $E_{ii} = 0$. Then $\Delta(W) = \operatorname{diag}(E\mathbf{1}) - E$. Fix a row index i. Since $\Delta_{ii} = \sum_{j\ne i} E_{ij}$ and $\Delta_{ij} = -E_{ij}$ for $j \ne i$,

$$\sum_{j=1}^n |\Delta_{ij}| = \left|\sum_{j\ne i} E_{ij}\right| + \sum_{j\ne i}|E_{ij}| \le 2\sum_{j\ne i}|E_{ij}| \le 2\sum_{j\ne i} R_{ij} = 2r_i.$$

Taking the maximum over i yields $\|\Delta(W)\|_\infty \le 2r^*$. Because $\Delta(W)$ is symmetric,

$$\|\Delta(W)\|_2 \le \|\Delta(W)\|_\infty \le 2r^*.$$

Weyl's inequality for Hermitian matrices therefore implies

$$|\lambda_k(L(W)) - \lambda_k(L(M))| \le \|\Delta(W)\|_2 \le 2r^*$$

for all k. □

Theorem 4.1 gives a computable uncertainty band around the midpoint spectrum. The next corollary controls the width of the exact interval itself.

Corollary 4.2 (Width of the exact eigenvalue interval). *For each $k = 1, \dots, n$,*

$$0 \le \lambda_k(L(W^+)) - \lambda_k(L(W^-)) \le 2\lambda_n(L(R)) \le 4r^*.$$

Equivalently, the half-width of the exact interval satisfies

$$\frac{\lambda_k(L(W^+)) - \lambda_k(L(W^-))}{2} \le \lambda_n(L(R)) \le 2r^*.$$

Proof. Since $L(W^+) - L(W^-) = 2L(R)$, we have $L(W^+) = L(W^-) + 2L(R)$. By Proposition 2.3, $L(R)$ is positive semidefinite. Applying Weyl's inequality to the Hermitian sum above gives

$$\lambda_k(L(W^+)) \le \lambda_k(L(W^-)) + 2\lambda_n(L(R)).$$

Subtracting $\lambda_k(L(W^-))$ proves the first inequality. The second follows from Gershgorin's theorem applied to $L(R)$:

$$\lambda_n(L(R)) \le 2 \max_i r_i = 2r^*.$$

□

We now record a global spectral box that depends only on the upper endpoint degrees.

Theorem 4.3 (Gershgorin-type enclosure). *For every admissible realization W,*

$$\operatorname{spec}(L(W)) \subseteq \bigcup_{i=1}^{n} [0, 2d_i(W)] \subseteq [0, 2\Delta^+].$$

In particular,

$$0 = \lambda_1(L(W)) \le \lambda_k(L(W)) \le 2\Delta^+ \quad \textit{for every } k.$$

Proof. In row i of $L(W)$ the diagonal entry is $d_i(W)$ and the sum of absolute values of the off-diagonal entries is $\sum_{j \ne i} |-w_{ij}| = d_i(W)$. Since the matrix is symmetric, Gershgorin's theorem places every eigenvalue in the real interval

$$[d_i(W) - d_i(W), d_i(W) + d_i(W)] = [0, 2d_i(W)]$$

for some i. Because $d_i(W) \le d_i^+ \le \Delta^+$, the global box follows. □

Theorems 4.1 and 4.3 are computationally light and useful when one wants certified bounds without diagonalizing both endpoint Laplacians. The price of this efficiency is conservatism, which will be quantified numerically in Section 6.

5. Homogeneous interval uncertainty on a fixed support graph

A particularly transparent case occurs when all present edges of a fixed simple support graph have the same uncertainty interval. In that regime, classical Laplacian spectra immediately transfer to interval-valued fuzzy graphs.

Let $G_0 = (V, E_0)$ be a simple graph on n vertices with adjacency matrix A_0 and Laplacian $L_0 = \operatorname{diag}(A_0\mathbf{1}) - A_0$. Assume that every edge in E_0 receives a weight in a common interval $[\alpha, \beta]$ with $0 < \alpha \le \beta$, while every nonedge has weight 0. Then every admissible realization has the form $W = (w_{ij})$ with $w_{ij} \in [\alpha, \beta]$ if $ij \in E_0$ and $w_{ij} = 0$ if $ij \notin E_0$.

Theorem 5.1 (Exact scaling interval)**.** *In the homogeneous model above,*

$$\alpha L_0 \preceq L(W) \preceq \beta L_0$$

for every admissible realization. Consequently,

$$\alpha\lambda_k(L_0) \leq \lambda_k(L(W)) \leq \beta\lambda_k(L_0), \quad k = 1, \ldots, n,$$

and each interval is exact.

Proof. Because A_0 is the support matrix, $\alpha A_0 \leq W \leq \beta A_0$ entrywise. Lemma 2.5 therefore yields $L(\alpha A_0) \preceq L(W) \preceq L(\beta A_0)$. Since $L(\alpha A_0) = \alpha L_0$ and $L(\beta A_0) = \beta L_0$, the conclusion follows from Theorem 3.1. Exactness is attained by the constant weight realizations $W = \alpha A_0$ and $W = \beta A_0$. □

Theorem 5.1 converts classical Laplacian formulas into exact uncertainty intervals. For convenience we list four canonical families. Their deterministic Laplacian spectra are classical; see [5, 13, 23, 12].

Table 1 shows that homogeneous interval uncertainty preserves the exact spectral shape of the support graph: only the scale changes. In particular for a path graph the algebraic connectivity interval is

$$\left[\alpha\left(2 - 2\cos\left(\frac{\pi}{n}\right)\right), \beta\left(2 - 2\cos\left(\frac{\pi}{n}\right)\right)\right],$$

whereas for a complete graph all nonzero eigenvalues lie in the same interval $[\alpha n, \beta n]$.

TABLE 1. Exact eigenvalue intervals in the homogeneous model of Theorem 5.1. Here $k = 1, \ldots, n$ and all present edges of the support graph have weights in $[\alpha, \beta]$.

Support graph	Deterministic Laplacian spectrum of L_0	Exact interval-valued fuzzy spectrum
P_n	$\lambda_k(L_0) = 2 - 2\cos\frac{(k-1)\pi}{n}$	$\lambda_k(L(W)) \in \left[\alpha\left(2 - 2\cos\frac{(k-1)\pi}{n}\right), \beta\left(2 - 2\cos\frac{(k-1)\pi}{n}\right)\right]$
C_n	$\lambda_k(L_0) = 2 - 2\cos\frac{2(k-1)\pi}{n}$	$\lambda_k(L(W)) \in \left[\alpha\left(2 - 2\cos\frac{2(k-1)\pi}{n}\right), \beta\left(2 - 2\cos\frac{2(k-1)\pi}{n}\right)\right]$
S_n	$0, 1^{(n-2)}, n$	0 fixed, middle $\in [\alpha, \beta]$, largest $\in [\alpha n, \beta n]$
K_n	$0, n^{(n-1)}$	0 fixed, remaining $n-1$ eigenvalues $\in [\alpha n, \beta n]$

6. Numerical illustrations

We now illustrate the theory on a six-vertex interval-valued fuzzy graph with crisp vertex memberships and uncertain edge weights. Its midpoint weights are

$$(0.74, 0.52, 0.63, 0.47, 0.66, 0.58, 0.71, 0.49, 0.41)$$

and the corresponding uncertainty radii are

$$(0.08, 0.06, 0.05, 0.07, 0.04, 0.06, 0.08, 0.05, 0.04).$$

Thus, for an uncertainty level $\tau \in [0, 1]$, each edge weight varies in the interval $[m_e - \tau\delta_e, m_e + \tau\delta_e]$. At $\tau = 1$, the interval endpoints are precisely the edge labels.

The maximal uncertainty degree of this example is $r^*(\tau) = 0.24\tau$, so Theorem 4.1 predicts the midpoint uncertainty band $\lambda_k(L(M)) \pm 0.48\tau$.

Table 2 quantifies the comparison. We denote the exact half-width of the k-th interval by

$$h_k(\tau) = \frac{\lambda_k(L(W_\tau^+)) - \lambda_k(L(W_\tau^-))}{2}.$$

The theorem-based half-width $2r^* = 0.48\tau$ dominates both $h_2(\tau)$ and $h_6(\tau)$, as predicted by Corollary 4.2, but the gap is much larger for λ_2 than for λ_6.

Table 2. Comparison between exact spectral half-widths and the midpoint perturbation half-width $2r^*$. Here $h_k(\tau) = \frac{1}{2}(\lambda_k^+ - \lambda_k^-)$, with $\lambda_k^\pm$ computed from the endpoint Laplacians.

τ	$h_2(\tau)$	$h_6(\tau)$	$2r^*$	$2r^*/h_2$	$2r^*/h_6$
0.25	0.0178	0.0768	0.1200	6.7401	1.5632
0.50	0.0356	0.1532	0.2400	6.7396	1.5663
0.75	0.0534	0.2291	0.3600	6.7389	1.5712
1.00	0.0712	0.3043	0.4800	6.7378	1.5773

Two conclusions are visible. First, the exact interval for λ_2 is highly stable relative to the generic uncertainty radius, which is consistent with the fact that algebraic connectivity depends on a global bottleneck rather than on a single large degree. Second, the interval for λ_6 expands more rapidly and tracks the large-degree vertices more closely, which is consistent with classical Laplacian spectral heuristics [21, 22, 12]. In both cases, however, the exact endpoint enclosure and the midpoint perturbation band have the same linear growth in τ, and the theorem-driven bounds remain certified for all admissible realizations.

7. Conclusion

We developed a spectral uncertainty theory for interval-valued fuzzy graph Laplacians. The key observation is that the Laplacian depends monotonically on the edge weights in the Loewner order. This leads to exact eigenvalue intervals generated by the lower and upper endpoint graphs, robust connectivity criteria, computable midpoint perturbation bounds, and Gershgorin-type global boxes. For homogeneous interval uncertainty on a fixed support graph, classical Laplacian formulas immediately produce exact interval-valued fuzzy spectra.

Several extensions suggest themselves. One can study signless and normalized Laplacians, directed fuzzy graphs, vertex-uncertain models, and inverse problems in which spectral measurements constrain admissible membership intervals. Another natural direction is to combine the present exact endpoint method with interval matrix algorithms when the admissible set contains additional dependence relations between edge weights. These questions lie at the intersection of fuzzy graph theory, discrete spectral analysis, and rigorous uncertainty quantification.

References

1. M. Akram, *Interval-valued fuzzy line graphs*, Neural Comput. Appl. **21** (2012), 145–150.

2. M. Akram, N. O. Alshehri, and W. A. Dudek, *Certain types of interval-valued fuzzy graphs*, J. Appl. Math. **2013** (2013), Art. ID 857070, 11 pp.
3. M. Akram and W. A. Dudek, *Interval-valued fuzzy graphs*, Comput. Math. Appl. **61** (2011), 289–299.
4. M. Akram, S. Yousaf, and W. A. Dudek, *Self centered interval-valued fuzzy graphs*, Afr. Mat. **26** (2015), 887–898.
5. W. N. Anderson, Jr. and T. D. Morley, *Eigenvalues of the Laplacian of a graph*, Linear Multilinear Algebra **18** (1985), 141–145.
6. K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems **20** (1986), 87–96.
7. S. Bera, T. Mahapatra, B. K. Sahoo, M. Pal, and R. Mahapatra, *Connectivity status of intuitionistic fuzzy graph and its application to merging of banks*, Mathematics **11** (2023), 1949.
8. P. Bhattacharya, *Some remarks on fuzzy graphs*, Pattern Recognition Lett. **6** (1987), 297–302.
9. K. R. Bhutani and A. Battou, *On M-strong fuzzy graphs*, Inform. Sci. **155** (2003), 103–109.
10. K. R. Bhutani and A. Rosenfeld, *Strong arcs in fuzzy graphs*, Inform. Sci. **152** (2003), 319–322.
11. R. Cai, B. Rangasamy, S. P. Karuppusamy, and A. Khan, *Characterization of degree energies and bounds in spectral fuzzy graphs*, Symmetry **17** (2025), 644.
12. K. C. Das, *The Laplacian spectrum of a graph*, Comput. Math. Appl. **48** (2004), 715–724.
13. M. Fiedler, *Algebraic connectivity of graphs*, Czechoslovak Math. J. **23** (1973), 298–305.
14. R. Grone, R. Merris, and V. S. Sunder, *The Laplacian spectrum of a graph*, SIAM J. Matrix Anal. Appl. **11** (1990), 218–238.
15. R. Grone and R. Merris, *The Laplacian spectrum of a graph. II*, SIAM J. Discrete Math. **7** (1994), 221–229.
16. M. Hladík, D. Daney, and E. P. Tsigaridas, *Bounds on real eigenvalues and singular values of interval matrices*, SIAM J. Matrix Anal. Appl. **31** (2010), 2116–2129.
17. M. Hladík, D. Daney, and E. P. Tsigaridas, *Characterizing and approximating eigenvalue sets of symmetric interval matrices*, Comput. Math. Appl. **62** (2011), 3152–3163.
18. F. Karaaslan, *Hesitant fuzzy graphs and their applications in decision making*, J. Intell. Fuzzy Systems **36** (2019), 2729–2741.
19. S. Krishnaraj, O. V. Shanmuga Sundaram, P. B. Dhandapani, and T. Radwan, *Seidel Laplacian energy of bipolar fuzzy graphs and enhanced score functions for decision-making applications*, AIMS Math. **10** (2025), 16865–16888.
20. H. Leng and Z. He, *Eigenvalue bounds for symmetric matrices with entries in one interval*, Appl. Math. Comput. **299** (2017), 58–65.
21. J.-S. Li and Y.-L. Pan, *A new upper bound for eigenvalues of the Laplacian matrix of a graph*, Linear Algebra Appl. **265** (1997), 93–100.
22. J.-S. Li and Y.-L. Pan, *On the Laplacian eigenvalues of a graph*, Linear Algebra Appl. **285** (1998), 305–307.
23. R. Merris, *Laplacian matrices of graphs: a survey*, Linear Algebra Appl. **197/198** (1994), 143–176.
24. R. Merris, *A survey of graph Laplacians*, Linear Multilinear Algebra **39** (1995), 19–31.
25. J. N. Mordeson, *Fuzzy line graphs*, Pattern Recognition Lett. **14** (1993), 381–384.
26. J. N. Mordeson and C.-S. Peng, *Operations on fuzzy graphs*, Inform. Sci. **79** (1994), 159–170.
27. K. R. Nair and M. S. Sunitha, *Strong domination index in fuzzy graphs*, Fuzzy Information and Engineering **16** (2024), 1–23.
28. A. M. Philip, S. J. Kalayathankal, and J. V. Kureethara, *On the Laplacian energy of interval valued fuzzy graphs*, AIP Conf. Proc. **2261** (2020), 030142.
29. Y. Rao, S. Lei, A. A. Talebi, and M. Mojahedfar, *A novel concept of level graph in interval-valued fuzzy graphs with application*, Symmetry **15** (2023), 2106.
30. N. R. Reddy, M. Z. Khan, S. Sharief Basha, A. Alahmadi, A. H. Alahmadi, and C. L. Chowdhary, *The Laplacian energy of hesitancy fuzzy graphs in decision-making problems*, Comput. Syst. Sci. Eng. **44** (2023), 2637–2653.

31. M. U. Romdhini, F. Al-Sharqi, A. Nawawi, A. Al-Quran, and H. Rashmanlou, *Signless Laplacian energy of interval-valued fuzzy graph and its applications*, Sains Malaysiana **52** (2023), 2127–2137.
32. A. Rosenfeld, *Fuzzy graphs*, in: Fuzzy Sets and Their Applications to Cognitive and Decision Processes, Academic Press, New York, 1975, pp. 77–95.
33. X. Shi, S. Kosari, A. A. Talebi, S. H. Sadati, and H. Rashmanlou, *Investigation of the main energies of picture fuzzy graph and its applications*, Int. J. Comput. Intell. Syst. **15** (2022), 31.
34. M.-H. Shih and Y.-D. Lur, *An inequality for the spectral radius of an interval matrix*, Linear Algebra Appl. **274** (1998), 27–36.
35. Q. Yuan and Z. Yang, *A fast algorithm for the eigenvalue bounds of a class of symmetric tridiagonal interval matrices*, Applied Math **3** (2023), 90–97.
36. L. A. Zadeh, *Fuzzy sets*, Inform. Control **8** (1965), 338–353.

Asokan Vasudevan: Faculty of Business and Communications, INTI International University, Nilai 71800, Malaysia
Email address: asokan.vasudevan@newinti.edu.my

Yogeesh N: Research Fellow, INTI International University, 71800 Nilai, Negeri Sembilan, Malaysia; Mathematics & Natural Sciences, Gulf University for Science & Technology, Kuwait; Department of Mathematics, Government First Grade College, Tumkur-572102, India
Email address: yogeesh.r@gmail.com

Koyel Roy: Symbiosis Law School Hyderabad, Symbiosis International (Deemed University), Nandigaon-509217, Hyderabad, Telangana, India
Email address: koyelroy847@gmail.com

P. William: School of Computer Science and Technology, Karunya Institute of Technology and Sciences (Deemed University), Coimbatore, India; Centre for Research and Consultancy, UNITAR International University, Selangor, Malaysia
Email address: william160891@gmail.com

Zetty Pakir Mastan: Faculty of Engineering and Quantity Surveying, INTI International University, Malaysia
Email address: srzetty.pakirmastan@newinti.edu.my

Thulasi Manoharan: Faculty of Business and Communications, INTI International University, Persiaran Perdana BBN Putra Nilai, 71800 Nilai, Negeri Sembilan, Malaysia
Email address: thulasi.manoharan@newinti.edu.my