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ASYMPTOTIC ANALYSIS OF THE STOCHASTIC HTLV-I MODEL WITH BLACK-KARASINSKI PROCESS

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Abstract. In this paper we focus on the asymptotic analysis of a stochastic HTLV-I infection model using Black-Karasinski process. As a first step, we start by mentioning that there is a unique global solution to the said stochastic model for any initial value. Then we find out an appropriate hypothetical framework leading to the existence of an ergodic stationary distribution. After that, we provide certain sufficient condition for the disease's extinction. Finaly, we present some numerical simulation example to back up our theoretical outcomes.

1. Introduction

Human T-cell leukemia virus type 1 (HTLV-1) is a retrovirus that predominantly infects human T lymphocytes: white blood cells that play a crucial role in immune defense. Identified as the first human retrovirus linked to cancer [1], HTLV-1 is associated with several serious diseases, including adult T-cell leukemia/lymphoma (ATLL) and HTLV-1-associated myelopathy/tropical spastic paraparesis (HAM/TSP) [7, 8]. ATLL is a rare but highly aggressive malignancy of T-cells, marked by uncontrolled proliferation and generally poor clinical outcomes. On the other hand, HAM/TSP is a chronic neuroinflammatory condition of the spinal cord, often leading to progressive muscle weakness, spasticity, and paralysis. The global distribution of HTLV-1 is uneven, with higher prevalence reported in regions such as southwestern Japan, sub-Saharan Africa, the Caribbean, South America, and parts of the Middle East [9]. Transmission primarily occurs through the transfer of infected T-cells via breastfeeding, sexual contact, and blood transfusion. Due to its chronic nature and the severity of associated conditions, HTLV-1 represents a major public health concern in endemic regions. This underscores the importance of effective prevention and treatment strategies. In this context, mathematical modeling and asymptotic analysis serve as powerful tools for understanding and combating such infections. Numerous models have been proposed to describe the dynamics of HTLV-1 infection [4, 3, 2]. Among them, the recent contribution by Shi and Jiang [5] is particularly noteworthy. In their study, the

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authors introduced the following mathematical model for HTLV-1 dynamics:

$$\begin{cases} \mathrm{d}\mathcal{T} = (\Lambda - m_{\mathcal{T}}\mathcal{T} - \kappa\Psi(\mathcal{T}_A)\mathcal{T})\,\mathrm{d}t, \\ \mathrm{d}\mathcal{T}_L = (\kappa\Psi(\mathcal{T}_A)\mathcal{T} - (m_L+\alpha)\mathcal{T}_L)\,\mathrm{d}t, \\ \mathrm{d}\mathcal{T}_A = (\alpha\mathcal{T}_L - (m_A+\rho)\mathcal{T}_A)\,\mathrm{d}t, \\ \mathrm{d}\mathcal{T}_M = \left(\rho\mathcal{T}_A + \beta\left(-\frac{\mathcal{T}_M}{\mathcal{T}_{M_{\max}}}+1\right)\mathcal{T}_M - m_M\mathcal{T}_M\right)\mathrm{d}t. \end{cases} \tag{1.1}$$

Where $\mathcal{T}$denotes the Uninfected CD_4^+T cells , $\mathcal{T}_L$represents the Latently infected CD_4^+T cells , $\mathcal{T}_A$ represents the Actively infected CD_4^+T cells and $\mathcal{T}_M$ denotes the Leukemia cells. All parameters are assumed to be positive constants, with their respective descriptions provided in Table 4. Since the fourth equation in system 1.1 is independent of the others, it can be omitted, simplifying the system to:

$$\begin{cases} \dfrac{\mathrm{d}\mathcal{T}}{\mathrm{d}t} = \Lambda - m_{\mathcal{T}}\mathcal{T} - \kappa\Psi(\mathcal{T}_A)\mathcal{T}, \\ \dfrac{\mathrm{d}\mathcal{T}_L}{\mathrm{d}t} = \kappa\Psi(\mathcal{T}_A)\mathcal{T} - (m_L+\alpha)\mathcal{T}_L, \\ \dfrac{\mathrm{d}\mathcal{T}_A}{\mathrm{d}t} = \alpha\mathcal{T}_L - (m_A+\rho)\mathcal{T}_A. \end{cases} \tag{1.2}$$

According to [5] the basic reproduction number of system is given by:

$$\mathfrak{R}_0 = \frac{\Lambda\alpha\kappa\Psi'(0)}{m_{\mathcal{T}}(m_L+\alpha)(m_A+\rho)}.$$

For more details on asymptotic analysis and disease equilibrium(see [5]).
In epidemiological models, key parameters are often influenced by environmental fluctuations. Recently, increasing attention has been directed toward exploring the dynamical behaviour of epidemiological models under stochastic perturbations. This interest arises because stochastic modeling has proven to offer a more accurate representation of the spread of infectious diseases [10][13][11]. Several methods have been developed to introduce stochastic effects into deterministic models. One of the most widely adopted approaches is to assume that system parameters follow an Ornstein–Uhlenbeck process, a type of Ito process. This stochastic process has gained considerable popularity in the modeling of both population and epidemic dynamics [14][12][19] but the problem with Ornstein Uhlenbeck process is that cannot guarantee the nonnegativity of epidemiological parameters. To this end, we further consider the Black–Karasinski process [17, 6, 18]. By perturbing the infection rate κ by this latter we have :

$$d\ln\kappa(t) = r[\ln\widehat{\kappa} - \ln\kappa(t)]\,\mathrm{d}t + \sigma dB(t),$$

where $\widehat{\kappa}$ denote the long-run mean level of the infection rate, $r > 0$ represents the speed of reversion, and σ denotes the intensity of volatility. $B(t)$ is a standard Brownian motion defined on a complete probability space $\{\Omega, \mathcal{F}, \mathcal{P}\}$ with a filtration$\{\mathcal{F}_t\}_{t\geq 0}$ that satisfies the usual conditions (i.e., it is right-continuous and $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets). Then by this denotation a stochastic version of 1 is given by:

$$\begin{cases} d\ln\kappa\left(t\right)=r\left[\ln\widehat{\kappa}-\ln\kappa\left(t\right)\right]\mathrm{d}t+\sigma dB\left(t\right), \\ d\mathcal{T}\left(t\right)=\left[\Lambda-m_{\mathcal{T}}\mathcal{T}\left(t\right)-\tilde{\kappa}\left(t\right)\Psi\left(\mathcal{T}_{A}\left(t\right)\right)\mathcal{T}\left(t\right)\right]\mathrm{d}t, \\ d\mathcal{T}_{L}\left(t\right)=\left[\tilde{\kappa}\left(t\right)\Psi\left(\mathcal{T}_{A}\left(t\right)\right)\mathcal{T}\left(t\right)-\left(m_{L}+\alpha\right)\mathcal{T}_{L}\left(t\right)\right]\mathrm{d}t, \\ d\mathcal{T}_{A}\left(t\right)=\left[\alpha\mathcal{T}_{L}\left(t\right)-\left(m_{A}+\rho\right)\mathcal{T}_{A}\left(t\right)\right]\mathrm{d}t. \end{cases} \tag{1.3}$$

In fact, our main aim is to study the asymptotic behavior of infectious disease, i.e., the long-term properties of $(\mathcal{T}(t),\mathcal{T}_l(t),\mathcal{T}_A(t))$. By letting $z=\ln\kappa-\ln\widehat{\kappa}$ we can equivalently transform system 1.4 into

$$\begin{cases} dz\left(t\right)=-rz(t)\mathrm{d}t+\sigma dB\left(t\right) \\ d\mathcal{T}\left(t\right)=\left[\Lambda-m_{\mathcal{T}}\mathcal{T}\left(t\right)-\widehat{\kappa}\mathrm{e}^{z(t)}\left(t\right)\Psi\left(\mathcal{T}_{A}\left(t\right)\right)\mathcal{T}\left(t\right)\right]\mathrm{d}t, \\ d\mathcal{T}_{L}\left(t\right)=\left[\widehat{\kappa}\mathrm{e}^{z(t)}\Psi\left(\mathcal{T}_{A}\left(t\right)\right)\mathcal{T}\left(t\right)-\left(m_{L}+\alpha\right)\mathcal{T}_{L}\left(t\right)\right]\mathrm{d}t, \\ d\mathcal{T}_{A}\left(t\right)=\left[\alpha\mathcal{T}_{L}\left(t\right)-\left(m_{A}+\rho\right)\mathcal{T}_{A}\left(t\right)\right]\mathrm{d}t. \end{cases} \tag{1.4}$$

The rest of this paper is organized as follows: In Section 2 we show the existence and uniqueness of the global solution. In Section 3 we identify a key value $\mathfrak{R}_0^S$ related to the existence of the stationary distribution through mathematical frameworks. Section 4 establishes the sufficient conditions for the extinction of the epidemic disease. Section 5 presents computer simulations that support our findings, and Section 6 summarizes the main conclusions of the article.

2. Existence and uniqueness of the global solution

Theorem 2.1. *For any initial condition* $(\mathcal{T}(0),\mathcal{T}_L(0),\mathcal{T}_L(0),\kappa(0))\in\mathbb{R}_+^3\times\mathbb{R}$ *the system 1.4 admits a unique global solution almost surely , and the solution remains forever in the invariant set* $\Xi=\left\{(\mathcal{T},\mathcal{T}_L,\mathcal{T}_A,\kappa)\in\mathbb{R}_+^3\times\mathbb{R}|\mathcal{T}+\mathcal{T}_L+\mathcal{T}_A<\frac{\Lambda}{\gamma}\right\}$ *with* $\gamma=\min\left(m_{\mathcal{T}},m_L,m_A\right)$.

Remark 2.2. By defining a desirable non-negative C^2-function $W=[\mathcal{T}-1-\ln\mathcal{T}]+[\mathcal{T}_L-1-\ln\mathcal{T}_L]+[\mathcal{T}_A-1-\ln\mathcal{T}_A]+\mathrm{e}^z-1-z$ The remainder of the proof is almost the same as Theorem 3.1 in [18] and is thus omitted.

3. Stationnary distibution

In this section, we investigate the existence of a stationary distribution for the stochastic model, as its presence provides evidence of the long-term persistence of the disease within the population.

Lemma 3.1. [20, 21, 22] *Suppose that there exists a bounded closed domain* $\mathbb{H}\in\mathbb{R}^d$ *with a regular boundary* L_0*, for every starting value* $X(0)\in\mathbb{R}^d$*, if*

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t\mathbb{P}(s,X(0),\mathbb{H})\mathrm{d}\tau>0\ \textit{a.s,}$$

where $\mathbb{P}(s,X(s),\mathbb{H})$ *is the transition probability of* $X(t)$*. Then system will posesesses a solution which has the Feller property. And system 1.4 admits at least one stationary distribution* $\eta(\cdot)$ *on* $\mathbb{R}^d$*.*

Define

$$\mathfrak{R}_0^s=\frac{\alpha\Lambda\hat{\kappa}\Psi'\left(0\right)\mathrm{e}^{\frac{\sigma^2}{16r}}}{m_{\mathcal{T}}\left(\rho+m_A\right)\left(m_L+\alpha\right)},$$

Theorem 3.2. *Assume that $\mathfrak{R}_0^s > 1$, then the stochastic system admits at least one ergodic stationary distribution $\eta(.)$ on Ξ.*

Proof. Let defining the following C^2-real valued functions:

$$\begin{aligned}
\Upsilon_1 &= \ln \mathcal{T}_L - a_1 \ln \mathcal{T} - a_2 \ln \mathcal{T}_A,\\
\Upsilon_2 &= \Upsilon_1 + \frac{\left(a_1 \Psi'(0)\hat{\kappa} e^{\frac{\sigma^2}{4r}} + a_3 p\right)}{\mu_A + \rho}\mathcal{T}_A,\\
\Upsilon_3 &= -\ln \mathcal{T} - \ln \mathcal{T}_A - \ln\left(\frac{\Lambda}{\gamma} - \mathcal{T} - \mathcal{T}_L - \mathcal{T}_A\right) + e^z - z - 1,\\
\Upsilon_4 &= M\Upsilon_2 + \Upsilon_3.
\end{aligned}$$

Applying Itô's formula for the functions above, we have for Υ_1 :

$$\begin{aligned}
\mathcal{L}\Upsilon_1 =& -\mathcal{L}\ln \mathcal{T}_L - a_1\mathcal{L}\ln\mathcal{T} - a_2\mathcal{L}\ln\mathcal{T}_A\\
=& -\frac{\hat{\kappa} e^z \Psi(\mathcal{T}_A)\mathcal{T}}{\mathcal{T}_L} + m_L + \alpha - \frac{a_1\Lambda}{\mathcal{T}} + a_1\hat{\kappa} e^z \Psi(\mathcal{T}_A) + a_1 m_{\mathcal{T}} - a_2\frac{\alpha \mathcal{T}_L}{\mathcal{T}_A}\\
&+ a_2 m_A + a_2\rho + \frac{a_3\mathcal{T}_A}{\Psi(\mathcal{T}_A)} - \frac{a_3\mathcal{T}_A}{\Psi(\mathcal{T}_A)}.
\end{aligned}$$

Using the conditions (2.1) in [5]

$$\frac{\mathcal{T}_A}{\Psi(\mathcal{T}_A)} \le \left(p\mathcal{T}_A + \frac{1}{\Psi'(0)}\right), \Psi(\mathcal{T}_A) \le \Psi'(0)\mathcal{T}_A$$

and arithmetic geometric mean inequality we have:

$$\begin{aligned}
\mathcal{L}\Upsilon_1 \le& -4\sqrt[4]{a_1 a_2 a_3 \alpha\Lambda\hat{\kappa} e^{\frac{\sigma^2}{16r}}} + m_L + \alpha + a_1 m_{\mathcal{T}} + a_2(m_A + \rho) + \frac{a_3}{\Psi'(0)}\\
&+ \left(a_1\Psi'(0)\hat{\kappa} e^z + a_3 p\right)\mathcal{T}_A + 4\left(\sqrt[4]{a_1 a_2 a_3\alpha\Lambda\hat{\kappa} e^{\frac{\sigma^2}{16r}}} - \sqrt[4]{a_1 a_2 a_3 \alpha\Lambda\hat{\kappa} e^z}\right),
\end{aligned}$$

denotes that

$$a_1 = \frac{\alpha\Lambda\hat{\kappa}\Psi'(0) e^{\frac{\sigma^2}{16\theta}}}{m_{\mathcal{T}}{}^2 (m_A + \rho)}, a_2 = \frac{\alpha\Lambda\hat{\kappa}\Psi'(0) e^{\frac{\sigma^2}{16\theta}}}{m_{\mathcal{T}}(m_A + \rho)^2}, a_3 = \frac{\alpha\Lambda\hat{\kappa}(\Psi'(0))^2 e^{\frac{\sigma^2}{16\theta}}}{m_{\mathcal{T}}(m_A + \rho)}.$$

Then we have:

$$\begin{aligned}
\mathcal{L}\Upsilon_1 \le& -\frac{\alpha\Lambda\hat{\kappa}\Psi'(0) e^{\frac{\sigma^2}{16r}}}{m_{\mathcal{T}}(m_A + \rho)} + m_L + \alpha + \left(a_1\Psi'(0)\hat{\kappa} e^{\frac{\sigma^2}{4r}} + a_3 p\right)\mathcal{T}_A\\
&+ a_1\hat{\kappa}\Psi'(0)\mathcal{T}_A\left(e^z - e^{\frac{\sigma^2}{4r}}\right) + 4\left(\sqrt[4]{a_1 a_2 a_3 \alpha\Lambda\hat{\kappa} e^{\frac{\sigma^2}{16r}}} - \sqrt[4]{a_1 a_2 a_3\alpha\Lambda\hat{\kappa} e^z}\right).
\end{aligned}$$

Note that we have:

$$\mathcal{T}_A \le \mathcal{T} + \mathcal{T}_L + \mathcal{T}_A \le \frac{\Lambda}{\gamma}$$

$$
\begin{aligned}
\mathcal{L}\Upsilon_1 \leq & -\frac{\alpha\Lambda\hat{\kappa}\Psi'(0)\,\mathrm{e}^{\frac{\sigma^2}{16r}}}{m_{\mathcal{T}}(m_A+\rho)} + m_L + \alpha + \left(a_1\Psi'(0)\,\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{4r}} + a_3p\right)\mathcal{T}_A \\
& + a_1\hat{\kappa}\Psi'(0)\,\frac{\Lambda}{\gamma}\left(\mathrm{e}^z - \mathrm{e}^{\frac{\sigma^2}{4r}}\right) + 4\left(\sqrt[4]{a_1a_2a_3\alpha\Lambda\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{16r}}} - \sqrt[4]{a_1a_2a_3\alpha\Lambda\hat{\kappa}\mathrm{e}^z}\right), \\
\leq & -(m_L+\alpha)\left(\frac{\alpha\Lambda\hat{\kappa}\Psi'(0)\,\mathrm{e}^{\frac{\sigma^2}{16r}}}{m_{\mathcal{T}}(m_A+\rho)(m_L+\alpha)} - 1\right) + \left(a_1\Psi'(0)\,\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{4r}} + a_3p\right)\mathcal{T}_A \\
& + a_1\hat{\kappa}\Psi'(0)\,\frac{\Lambda}{\gamma}\left(\mathrm{e}^z - \mathrm{e}^{\frac{\sigma^2}{4r}}\right) + 4\left(\sqrt[4]{a_1a_2a_3\alpha\Lambda\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{16r}}} - \sqrt[4]{a_1a_2a_3\alpha\Lambda\hat{\kappa}\mathrm{e}^z}\right), \\
\leq & -(m_L+\alpha)(\mathfrak{R}_0^s - 1) + \left(a_1\Psi'(0)\,\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{4r}} + a_3p\right)\mathcal{T}_A \\
& + a_1\hat{\kappa}\Psi'(0)\,\frac{\Lambda}{\gamma}\left(\mathrm{e}^z - \mathrm{e}^{\frac{\sigma^2}{4r}}\right) + 4\left(\sqrt[4]{a_1a_2a_3\alpha\Lambda\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{16r}}} - \sqrt[4]{a_1a_2a_3\alpha\Lambda\hat{\kappa}\mathrm{e}^z}\right),
\end{aligned}
$$

with

$$
\mathfrak{R}_0^s = \frac{\alpha\Lambda\hat{\kappa}\Psi'(0)\,\mathrm{e}^{\frac{\sigma^2}{16r}}}{m_{\mathcal{T}}(m_A+\rho)(m_L+\alpha)}.
$$

For Υ_2 we have

$$
\begin{aligned}
\mathcal{L}\Upsilon_2 \leq & \mathcal{L}\Upsilon_1 + \alpha\frac{\left(a_1\Psi'(0)\,\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{4r}} + a_3p\right)}{m_A+\rho}\mathcal{T}_L \\
\mathcal{L}\Upsilon_2 \leq & -(m_L+\alpha)(\mathfrak{R}_0^s - 1) + \alpha\frac{\left(a_1\Psi'(0)\,\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{4r}} + a_3p\right)}{m_A+\rho}\mathcal{T}_L \\
& + g_1(z) + g_2(z),
\end{aligned}
$$

where

$$
g_1(z) = a_1\hat{\kappa}\Psi'(0)\,\frac{\Lambda}{\gamma}\left(\mathrm{e}^z - \mathrm{e}^{\frac{\sigma^2}{4r}}\right)
$$

and

$$
g_2(z) = 4\left(\sqrt[4]{a_1a_2a_3\alpha\Lambda\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{16r}}} - \sqrt[4]{a_1a_2a_3\alpha\Lambda\hat{\kappa}\mathrm{e}^z}\right).
$$

For Υ_3 we have

$$
\begin{aligned}
\mathcal{L}\Upsilon_3 \leq & -\frac{\Lambda}{\mathcal{T}} - \frac{\alpha\mathcal{T}_L}{\mathcal{T}_A} + \frac{\Lambda - \gamma(\mathcal{T}+\mathcal{T}_L+\mathcal{T}_A) - \rho\mathcal{T}_A}{\frac{\Lambda}{\gamma} - \mathcal{T} - \mathcal{T}_L - \mathcal{T}_A} - rz(\mathrm{e}^z - 1) \\
& + \frac{\sigma^2\mathrm{e}^z}{2} + \Psi'(0)\hat{\kappa}\mathrm{e}^z\mathcal{T}_A + m_{\mathcal{T}} + m_A + \rho \\
\leq & -\frac{\Lambda}{\mathcal{T}} - \frac{\alpha\mathcal{T}_L}{\mathcal{T}_A} - \frac{\rho\mathcal{T}_A}{\frac{\Lambda}{\gamma} - T - T_L - T_A} - \frac{rz(\mathrm{e}^z-1)}{2} + Q_1,
\end{aligned}
$$

where

$$
Q_1 = \sup_{z\in\mathbb{R}}\left\{m_{\mathcal{T}} + m_A + \rho + \gamma + (\Psi'(0)\hat{\kappa}\frac{\Lambda}{\gamma} + \frac{\sigma^2}{2})\mathrm{e}^z - \frac{rz(\mathrm{e}^z-1)}{2}\right\}.
$$

For Υ_4 we have

$$\begin{aligned}\mathcal{L}\Upsilon_4 \leq & -2 + M\alpha\frac{\left(a_1\Psi'(0)\,\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{4r}} + a_3 p\right)}{m_A+\rho}\mathcal{T}_L - \frac{\Lambda}{\mathcal{T}} - \frac{\alpha\mathcal{T}_L}{\mathcal{T}_A} - \frac{\rho\mathcal{T}_A}{\frac{\Lambda}{\gamma} - \mathcal{T} - \mathcal{T}_L - \mathcal{T}_A} \\ & - \frac{rz(\mathrm{e}^z-1)}{2} + Mg_1(z) + Mg_2(z),\end{aligned}$$

for a sufficiently large constant M such that

$$-M\left(m_L+\alpha\right)\left(\mathfrak{R}_0^s - 1\right) + Q_1 \leq -2$$

taking

$$\begin{aligned}G(\mathcal{T},\mathcal{T}_L,\mathcal{T}_A,z) = & -2 + M\alpha\frac{\left(a_1\Psi'(0)\,\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{4r}} + a_3 p\right)}{m_A+\rho}\mathcal{T}_L - \frac{\Lambda}{\mathcal{T}} - \frac{\alpha\mathcal{T}_L}{\mathcal{T}_A} \\ & - \frac{\rho\mathcal{T}_A}{\frac{\Lambda}{\gamma} - \mathcal{T} - \mathcal{T}_L - \mathcal{T}_A} - \frac{rz(\mathrm{e}^z-1)}{2},\end{aligned}$$

then we have :

$$\mathcal{L}\Upsilon_4 \leq: G(\mathcal{T},\mathcal{T}_L,\mathcal{T}_A,z) + Mg_1(z) + Mg_2(z).$$

Let now construct a compact set :

$$\begin{aligned}\mathbb{H} = \big\{ & (\mathcal{T},\mathcal{T}_L,\mathcal{T}_A,\kappa) \in \Xi \mid \epsilon \leq \mathcal{T}, \epsilon \leq \mathcal{T}_L, \epsilon^2 \leq \mathcal{T}_A, \mathcal{T} \\ & + \mathcal{T}_L + \mathcal{T}_A \leq \frac{\Lambda}{\gamma} - \epsilon^3, |z| \leq \frac{1}{\epsilon}\bigg\}\end{aligned}$$

such that $G(\mathcal{T},\mathcal{T}_L,\mathcal{T}_A,z) \leq -1$ for any $(\mathcal{T},\mathcal{T}_L,\mathcal{T}_A,z) \in \Xi\backslash\mathbb{H} := \mathbb{H}^c$
let $\mathbb{H}^c = \bigcup_{i=1}^5 \mathbb{H}_i^c$, where

$$\begin{aligned}
\mathbb{H}_1^c &= \left\{(\mathcal{T},\mathcal{T}_L,\mathcal{T}_A,z) \in \Xi \mid \mathcal{T}_L \in (0,\epsilon)\right\},\\
\mathbb{H}_2^c &= \left\{(\mathcal{T},\mathcal{T}_L,\mathcal{T}_A,z) \in \Xi \mid \mathcal{T} \in (0,\epsilon)\right\},\\
\mathbb{H}_3^c &= \left\{(\mathcal{T},\mathcal{T}_L,\mathcal{T}_A,z) \in \Xi \mid \mathcal{T}_A \in (0,\epsilon^2), \mathcal{T}_L \in [\epsilon,\infty)\right\},\\
\mathbb{H}_4^c &= \left\{(\mathcal{T},\mathcal{T}_L,\mathcal{T}_A,z) \in \Xi \mid \ \mathcal{T}+\mathcal{T}_L+\mathcal{T}_A \in (\frac{\Lambda}{\gamma}-\epsilon^3,\infty), \mathcal{T}_A \in [\epsilon^2,\infty)\right\},\\
\mathbb{H}_5^c &= \left\{(\mathcal{T},\mathcal{T}_L,\mathcal{T}_A,z) \in \Xi \mid z \in (\frac{1}{\epsilon},\infty) or (-\infty,-\frac{1}{\epsilon})\right\},
\end{aligned}$$

In view of

$$\inf_{|z|>\epsilon^{-1}}\left\{z\left(e^z-1\right)\right\} = \frac{1}{\epsilon}\left(1-e^{-\frac{1}{\epsilon}}\right), \text{ we obtain}$$

$$\begin{cases} -2+M\alpha\frac{\left(a_1\Psi'(0)\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{4\theta}}+a_3p\right)}{m_A+\rho}\epsilon\leq -1, \\ -2+M\alpha\frac{\left(a_1\Psi'(0)\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{4\theta}}+a_3p\right)}{m_A+\rho}\frac{\Lambda}{\gamma}-\min\left\{\frac{\Lambda}{\epsilon},\frac{\alpha}{\epsilon},\frac{\rho}{\epsilon}\right\}\leq -1, \\ -2+M\alpha\frac{\left(a_1\Psi'(0)\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{4\theta}}+a_3p\right)}{m_A+\rho}\frac{\Lambda}{\gamma}-\frac{r}{2\epsilon}\left(1-e^{-\frac{1}{\epsilon}}\right)\leq -1. \end{cases} \tag{3.1}$$

$\mathbf{1}^{st}$ **case** : Whenever $(\mathcal{T},\mathcal{T}_L,\mathcal{T}_A,z)\in\mathbb{H}_1^c$, we have

$$\begin{aligned} G(\mathcal{T},\mathcal{T}_L,\mathcal{T}_A,z) &\leq -2+M\alpha\frac{\left(a_1\Psi'(0)\,\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{4\theta}}+a_3p\right)}{m_A+\rho}\mathcal{T}_L \\ &\leq -2 \quad +M\alpha\frac{\left(a_1\Psi'(0)\,\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{4\theta}}+a_3p\right)}{m_A+\rho}\epsilon \\ &\leq -1. \end{aligned}$$

$\mathbf{2}^{nd}$ **case** : Whenever $(\mathcal{T},\mathcal{T}_L,\mathcal{T}_A,\kappa)\in\mathbb{H}_2^c$, we have

$$\begin{aligned} G(\mathcal{T},\mathcal{T}_L,\mathcal{T}_A,\kappa) &\leq -\frac{\Lambda}{\mathcal{T}}-2+M\alpha\frac{\left(a_1\Psi'(0)\,\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{4\theta}}+a_3p\right)}{m_A+\rho}\mathcal{T}_L \\ &\leq -\frac{\Lambda}{\epsilon}-2+M\alpha\frac{\left(a_1\Psi'(0)\,\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{4\theta}}+a_3p\right)}{m_A+\rho}\frac{\Lambda}{\gamma} \\ &\leq -1. \end{aligned}$$

$\mathbf{3}^{st}$ **case** : Whenever $(\mathcal{T},\mathcal{T}_L,\mathcal{T}_A,z)\in\mathbb{H}_3^c$, we have

$$\begin{aligned} G(\mathcal{T},\mathcal{T}_L,\mathcal{T}_A,z) &\leq -\frac{\alpha\mathcal{T}_L}{\mathcal{T}_A}-2+M\alpha\frac{\left(a_1\Psi'(0)\,\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{4\theta}}+a_3p\right)}{m_A+\rho}\mathcal{T}_L \\ &\leq -\frac{\alpha}{\epsilon}-2 \quad +M\alpha\frac{\left(a_1\Psi'(0)\,\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{4\theta}}+a_3p\right)}{m_A+\rho}\frac{\Lambda}{\gamma} \\ &\leq -1. \end{aligned}$$

$\mathbf{4}^{th}$ **case**: Whenever $(\mathcal{T},\mathcal{T}_L,\mathcal{T}_A,z)\in\mathbb{H}_4^c$, we have

$$\begin{aligned} G(\mathcal{T},\mathcal{T}_L,\mathcal{T}_A,z) &\leq -\frac{\rho\mathcal{T}_A}{\frac{\Lambda}{\gamma}-\mathcal{T}-\mathcal{T}_L-\mathcal{T}_A}-2+M\alpha\frac{\left(a_1\Psi'(0)\,\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{4\theta}}+a_3p\right)}{m_A+\rho}\mathcal{T}_L \\ &\leq -\frac{\rho}{\epsilon}-2+M\alpha\frac{\left(a_1\Psi'(0)\,\hat{\kappa}\mathrm{e}^{\frac{\sigma^2}{4\theta}}+a_3p\right)}{m_A+\rho}\frac{\Lambda}{\gamma} \\ &\leq -1. \end{aligned}$$

$\mathbf{5}^{th}$ **case** : Whenever $(\mathcal{T}, \mathcal{T}_L, \mathcal{T}_A, \kappa) \in \mathbb{H}_5^c$, we have

$$\begin{aligned} G(\mathcal{T}, \mathcal{T}_L, \mathcal{T}_A, z) &\leq -\frac{rz(\mathrm{e}^z - 1)}{2} - 2 + M\alpha \frac{\left(a_1 \Psi'(0)\, \hat{\kappa} \mathrm{e}^{\frac{\sigma^2}{4\theta}} + a_3 p\right)}{m_A + \rho} \mathcal{T}_L \\ &\leq -\frac{r}{2\epsilon}\left(1 - e^{-\frac{1}{\epsilon}}\right) - 2 \quad + M\alpha \frac{\left(a_1 \Psi'(0)\, \hat{\kappa} \mathrm{e}^{\frac{\sigma^2}{4\theta}} + a_3 p\right)}{m_A + \rho} \frac{\Lambda}{\gamma} \\ &\leq -1. \end{aligned}$$

In summury we have $G(\mathcal{T}, \mathcal{T}_L, \mathcal{T}_A, z) \leq -1$ for any $(\mathcal{T}, \mathcal{T}_L, \mathcal{T}_A, z) \in \Gamma \backslash \mathbb{H} := \mathbb{H}^c$. Following a standard argument presented in [18], we find that the process $\{z(t)\}_{t\geq 0}$ converges weakly to the unique stationary distribution, which is the normal distribution $\mathbb{N}\left(0, \frac{\sigma^2}{2r}\right)$. This stationary distribution has a density function given by $\pi^*(z) = \frac{\sqrt{r}}{\sigma\sqrt{\pi}} e^{-\frac{r}{\sigma^2} z^2}, \forall z \in \mathbb{R}$, and we have for any $a > 0$:

$$\int_{\mathbb{R}} e^{ax} \pi^*(z) dz = \frac{\sqrt{r}}{\sigma\sqrt{\pi}} \int_{\mathbb{R}} e^{-\left(\frac{\sqrt{\sigma} z}{\sigma} - \frac{a\sigma}{2\sqrt{r}}\right)^2 + \frac{(a\sigma)^2}{4\theta}} dz = e^{\frac{(a\sigma)^2}{4r}}, \quad a.s.$$

So and by using the Itô's integral, we have:

$$\begin{aligned} 0 \leq \frac{\mathbb{E}\Upsilon(\mathcal{T}, \mathcal{T}_L, \mathcal{T}_A, z)}{T} =& \frac{\mathbb{E}\Upsilon(\mathcal{T}(0), \mathcal{T}_L(0), \mathcal{T}_A(0), z(0))}{T} \\ &+ \frac{1}{T}\int_0^T \mathbb{E}(G(\mathcal{T}(t), \mathcal{T}_L(t), \mathcal{T}_A(t), z(t)))dt \\ &+ M\mathbb{E}\left(\frac{1}{T}\int_0^T g_1(z(t))dt\right) + M\mathbb{E}\left(\frac{1}{T}\int_0^T g_2(z(t))dt\right). \end{aligned} \tag{3.2}$$

Using the fact that

$$G(\mathcal{T}(t), \mathcal{T}_L(t), \mathcal{T}_A(t), z(t)) \leq M\alpha \frac{\left(a_1 \Psi'(0)\, \hat{\kappa} \mathrm{e}^{\frac{\sigma^2}{4\theta}} + a_3 p\right)}{m_A + \rho} \frac{\Lambda}{\gamma} := K$$

$$\text{for all}\, (\mathcal{T}, \mathcal{T}_L, \mathcal{T}_A, z) \in \Xi$$

$$\begin{aligned} \frac{1}{T}\int_0^T \mathbb{E}(G(\mathcal{T}(t), \mathcal{T}_L(t), \mathcal{T}_A(t), z(t)))dt \leq& \frac{K}{T}\int_0^T \mathbf{1}_{\{(\mathcal{T}(t), \mathcal{T}_L(t), \mathcal{T}_A(t), z(t)) \in \mathbb{H}\}} dt \\ &- \frac{1}{T}\int_0^T \mathbf{1}_{\{(\mathcal{T}(t), \mathcal{T}_L(t), \mathcal{T}_A(t), z(t)) \in \Gamma\backslash\mathbb{H}\}} dt \\ =& -1 \\ &+ \frac{K+1}{T}\int_0^T \mathbf{1}_{\{(\mathcal{T}(t), \mathcal{T}_L(t), \mathcal{T}_A(t), z(t)) \in \mathbb{H}\}} dt. \end{aligned} \tag{3.3}$$

In addition and by the ergodicity of $\{z(t)\}_{t\geq 0}$ we have
$\lim_{T\to\infty}\mathbb{E}\left(\frac{1}{T}\int_0^T g_1(z(t))dt\right)=0$, a.s and $\lim_{T\to\infty}\mathbb{E}\left(\frac{1}{T}\int_0^T g_2(z(t))dt\right)=0$,a.s
By combining we get

$$\liminf_{T\to\infty}\frac{1}{T}\int_0^T \mathbf{1}_{\{(\mathcal{T}(t),\mathcal{T}_L(t),\mathcal{T}_A(t),z(t))\in\mathbb{H}\}}dt\geq\frac{1}{K+1}>0,\quad \text{a.s.}$$

Then by Fatou's lemma, it implies

$$\liminf_{T\to\infty}\frac{1}{T}\int_0^T \mathbb{P}\left((\mathcal{T}(0),\mathcal{T}_L(0),\mathcal{T}_A(0),z(0))\,,\mathbb{H},t\right)dt\geq\frac{1}{K+1},$$

for all $(\mathcal{T}(0),\mathcal{T}_L(0),\mathcal{T}_A(0),z(0))\in\Xi$. Where $\mathbb{P}\left(S(0),V(0),E(0),I(0),z(0),\mathbb{H},t\right)$ denotes the transition probability of $(S(t),V(t),E(t),I(t),z(t))\in\mathbb{H}$ with initial value

$$(S(0),V(0),E(0),I(0),z(0)).$$

According to Lemma 2 3.1, system 1 has at least one stationary distribution $\eta(\cdot)$ on Ξ if $\mathfrak{R}_0^s>1$. Which completes the proof. □

4. Extinction

In this section, we investigate sufficient conditions for disease extinction.

Theorem 4.1. *For any initial value $(\mathcal{T}(0),\mathcal{T}_L(0),\mathcal{T}_A(0),z(0))\in\Xi$, the solution $(\mathcal{T}(t),\mathcal{T}_L(t),\mathcal{T}_A(t),z(t))$ of system has the following property*

$$\limsup_{t\to+\infty}\frac{\ln\left(\frac{\omega_1}{m_L+\alpha}\mathcal{T}_L+\frac{\omega_2}{m_A+\rho}\mathcal{T}_A\right)}{t}\leq\min\{m_L+\alpha,m_A+\rho\}\left(\mathfrak{R}_0^e-1\right)a.s.$$

Where

$$\omega_1=\frac{\alpha}{(m_A+\rho)\sqrt{\mathfrak{R}_0}},\omega_2=1,\mathfrak{R}_0^e=\sqrt{\mathfrak{R}_0}+\frac{\alpha\hat{\kappa}\Psi'(0)\,\mathcal{T}_0\sqrt{e^{\frac{\sigma^2}{r}}-2e^{\frac{\sigma^2}{4r}}+1}}{\sqrt{\mathfrak{R}_0}\,(m_L+\alpha)\min(m_L+\alpha,m_A+\rho)}.$$

Especially, if $\mathfrak{R}_0^e<1$, $\lim_{t\to+\infty}\mathcal{T}_L(t)=0$, $\lim_{t\to+\infty}\mathcal{T}_A(t)=0$, a.s.

Proof. Let $\mathcal{T}_0=\frac{\Lambda}{m_{\mathcal{T}}}$, according to [16] there is a left eigenvector (ω_1,ω_2) of

$$M=\begin{pmatrix}0 & \frac{\hat{\kappa}\Psi'(0)\mathcal{T}_0}{m_L+\alpha}\\ \frac{\alpha}{m_A+\rho} & 0\end{pmatrix}$$

corresponding to $\sqrt{\mathfrak{R}_0}$ i.e. $\sqrt{\mathfrak{R}_0}\,(\omega_1,\omega_2)=(\omega_1,\omega_2)\,M$.
Taking $\omega_1=\frac{\alpha}{(m_A+\rho)\sqrt{\mathfrak{R}_0}}$, $\omega_2=1$, and denoting

$$V_e=\frac{\omega_1}{m_L+\alpha}\mathcal{T}_L+\frac{\omega_2}{m_A+\rho}\mathcal{T}_A$$

we have

$$
\begin{aligned}
\mathcal{L}(\ln V_e) =& \frac{1}{V_e}\left[\frac{\omega_1}{m_L+\alpha}\hat{\kappa}\mathrm{e}^{z}\Psi\left(\mathcal{T}_A\right)\mathcal{T}-\omega_1\mathcal{T}_L+\frac{\omega_2\alpha}{m_A+\rho}\mathcal{T}_L-\omega_2\mathcal{T}_A\right],\\
\leq& \frac{1}{V_e}\left[\frac{\omega_1}{m_L+\alpha}\hat{\kappa}\mathrm{e}^{z}\Psi'\left(0\right)\mathcal{T}_A\mathcal{T}_0-\omega_1\mathcal{T}_L+\frac{\omega_2\alpha}{m_A+\rho}\mathcal{T}_L-\omega_2\mathcal{T}_A\right],\\
=& \frac{1}{V_e}\left[\frac{\omega_1}{m_L+\alpha}\hat{\kappa}\Psi'\left(0\right)\mathcal{T}_A\mathcal{T}_0-\omega_1\mathcal{T}_L+\frac{\omega_2\alpha}{m_A+\rho}\mathcal{T}_L-\omega_2\mathcal{T}_A\right]\\
&+\frac{1}{V_e}\frac{\omega_1}{m_L+\alpha}\hat{\kappa}\Psi'\left(0\right)\left(\mathrm{e}^{z}-1\right)\mathcal{T}_0\mathcal{T}_A,\\
\leq& \frac{1}{V_e}\left[\frac{\omega_1}{m_L+\alpha}\hat{\kappa}\Psi'\left(0\right)\mathcal{T}_A\mathcal{T}_0-\omega_1\mathcal{T}_L+\frac{\omega_2\alpha}{m_A+\rho}\mathcal{T}_L-\omega_2\mathcal{T}_A\right]\\
&+\frac{1}{V_e}\frac{\omega_1}{m_L+\alpha}\hat{\kappa}\Psi'\left(0\right)\mid\mathrm{e}^{z}-1\mid\mathcal{T}_0\mathcal{T}_A,\\
\leq& \frac{1}{V_e}\left(\omega_1,\omega_2\right)\left(M-I_2\right)\left[\begin{smallmatrix}\mathcal{T}_L\\ \mathcal{T}_A\end{smallmatrix}\right]+\frac{\mathcal{T}_A}{V_e}\frac{\omega_1}{m_L+\alpha}\hat{\kappa}\Psi'\left(0\right)\left|\mathrm{e}^{z}-1\right|\mathcal{T}_0,\\
=& \frac{1}{\frac{\omega_1}{m_L+\alpha}\mathcal{T}_L+\frac{\omega_2}{m_A+\rho}\mathcal{T}_A}\left(\sqrt{\mathfrak{R}_0}-1\right)\left(\omega_1\mathcal{T}_L+\omega_2\mathcal{T}\right)\\
&+\frac{\mathcal{T}_A}{\frac{\omega_1}{m_L+\alpha}\mathcal{T}_L+\frac{\omega_2}{m_A+\rho}\mathcal{T}_A}\frac{\omega_1}{m_L+\alpha}\hat{\kappa}\Psi'\left(0\right)\left|\mathrm{e}^{z}-1\right|\mathcal{T}_0,\\
\leq& \min\left(m_L+\alpha,m_A+\rho\right)\left(\sqrt{\mathfrak{R}_0}-1\right)+\frac{\alpha\hat{\kappa}\Psi'\left(0\right)\mathcal{T}_0}{\left(m_L+\alpha\right)\sqrt{\mathfrak{R}_0}}\left|\mathrm{e}^{z}-1\right|.
\end{aligned}
$$

By the ergodicity of $\{z(t)\}_{t\geq 0}$ and the Hölder's inequality, we have :

$$
\begin{aligned}
\lim_{t\to\infty}\frac{1}{t}\int_0^t\left|e^{z(s)}-1\right|ds=\int_{\mathbb{R}}\left|e^{z}-1\right|\pi^*(z)dz\leq&\left(\int_{\mathbb{R}}\left(e^{z}-1\right)^2\pi^*(z)dz\right)^{\frac{1}{2}}\\
=&\sqrt{e^{\frac{\sigma^2}{r}}-2e^{\frac{\sigma^2}{4r}}+1},\quad\text{a.s.}
\end{aligned}
$$

Integrating from 0 to t and then dividing by t on both sides, and combining, we get

$$
\begin{aligned}
\limsup_{t\to+\infty}\frac{\ln\left(\frac{\omega_1}{m_L+\alpha}\mathcal{T}_L+\frac{\omega_2}{m_A+\rho}\mathcal{T}_A\right)}{t}\leq&\min\left(m_L+\alpha,m_A+\rho\right)\left(\sqrt{\mathfrak{R}_0}-1\right)\\
&+\frac{\alpha\Psi'\left(0\right)\mathcal{T}_0\hat{\kappa}}{\left(m_L+\alpha\right)\sqrt{\mathfrak{R}_0}}\sqrt{e^{\frac{\sigma^2}{r}}-2e^{\frac{\sigma^2}{4r}}+1}\\
=&\min\left\{m_L+\alpha,m_A+\rho\right\}\left(\mathfrak{R}_0^e-1\right).
\end{aligned}
$$

Therefore, if $\mathfrak{R}_0^e<1$, then the disease will go to extinction. this completes the proof. □

5. Numerical simulations

This section is devoted to the numerical illustration of the theoretical results. In the following examples, we implement the algorithm presented in section 5

Parameters	Description	Data 1	Data 2
Λ	The source of CD_4^+T cells from precursors	6	30
$m_{\mathcal{T}}$	The removal or death rate of $\mathcal{T}$	0.2	0.1
m_L	The removal or death rate of $\mathcal{T}_L$	0.2	0.1
m_A	The removal or death rate of $\mathcal{T}_A$	0.2	0.2
m_M	The removal or death rate of $\mathcal{T}_M$	0.2	0.2
α	The transmission rate from $\mathcal{T}_L$ to $\mathcal{T}_A$	0.6	0.1
ρ	The transmission rate from $\mathcal{T}_A$ to $\mathcal{T}_M$	0.4	0.1
$\widehat{\kappa}$	he long-run mean of the infection rate	0.03	0.1
$\Psi(\mathcal{T}_A)$	The non-linear incidence	-	-
$\mathcal{T}_{M_{\max}}$	The maximal number that$\mathcal{T}_M$ cells proliferate	-	-
β	The logistic growth rate of $\mathcal{T}_M$	-	-

Table 1. List of parameters

[15] to analyze the stochastic system 1.4, we numerically simulate the solution of the system 1.4 starting from the initial condition $(\mathcal{T}(0), \mathcal{T}_L(0), \mathcal{T}_A(0), z(0)) = (30, 1, 1, 0.03)$ and $(\mathcal{T}(0), \mathcal{T}_L(0), \mathcal{T}_A(0), z(0)) = (300, 1, 1, 0.1)$.

Exemple 1. (Stationary Distribution).

Let $\Psi(\mathcal{T}_A) = \mathcal{T}_A$ in this example, we choose $\sigma = 0.3$, $r = 0.7$, the values of the remaining parameters are given in Data 1 of Table 4. By a simple computation, we obtain the desired results (See Figure 1).

$$\mathfrak{R}_0 = \frac{\Lambda\alpha k\Psi'(0)}{m_{\mathcal{T}}(m_L+\alpha)(m_A+\rho)} = 1.1250 > 1,$$

$$\mathfrak{R}_0^s = \frac{\alpha\Lambda\hat{\kappa}\Psi'(0)\mathrm{e}^{\frac{\sigma^2}{16r}}}{m_{\mathcal{T}}(\rho+m_A)(m_L+\alpha)} = 1.1294 > 1. \tag{5.1}$$

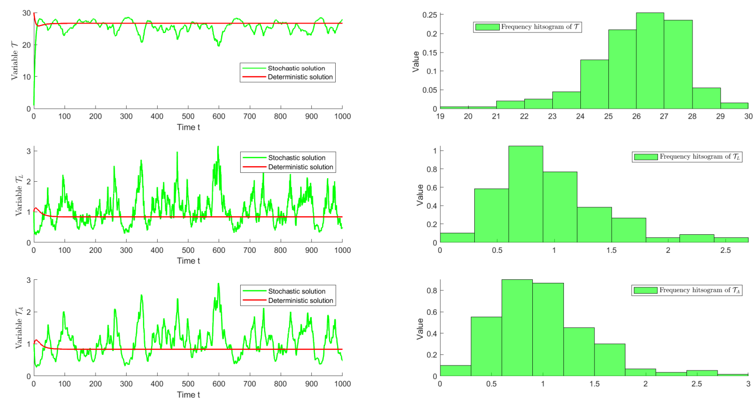

FIGURE 1. Computer simulations are performed for both the stochastic and deterministic versions of the model, along with histograms of the variables $\mathcal{T}$, $\mathcal{T}_L$, and $\mathcal{T}_A$.

Exemple 2. (Extinction)

In this example we choose $\Lambda = 1$, $\sigma = 0.4$, $r = 1.8, \widehat{\kappa} = 0.1$ the values of the remaining parameters are given in Data 1 of Table 4. By a simple computation, we obtain the desired results (See Figures 2 and 3).

$$\mathfrak{R}_0^e = \sqrt{\mathfrak{R}_0} + \frac{\alpha\hat{\kappa}\Psi'(0)\,\mathcal{T}_0\sqrt{e^{\frac{\sigma^2}{r}} - 2e^{\frac{\sigma^2}{4r}} + 1}}{\sqrt{\mathfrak{R}_0}\,(m_L + \alpha)\min(m_L + \alpha, m_A + \rho)} = 0.9087 < 1.$$

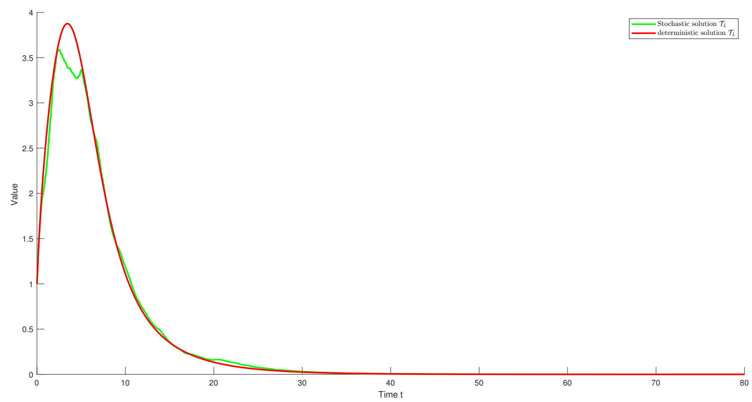

FIGURE 2. Numerical simulations of the $\mathcal{T}_L$ model with bilinear incidence rate are conducted in both deterministic and stochastic frameworks.

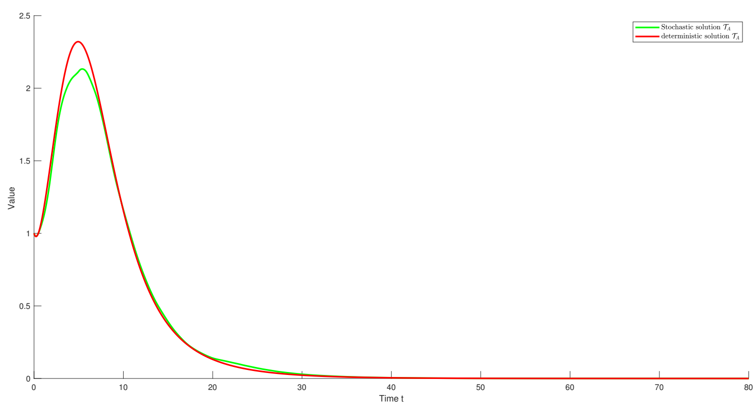

FIGURE 3. Numerical simulations of the $\mathcal{T}_A$ model with bilinear incidence rate are conducted in both deterministic and stochastic frameworks.

Exemple 3. (Stationary Distribution).

Let $\Psi\left(\mathcal{T}_A\right) = \frac{a\mathcal{T}_A}{\mathcal{T}_A+b}$ in this example, we choose $\sigma = 0.2$, $r = 2$, $a = 0.1$, $b = 6$ the values of the remaining parameters are given in Data 2 of Table 4. By a simple computation, we obtain the desired results (See Figure 4):

$$\begin{aligned} \Re_0 &= \frac{\Lambda \alpha k \Psi'(0)}{m_{\mathcal{T}}\left(m_L+\alpha\right)\left(m_A+\rho\right)} = 1.3333 > 1, \\ \Re_0^s &= \frac{\alpha \Lambda \hat{\kappa} \Psi'\left(0\right) \mathrm{e}^{\frac{\sigma^2}{16r}}}{m_{\mathcal{T}}\left(\rho+m_A\right)\left(m_L+\alpha\right)} = 13.4002 > 1. \end{aligned} \tag{5.2}$$

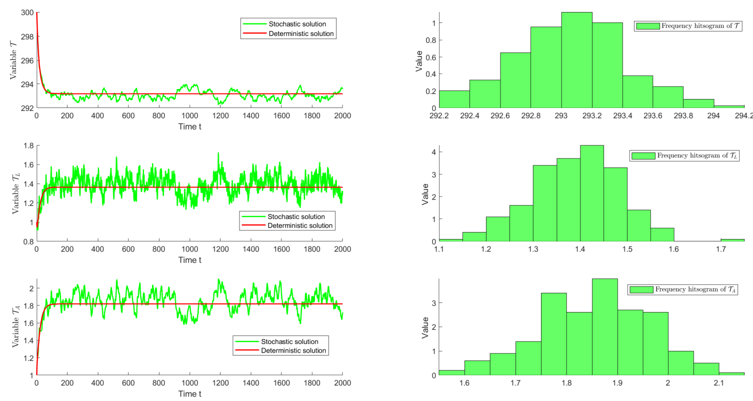

FIGURE 4. Computer simulations are performed for both the stochastic and deterministic versions of the model, along with histograms of the variables $\mathcal{T}$, $\mathcal{T}_L$, and $\mathcal{T}_A$.

Exemple 4. (Extinction)

In this example we choose $\Lambda = 1.2$, $a = 0.1$, $b = 5$ $\sigma = 0.5$, $r = 1.8$ the values of the remaining parameters are given in Data 2 of Table 4. By a simple computation, we obtain the desired results (See Figures 5 and 6).

$$\mathfrak{R}_0^e = \sqrt{\mathfrak{R}_0} + \frac{\alpha\hat{\kappa}\Psi'(0)\,\mathcal{T}_0\sqrt{e^{\frac{\sigma^2}{r}} - 2e^{\frac{\sigma^2}{4r}} + 1}}{\sqrt{\mathfrak{R}_0}\,(m_L + \alpha)\min(m_L + \alpha, m_A + \rho)} = 0.9610 < 1.$$

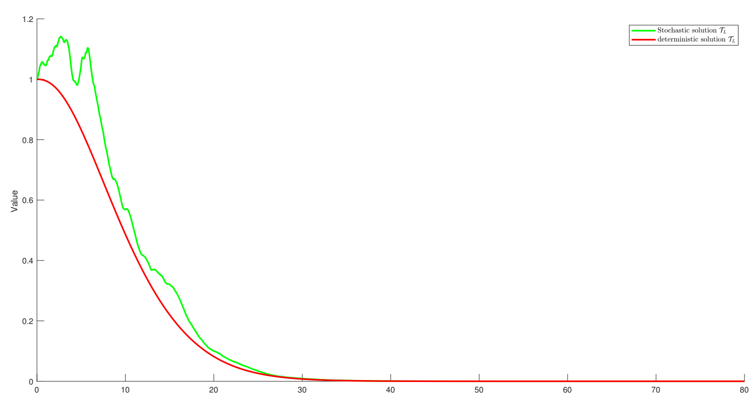

FIGURE 5. Numerical simulations of $\mathcal{T}_L$ with a saturation incidence rate are conducted in both deterministic and stochastic frameworks.

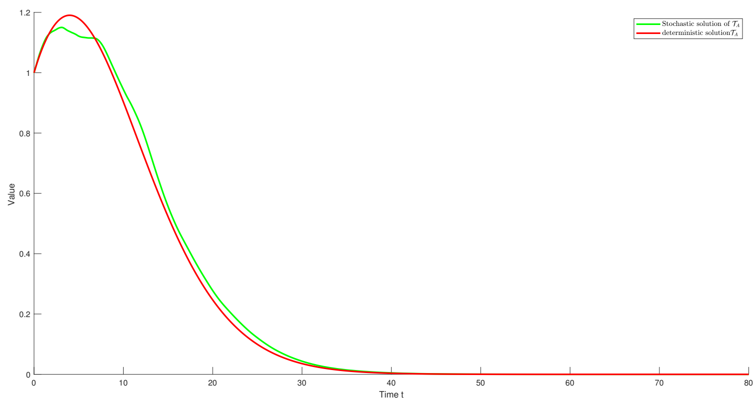

FIGURE 6. Numerical simulations of $\mathcal{T}_A$ with a saturation incidence rate are conducted in both deterministic and stochastic frameworks.

6. Conclusion

In this work, we have investigated the asymptotic analysis of the stochastic HTLV-I infection model using Black-Karasinski process . we first mentioned the existence and uniqueness of the global solution. After that, we have demonstrated the existence of an ergodic stationary distribution.Then, we established the sufficient conditions for the disease to be extinct and we support our finding by some numirecal simulations. Finaly, unlike the ornstein-uhlenbeck process, we have introduced the Black-Karasinski process to perturbe the infection rate of HTLV-I transmission dynamics and this serve as a solid foundation for future studies on the HTLV-I epidemic model.

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