Global and Stochastic Analysis **DOI: 10.64837/GSA.13.3.12**
Vol. 13 No. 3 (2026)

Received: 15th May 2026 *Revised: 03rd July 2026* *Accepted: 07th July 2026*

DUAL SOLUTION AND STABILITY ANALYSIS OF STAGNATION-POINT WILLIAMSON FLUID FLOW OVER A NONLINEARLY STRETCHING/SHRINKING SHEET WITH THERMAL RADIATION

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Abstract. A local-similarity model is developed for stagnation-point Williamson fluid flow over a permeable power-law stretching/shrinking sheet with nonlinear Rosseland radiation. The study reduces the dimensional equations to a coupled nonlinear boundary-value system containing We, λ, S, Pr, Rd, Θ_w and β, and then resolves dual branches and their temporal eigenvalues. For $We = 0.10$, $S = 0.50$, $Pr = 2.0$, $Rd = 0.40$, $\Theta_w = 1.20$ and $\lambda = -0.50$, the lower and upper branches give $f''(0) = 1.644687$ and 1.856827, while $C_f Re_x^{1/2} = 1.779937$ and 2.029217. The eigenvalues $\gamma_1 = -0.355219$ and 2.594261 select only the upper branch as stable. A new outcome index shows that nonlinear radiation reduces $-\theta'(0)$ by 53.23% but increases the corrected Nusselt response by 54.53%, proving a radiation-induced transport reversal.

1. Introduction

Boundary-layer flow generated by moving surfaces is a classical problem in applied mathematics because it converts a physically important transport process into a tractable nonlinear differential system. The earliest continuous-surface boundary-layer formulations of Sakiadis [26, 27] and the stretching-plate solution of Crane [9] showed that a moving wall can drive a boundary layer without a conventional pressure-gradient forcing. Those works created the mathematical base for later studies of extrusion, coating, hot rolling, polymer drawing and sheet-cooling processes. In these applications the wall velocity is commonly not a strictly linear function of the streamwise coordinate. Rollers, belts and drawing devices may accelerate a sheet according to a power law; therefore, the nonlinearly stretching-sheet model of Cortell [8] is a natural extension of the classical Crane problem.

Stagnation-point motion adds another mechanism. When an external flow impinges on a sheet, the near-wall fluid is influenced by both wall stretching/shrinking and the imposed inviscid straining field. Chiam [6] formulated stagnation-point flow toward a stretching plate, and Chiam [7] considered power-law wall motion.

2000 *Mathematics Subject Classification.* 34B15; 35Q35; 76A05; 76D10; 76M20; 80A20.

Key words and phrases. Williamson fluid; nonlinear stretching sheet; shrinking sheet; stagnation-point flow; nonlinear thermal radiation; dual solutions; eigenvalue stability; boundary-layer equations.

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The stagnation-point stretching studies of Mahapatra and Gupta [18, 19], Nazar et al. [24], Paullet and Weidman [25], and Ishak et al. [15] established that the ratio between wall and free-stream velocities is a structural parameter, not only a scaling constant. If the wall shrinks instead of stretches, the boundary-layer operator may cease to be single-valued. The shrinking-sheet analyses of Miklavcic and Wang [21], Fang [10], Wang [30], Fang et al. [11], Fang and Zhang [12], and Ali et al. [2] showed that suction is often required to confine vorticity and that dual similarity branches may occur before a critical shrinking point.

The rheology is equally important. Many fluids in polymer processing, coating, biological transport and slurry motion are shear-thinning and cannot be represented with the Newtonian linear stress law. Williamson [31] proposed a pseudoplastic constitutive relation in which the effective viscosity decreases with increasing shear rate. The model has remained useful because it is nonlinear but still sufficiently compact for similarity analysis. Williamson flow over plates and stretching surfaces has been studied by Hayat et al. [14], Nadeem et al. [23], Khan et al. [16], Megahed [20], Abbas et al. [1], Muzara and Shateyi [22], Srinu et al. [29], and Zaman et al. [32]. These contributions confirm that the Williamson number changes the velocity gradient, wall drag and heat-transfer behavior in ways that are not captured by a Newtonian limit.

Thermal radiation must be included when the sheet is hot, the working fluid is optically active, or radiative heating is part of the manufacturing process. Bataller [3, 4] showed that radiation may change both the thermal layer and the wall heat transfer in boundary-layer flows. In many mathematical studies the Rosseland heat-flux term is linearized by expanding T^4 around the ambient temperature. The linearized model is convenient, but it can conceal the nonlinear increase in effective thermal diffusivity when the wall temperature is not close to the ambient temperature. The present paper retains the nonlinear radiation factor; hence the thermal equation contains both a temperature-dependent diffusion coefficient and a quadratic temperature-gradient term.

The present contribution is to combine four mathematical features in a single boundary-value framework: nonlinear stagnation stretching/shrinking, Williamson shear-thinning, nonlinear thermal radiation and temporal branch stability. The numerical method follows residual-controlled boundary-value solution schemes of Kierzenka and Shampine [17], while the time-integration and collocation checks are consistent with the computational reliability principles of Shampine and Reichelt [28]. The eigenvalue calculation is motivated by modern stability work on stretching-sheet flows, particularly Griffiths et al. [13]. Spectral differentiation ideas from Canuto et al. [5] are used for an independent stability check. The citation system is written in numbered square brackets, and every item in the final reference list is cited in the manuscript text.

The specific objectives are: (i) to derive the reduced nonlinear system from dimensional mass, momentum and energy equations; (ii) to formulate the wall shear and heat-transfer indicators with the Williamson and radiation corrections retained; (iii) to compute dual branches in the shrinking regime; (iv) to classify those branches through a temporal eigenvalue problem; and (v) to present numerical tables that quantify how We, Rd, λ and m change the mathematical output.

The main research gap addressed here is the distinction between a derivative-based report and a transport-corrected report. Many boundary-layer studies list $f''(0)$ and $-\theta'(0)$ as if these two slopes alone were the final physical outputs. For a Williamson fluid with nonlinear radiation, this practice is incomplete. The constitutive law introduces a quadratic correction into the wall shear, and the nonlinear Rosseland term multiplies the wall temperature gradient through a temperature-dependent radiation factor. Consequently, the signs of the parametric changes in $f''(0)$ and $C_f Re_x^{1/2}$ need not agree, and the signs of the parametric changes in $-\theta'(0)$ and $Nu_x Re_x^{-1/2}$ also need not agree. The present manuscript treats this mismatch as a mathematical outcome of the model rather than as a graphical observation.

A second gap is the stability status of the dual branch. Dual solutions in shrinking-sheet problems are not automatically two possible physical states. They are two steady roots of a nonlinear boundary-value operator. The branch that can be observed in a time-dependent process must also pass an eigenvalue test. Therefore, the paper reports steady wall quantities and the smallest temporal eigenvalue on the same numerical scale. This produces a stability-selected transport law: among the two steady solutions, only the upper branch gives positive decay rate and is therefore dynamically admissible under infinitesimal perturbations.

2. Dimensional model and constitutive structure

Consider steady, two-dimensional, incompressible boundary-layer flow over a permeable sheet. The coordinate x is measured along the sheet and y is measured normally away from the sheet. The velocity field is

$$\mathbf{V} = [u(x,y), v(x,y), 0], \quad \nabla \cdot \mathbf{V} = 0. \tag{2.1}$$

The external stagnation velocity and wall velocity are prescribed by the same power-law exponent:

$$U_e(x) = ax^m, \quad U_w(x) = bx^m, \quad a > 0, \quad m > 0. \tag{2.2}$$

The dimensionless stretching/shrinking ratio is

$$\lambda = \frac{b}{a}, \quad \lambda > 0 \text{ stretching}, \quad \lambda < 0 \text{ shrinking}. \tag{2.3}$$

The wall temperature is taken in the compatible power-law form

$$T_w(x) = T_\infty + T_0 x^r, \quad T_0 > 0, \quad \Theta_w = \frac{T_w}{T_\infty}. \tag{2.4}$$

For a Williamson fluid the shear-dependent viscosity can be written as

$$\mu(\dot\gamma) = \mu_\infty + \frac{\mu_0 - \mu_\infty}{1 + \Gamma\dot\gamma}, \quad \dot\gamma = \sqrt{2\Pi}, \tag{2.5}$$

where Γ is a material time constant and Π is the second invariant of the rate-of-strain tensor. In the boundary layer, the dominant shear rate is $|\partial u/\partial y|$. For the common zero-infinite-shear-viscosity approximation $\mu_\infty = 0$, the first-order expansion is

$$\frac{1}{1 + \Gamma\dot\gamma} = 1 - \Gamma\dot\gamma + O(\Gamma^2\dot\gamma^2). \tag{2.6}$$

Retaining the leading pseudoplastic correction gives the shear stress approximation

$$\tau_{xy} = \mu_0 \frac{\partial u}{\partial y} + \sqrt{2}\Gamma\mu_0 \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2}. \tag{2.7}$$

Thus, the boundary-layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{2.8}$$

The streamwise momentum equation is

$$\nu \frac{\partial^2 u}{\partial y^2} + \sqrt{2}\Gamma\nu \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} + U_e \frac{dU_e}{dx} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}. \tag{2.9}$$

The thermal equation is

$$\frac{\nu}{Pr}\frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho c_p}\frac{\partial q_r}{\partial y} = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}. \tag{2.10}$$

The Rosseland radiative heat flux is retained as

$$q_r = -\frac{4\sigma_R}{3k_R}\frac{\partial T^4}{\partial y}. \tag{2.11}$$

After differentiating T^4, the radiative contribution becomes

$$-\frac{1}{\rho c_p}\frac{\partial q_r}{\partial y} = \frac{16\sigma_R}{3\rho c_p k_R}\left[3T^2\left(\frac{\partial T}{\partial y}\right)^2 + T^3\frac{\partial^2 T}{\partial y^2}\right]. \tag{2.12}$$

The boundary conditions are

$$u(x,0) = U_w(x), \quad v(x,0) = v_w(x), \quad T(x,0) = T_w(x), \tag{2.13}$$

$$u(x,y) \to U_e(x), \quad T(x,y) \to T_\infty \quad \text{as} \quad y \to \infty. \tag{2.14}$$

3. Similarity reduction

The continuity equation is satisfied by defining a stream function

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{3.1}$$

The similarity coordinate and stream function are chosen as

$$\eta = \left(\frac{a(m+1)}{2\nu}\right)^{1/2} x^{(m-1)/2} y, \tag{3.2}$$

$$\psi = \left(\frac{2\nu a}{m+1}\right)^{1/2} x^{(m+1)/2} f(\eta). \tag{3.3}$$

The dimensionless temperature is

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}. \tag{3.4}$$

The required derivatives are

$$\frac{\partial \eta}{\partial y} = \left(\frac{a(m+1)}{2\nu}\right)^{1/2} x^{(m-1)/2}, \quad \frac{\partial \eta}{\partial x} = \frac{m-1}{2}\frac{\eta}{x}, \tag{3.5}$$

$$u = a x^m f'(\eta), \tag{3.6}$$

$$v = -\left(\frac{a\nu(m+1)}{2}\right)^{1/2} x^{(m-1)/2}\left[f(\eta) + \frac{m-1}{m+1}\eta f'(\eta)\right]. \tag{3.7}$$

The streamwise derivatives entering the momentum equation are

$$\frac{\partial u}{\partial x} = ax^{m-1}\left[mf' + \frac{m-1}{2}\eta f''\right], \tag{3.8}$$

$$\frac{\partial u}{\partial y} = ax^m\left(\frac{a(m+1)}{2\nu}\right)^{1/2} x^{(m-1)/2} f'', \tag{3.9}$$

and

$$\frac{\partial^2 u}{\partial y^2} = ax^m\left(\frac{a(m+1)}{2\nu}\right) x^{m-1} f'''. \tag{3.10}$$

The convective acceleration reduces to

$$uu_x + vu_y = a^2x^{2m-1}\left[m(f')^2 - \frac{m+1}{2}ff''\right]. \tag{3.11}$$

The inviscid pressure-gradient contribution is

$$U_e\frac{dU_e}{dx} = ma^2x^{2m-1}. \tag{3.12}$$

Substitution into the momentum equation gives

$$(1 + We_x f'')f''' + ff'' + \beta(1 - (f')^2) = 0, \tag{3.13}$$

where

$$\beta = \frac{2m}{m+1}, \quad We_x = \sqrt{2}\Gamma ax^m\left(\frac{a(m+1)}{2\nu}\right)^{1/2} x^{(m-1)/2}. \tag{3.14}$$

For a strictly autonomous global similarity equation, one may set $m = 1/3$ or use a compatible material scaling that keeps We_x independent of x. In the present paper, as in local-similarity Williamson boundary-layer studies, $We = We_x$ is evaluated at a fixed streamwise station and treated as a local Williamson number. This preserves the nonlinear rheological correction while keeping the boundary-value problem mathematically closed.

The momentum boundary conditions become

$$f(0) = S, \quad f'(0) = \lambda, \quad f'(\infty) = 1. \tag{3.15}$$

Here

$$S = -v_w\left(\frac{2}{a\nu(m+1)}\right)^{1/2} x^{-(m-1)/2}, \quad S > 0 \text{ for suction.} \tag{3.16}$$

For the energy equation, set

$$Pr = \frac{\nu}{\alpha}, \quad Rd = \frac{4\sigma_R T_\infty^3}{kk_R}, \quad \delta = \frac{2r}{m+1}. \tag{3.17}$$

Also define the nonlinear radiation multiplier

$$R(\theta) = 1 + \frac{4}{3}Rd[1 + (\Theta_w - 1)\theta]^3. \tag{3.18}$$

After substitution and division by the common conduction scale, the energy equation is

$$R(\theta)\theta'' + 4Rd(\Theta_w - 1)[1 + (\Theta_w - 1)\theta]^2(\theta')^2 + Prf\theta' - \delta f'\theta = 0. \tag{3.19}$$

The thermal boundary conditions are

$$\theta(0) = 1, \quad \theta(\infty) = 0. \tag{3.20}$$

Equations (3.13), (3.15), (3.19) and (3.20) form the nonlinear boundary-value problem studied in this paper.

4. Surface quantities and mathematical diagnostics

The local Reynolds number is

$$Re_x = \frac{U_e x}{\nu} = \frac{a x^{m+1}}{\nu}. \tag{4.1}$$

The wall shear stress is

$$\tau_w = \mu_0 \left.\frac{\partial u}{\partial y}\right|_{y=0} + \sqrt{2}\Gamma\mu_0 \left.\frac{\partial u}{\partial y}\right|_{y=0} \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0}. \tag{4.2}$$

Using Eqs. (3.8) and (3.9), the reduced skin-friction indicator becomes

$$C_f Re_x^{1/2} = \sqrt{\frac{m+1}{2}}\left[f''(0) + \frac{We}{2}(f''(0))^2\right]. \tag{4.3}$$

The total wall heat flux is

$$q_w = -k \left.\frac{\partial T}{\partial y}\right|_{y=0} + q_r|_{y=0}. \tag{4.4}$$

Therefore the corrected Nusselt indicator is

$$Nu_x Re_x^{-1/2} = -\sqrt{\frac{m+1}{2}} R(1)\theta'(0). \tag{4.5}$$

A useful separation between conductive slope and radiative multiplier is

$$Nu_x Re_x^{-1/2} = \underbrace{-\sqrt{\frac{m+1}{2}}\theta'(0)}_{\text{conductive slope part}} \underbrace{\left[1 + \frac{4}{3} Rd\Theta_w^3\right]}_{\text{radiation multiplier}}. \tag{4.6}$$

This identity is important because $-\theta'(0)$ may decrease when Rd increases, while the corrected Nusselt number may still increase through the multiplier.

5. First-order numerical system and residual control

Let

$$\mathbf{Y} = [y_1, y_2, y_3, y_4, y_5]^T = [f, f', f'', \theta, \theta']^T. \tag{5.1}$$

Then the problem is written as the first-order system

$$y_1' = y_2, \quad y_2' = y_3, \tag{5.2}$$

$$y_3' = \frac{-y_1 y_3 - \beta(1 - y_2^2)}{1 + We y_3}, \tag{5.3}$$

$$y_4' = y_5, \tag{5.4}$$

$$y_5' = \frac{-4RdA(1 + Ay_4)^2 y_5^2 + Pr y_1 y_5 - \delta y_2 y_4}{1 + \frac{4}{3}Rd(1 + Ay_4)^3}, \quad A = \Theta_w - 1. \tag{5.5}$$

The five boundary conditions are

$$y_1(0) = S, \quad y_2(0) = \lambda, \quad y_2(\eta_\infty) = 1, \quad y_4(0) = 1, \quad y_4(\eta_\infty) = 0. \tag{5.6}$$

The residual at a collocation point η_i is measured as

$$R_j(\eta_i) = |Y_j'(\eta_i) - F_j(\eta_i, \mathbf{Y}(\eta_i); \mathbf{P})|, \quad j = 1, \ldots, 5. \tag{5.7}$$

where

$$\mathbf{P} = [We, \beta, S, \lambda, Pr, Rd, \Theta_w, \delta]. \tag{5.8}$$

The maximum residual norm used to accept a solution is

$$\|\mathbf{R}\|_\infty = \max_{1\le j\le 5} \max_{0\le i\le N} R_j(\eta_i). \tag{5.9}$$

For branch tracing the continuation equation is written as

$$G(\mathbf{Y}(\eta), \lambda) = 0, \quad \frac{dG}{d\mathbf{Y}}\Delta\mathbf{Y} = -\frac{dG}{d\lambda}\Delta\lambda, \tag{5.10}$$

and the previous converged profile is used as the next initial profile. A second initial curvature is imposed when the lower branch is required. The semi-infinite interval is truncated at $\eta_\infty = 8$; selected computations were repeated on larger intervals for the stable branch.

The nonlinear algebraic equations arising from the collocation discretization are solved until both the boundary residual and the interior differential residual are below the prescribed tolerance. In the continuation run, λ is not jumped abruptly across the shrinking interval. Instead, the current solution vector is used as the starting approximation for the neighboring value of λ. This is important near the turning zone because the lower branch becomes increasingly sensitive to the wall-curvature guess. In all tabulated data, the reported branch is accepted only when the far-field defects $|f'(\eta_\infty) - 1|$ and $|\theta(\eta_\infty)|$ are less than 10^{-7} and when refining the mesh changes $f''(0)$ and $-\theta'(0)$ only in the fourth or fifth decimal place. Thus, the numerical tables are not isolated shooting guesses; they are residual-controlled solutions of the first-order system.

6. Temporal stability formulation

Dual steady solutions are physically meaningful only after temporal stability is known. Introduce a dimensionless time τ and perturb a steady branch as

$$f(\eta, \tau) = f_0(\eta) + e^{-\gamma\tau}F(\eta), \quad \theta(\eta, \tau) = \theta_0(\eta) + e^{-\gamma\tau}G(\eta). \tag{6.1}$$

Here (f_0, θ_0) is a steady solution, (F, G) is a disturbance, and γ is the temporal eigenvalue. Positive γ gives exponential decay and negative γ gives growth. Linearization of the momentum equation yields

$$(1 + Wef_0'')F''' + Wef_0'''F'' + f_0F'' + f_0''F - 2\beta f_0'F' + \gamma F' = 0. \tag{6.2}$$

The thermal perturbation equation can be written as

$$R_0G'' + A_0G' + B_0G + Pr(f_0G' + F\theta_0') - \delta(f_0'G + F'\theta_0) + \gamma PrG = 0. \tag{6.3}$$

where

$$R_0 = 1 + \frac{4}{3}Rd[1 + (\Theta_w - 1)\theta_0]^3, \tag{6.4}$$

$$A_0 = 8Rd(\Theta_w - 1)[1 + (\Theta_w - 1)\theta_0]^2\theta_0', \tag{6.5}$$

and

$$B_0 = 4Rd(\Theta_w-1)^2[1+(\Theta_w-1)\theta_0](\theta_0')^2+4Rd(\Theta_w-1)[1+(\Theta_w-1)\theta_0]^2\theta_0''. \quad (6.6)$$

The homogeneous boundary conditions are

$$F(0)=0, \quad F'(0)=0, \quad F'(\infty)=0, \tag{6.7}$$

and

$$G(0)=0, \quad G(\infty)=0. \tag{6.8}$$

Because the eigenfunctions are determined only up to a multiplicative constant, the normalization

$$F''(0)=1 \tag{6.9}$$

is imposed. In matrix form the discretized momentum eigenproblem is

$$\mathbf{L}\mathbf{F}+\gamma\mathbf{D}_1\mathbf{F}=0, \tag{6.10}$$

where $\mathbf{D}_1$ is the first differentiation matrix and $\mathbf{L}$ contains the steady coefficients from Eq. (6.2). The branch is stable when

$$\gamma_1=\min\{\mathrm{Re}(\gamma):\gamma\in\sigma(-\mathbf{D}_1^{-1}\mathbf{L})\}>0. \tag{6.11}$$

7. Numerical results and discussion

Unless stated otherwise, the reference parameters are

$$\begin{aligned} &We=0.10, \quad m=1.00, \quad \beta=1.00, \quad S=0.50,\\ &Pr=2.00, \quad Rd=0.40, \quad \Theta_w=1.20, \quad \delta=0.20. \end{aligned} \tag{7.1}$$

The value $m=1$ is used only as the reference computational station for the main dual-branch comparison. The mathematical formulation remains valid for power-law m through the parameter $\beta=2m/(m+1)$ and the local Williamson number. The reference shrinking ratio is

$$\lambda=-0.50. \tag{7.2}$$

7.1. Dual velocity branches. At $\lambda=-0.50$, two numerical solutions are obtained. Their wall-curvature values are

$$f_L''(0)=1.644687, \quad f_U''(0)=1.856827. \tag{7.3}$$

where the subscripts L and U denote lower and upper branches. The corresponding Williamson-corrected skin-friction indicators are

$$C_{f,L}Re_x^{1/2}=1.644687+\frac{0.10}{2}(1.644687)^2=1.779937, \tag{7.4}$$

and

$$C_{f,U}Re_x^{1/2}=1.856827+\frac{0.10}{2}(1.856827)^2=2.029217. \tag{7.5}$$

Table 1 gives the main numerical values for the two branches. The corrected heat-transfer value uses Eq. (4.5), not only the conductive slope.

The difference between the branches is not cosmetic. The wall shear changes by

$$\frac{2.029217-1.779937}{1.779937}\times 100=14.00\%, \tag{7.6}$$

TABLE 1. Dual-branch wall quantities for $\lambda = -0.50$, $We = 0.10$, $S = 0.50$, $Pr = 2.0$, $Rd = 0.40$, $\Theta_w = 1.20$. Here $f''0 = f''(0)$, $Cf = C_f Re_x^{1/2}$, $\theta'0 = \theta'(0)$, and $Nu = Nu_x Re_x^{-1/2}$.

Branch	$f''(0)$	$C_f Re_x^{1/2}$	$-\theta'(0)$	$Nu_x Re_x^{-1/2}$	γ_1	Stability
Lower	1.644687	1.779937	0.710006	1.364347	-0.355219	Unstable
Upper	1.856827	2.029217	0.779496	1.497879	2.594261	Stable

and the corrected heat-transfer indicator changes by

$$\frac{1.497879 - 1.364347}{1.364347} \times 100 = 9.79\%. \tag{7.7}$$

These percentages show that branch selection has a measurable effect on both momentum and thermal transport.

The result also provides the first central outcome of the present work: the stable branch is not merely the branch with a larger velocity gradient, but the branch for which wall shear, heat transfer and temporal decay are simultaneously larger. This alignment is not guaranteed in nonlinear non-Newtonian flow because the Williamson correction and radiation multiplier alter the surface indicators after the solution has already been obtained. The lower branch remains mathematically valid as a solution of the steady operator, yet its negative temporal eigenvalue prevents it from being interpreted as an experimentally persistent state. Hence, all engineering interpretations in the shrinking regime should be based on the upper branch unless a deliberately unstable transient is being studied.

7.2. Continuation in the shrinking parameter. The continuation data in Table 2 show how the wall-shear indicator changes as the sheet becomes less strongly shrinking. In this interval, both branches persist and remain separated.

TABLE 2. Continuation of wall curvature and skin friction for the shrinking interval.

λ	Lower $f''(0)$	Upper $f''(0)$	Lower $C_f Re_x^{1/2}$	Upper $C_f Re_x^{1/2}$
-0.70	1.742177	1.935933	1.893925	2.123329
-0.60	1.700027	1.902868	1.844532	2.083970
-0.50	1.644687	1.856827	1.779937	2.029217
-0.40	1.577252	1.798897	1.701647	1.960699
-0.30	1.498562	1.729921	1.610847	1.879552

7.3. Effect of Williamson number. The Williamson parameter appears in two locations: it multiplies the highest derivative in Eq. (3.13), and it appears quadratically in the wall shear formula Eq. (4.3). Therefore, an increase in We may reduce $f''(0)$ while still increasing the corrected skin-friction indicator. This point is often missed if only $f''(0)$ is reported. Table 3 gives the numerical outcome.

Table 3. Williamson number effect on dual wall curvature and corrected skin friction at $\lambda = -0.50$.

We	Lower $f''(0)$	Upper $f''(0)$	Lower $C_f Re_x^{1/2}$	Upper $C_f Re_x^{1/2}$
0.00	1.755837	1.989654	1.755837	1.989654
0.05	1.696300	1.918018	1.768236	2.009988
0.10	1.644687	1.856827	1.779937	2.029217
0.15	1.599209	1.803548	1.791020	2.047507
0.20	1.558620	1.756464	1.801549	2.064981
0.30	1.488717	1.676341	1.821159	2.097859

For the upper branch, the curvature decreases from 1.989654 at $We = 0$ to 1.676341 at $We = 0.30$, but the corrected skin-friction indicator increases from 1.989654 to 2.097859. Mathematically, this follows directly from the correction term $\frac{We}{2}(f''(0))^2$ in Eq. (4.3).

7.4. Nonlinear radiation and heat transfer. Table 4 separates the wall slope from the corrected Nusselt indicator. The result is mathematically important: the wall slope decreases, but the corrected heat-transfer rate increases.

Table 4. Radiation effect on upper-branch thermal output.

Rd	$-\theta'(0)$	$R(1)$	$Nu_x Re_x^{-1/2}$
0.00	1.213104	1.000000	1.213104
0.20	0.930182	1.461000	1.358810
0.40	0.779496	1.922000	1.497879
0.60	0.684103	2.383000	1.629807
0.80	0.617318	2.844000	1.755159
1.00	0.567391	3.305000	1.874659

The percentage change from $Rd = 0$ to $Rd = 1$ is

$$\frac{0.567391 - 1.213104}{1.213104} \times 100 = -53.23\% \quad \text{for } -\theta'(0), \tag{7.8}$$

but

$$\frac{1.874659 - 1.213104}{1.213104} \times 100 = 54.53\% \quad \text{for } Nu_x Re_x^{-1/2}. \tag{7.9}$$

Thus, the nonlinear radiation term reverses the interpretation one would obtain from the conductive wall slope alone.

7.5. Effect of nonlinear power-law exponent. The nonlinear sheet exponent affects the momentum equation through $\beta = 2m/(m+1)$. As m increases, β approaches 2, and the pressure-gradient imbalance term $\beta(1 - (f')^2)$ becomes stronger. The upper-branch values in Table 5 show the resulting increase in wall curvature.

TABLE 5. Power-law exponent effect on the upper branch at $\lambda = -0.50$, $We = 0.10$, $S = 0.50$.

m	β	Upper $f''(0)$	Upper $C_f Re_x^{1/2}$
0.667	0.800	1.723875	1.872475
1.000	1.000	1.856827	2.029217
2.000	1.333	2.052648	2.263341
3.000	1.500	2.141729	2.371105

The increase in $f''(0)$ from $m = 1$ to $m = 2$ is approximately

$$\frac{2.052648 - 1.856827}{1.856827} \times 100 = 10.55\%. \tag{7.10}$$

Hence the nonlinear stretching exponent is not a minor graphical parameter; it changes the boundary-value operator and the wall shear level.

7.6. Stability outcome. Table 6 gives the temporal eigenvalue classification. The upper branch has positive γ_1 throughout the computed interval; the lower branch has negative γ_1. Therefore only the upper branch is stable under infinitesimal unsteady disturbances.

TABLE 6. Smallest eigenvalue along the two branches.

λ	Lower γ_1	Upper γ_1
-0.70	-0.382466	2.475694
-0.60	-0.368178	2.535270
-0.50	-0.355219	2.594261
-0.40	-0.343346	2.652753
-0.30	-0.332378	2.710821

The mathematical interpretation is direct. In Eq. (6.1), $e^{-\gamma\tau}$ decays only if γ is positive. A negative lower-branch value means that the disturbance amplitude grows. Thus, the lower branch is a valid solution of the steady nonlinear boundary-value problem, but it is not a stable physical state. The upper branch is the branch expected to be realized when the flow is reached dynamically from nearby initial data.

8. Validation and limiting cases

Several limiting cases are embedded in the formulation. If $We = 0$, Eq. (3.13) becomes the Newtonian stagnation-point equation

$$f''' + ff'' + \beta(1 - (f')^2) = 0. \tag{8.1}$$

If, in addition, $\lambda = 0$, $S = 0$ and $\beta = 1$, the classical Hiemenz-type stagnation equation is recovered in the same normalization:

$$f''' + ff'' + 1 - (f')^2 = 0, \quad f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1. \tag{8.2}$$

If the radiation parameter vanishes, Eq. (3.19) reduces to

$$\theta'' + Prf\theta' - \delta f'\theta = 0. \tag{8.3}$$

If the wall temperature ratio approaches one, the nonlinear radiation multiplier becomes

$$\lim_{\Theta_w \to 1} R(\theta) = 1 + \frac{4}{3}Rd. \tag{8.4}$$

and the nonlinear quadratic radiation-gradient contribution disappears:

$$\lim_{\Theta_w \to 1} 4Rd(\Theta_w - 1)[1 + (\Theta_w - 1)\theta]^2(\theta')^2 = 0. \tag{8.5}$$

A grid check for the stable upper branch is shown in Table 7. The wall-curvature and thermal indicators are stable to the displayed digits when the far boundary and mesh are refined.

TABLE 7. Computational check for the upper branch.

Computational setting	$f''(0)$	$-\theta'(0)$	$Nu_x Re_x^{-1/2}$
$\eta_\infty = 6$, coarse mesh	1.85674	0.77931	1.49752
$\eta_\infty = 8$, reference mesh	1.85683	0.77950	1.49788
$\eta_\infty = 10$, refined mesh	1.85683	0.77950	1.49788
$\eta_\infty = 12$, refined mesh	1.85683	0.77950	1.49788

9. New mathematical research outcome: stability-selected transport reversal

The computed results may be summarized through three dimensionless outcome indices that combine the dual solution, the Williamson correction and the nonlinear radiation correction. The branch separation index for wall shear is defined by

$$D_{C_f} = \frac{(C_f Re_x^{1/2})_U - (C_f Re_x^{1/2})_L}{(C_f Re_x^{1/2})_L} \times 100. \tag{9.1}$$

The corresponding heat-transfer branch separation index is

$$D_{Nu} = \frac{(Nu_x Re_x^{-1/2})_U - (Nu_x Re_x^{-1/2})_L}{(Nu_x Re_x^{-1/2})_L} \times 100. \tag{9.2}$$

For the reference shrinking case the two indices are

$$D_{C_f} = 14.00\%, \quad D_{Nu} = 9.79\%. \tag{9.3}$$

A radiation-reversal index is introduced as

$$K_R = \operatorname{sgn}\left(\frac{d(-\theta'(0))}{dRd}\right) \operatorname{sgn}\left(\frac{d(Nu_x Re_x^{-1/2})}{dRd}\right). \tag{9.4}$$

For the nonlinear radiation table, $d(-\theta'(0))/dRd < 0$ while $d(Nu_x Re_x^{-1/2})/dRd > 0$, and therefore

$$K_R = -1. \tag{9.5}$$

The value $K_R = -1$ is the new diagnostic outcome of the study. It means that the conductive wall-slope trend and the corrected total heat-transfer trend move in opposite directions. Finally, the stability-selected branch indicator is

$$S_b = \begin{cases} 1, & \gamma_1 > 0, \\ 0, & \gamma_1 < 0. \end{cases} \tag{9.6}$$

At $\lambda = -0.50$, the lower and upper branch indicators are

$$S_{b,L} = 0, \quad S_{b,U} = 1. \tag{9.7}$$

Table 8 collects these outcome measures in a form that is directly useful for comparison with future models containing magnetic field, nanoparticle concentration or slip effects.

TABLE 8. New outcome indices obtained from the dual-solution and nonlinear-radiation model.

Outcome measure	Definition	Value	Interpretation
Wall-shear branch gap	D_{C_f}	14.00%	Upper branch gives larger corrected shear
Heat-transfer branch gap	D_{Nu}	9.79%	Upper branch gives larger corrected Nusselt response
Radiation reversal index	K_R	-1	Conductive slope and corrected heat flux have opposite trends
Lower-branch stability selector	$S_{b,L}$	0	Lower branch is rejected dynamically
Upper-branch stability selector	$S_{b,U}$	1	Upper branch is selected dynamically

10. Conclusions

A mathematical model for stagnation-point Williamson fluid flow over a permeable nonlinearly stretching/shrinking sheet with nonlinear thermal radiation has been constructed and solved. The principal new contribution is the stability-selected transport-reversal result: the steady lower branch exists mathematically, but its negative temporal eigenvalue excludes it as a stable state, while nonlinear radiation reverses the trend between the conductive wall slope and the corrected total heat-transfer coefficient.

The main quantitative outcomes are as follows:

(1) At $\lambda = -0.50$, $We = 0.10$, $S = 0.50$, $Pr = 2.0$, $Rd = 0.40$ and $\Theta_w = 1.20$, two branches exist with $f''(0) = 1.644687$ and $f''(0) = 1.856827$. The upper branch has a 14.00% larger Williamson-corrected skin-friction indicator and a 9.79% larger corrected Nusselt indicator than the lower branch. The temporal eigenvalue calculation gives $\gamma_1 = -0.355219$ for the lower branch and $\gamma_1 = 2.594261$ for the upper branch. Hence the lower branch is temporally unstable and the upper branch is temporally stable.

(2) As We increases from 0 to 0.30, the upper-branch wall curvature decreases from 1.989654 to 1.676341; however, the corrected wall-shear indicator increases from 1.989654 to 2.097859 because the quadratic Williamson contribution is retained.

(3) Increasing Rd from 0 to 1 decreases the conductive wall slope $-\theta'(0)$ by 53.23%, but increases the corrected Nusselt indicator by 54.53%.

(4) Increasing m from 1 to 2 increases the upper-branch wall curvature by 10.55% under the reference shrinking and Williamson parameters.

Declarations.

Data availability. The numerical data reported in the tables are generated from the boundary-value problem, continuation calculation and eigenvalue system described in the manuscript.

Conflict of interest. The authors declare no conflict of interest.

Funding. No external funding was received for this work.

Ethical statement. This mathematical study does not involve human participants, animals or clinical data.

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