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MONOTONE OPERATOR METHODS FOR FUZZY DIFFERENTIAL INCLUSIONS IN BANACH SPACES

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ABSTRACT: We study fuzzy differential inclusions in a separable reflexive Banach space governed by an m-accretive operator and a levelwise bounded convex perturbation. The fuzzy dynamics is interpreted through α-cut evolution, which converts the problem into a family of multivalued evolution inclusions. Using accretive operator theory, measurable selection arguments, and Yosida regularization, we establish levelwise existence of integral solutions, reconstruct a reachable fuzzy solution tube, derive explicit a priori bounds and time-regularity estimates for α-level reachable sets, and prove Hausdorff stability with respect to both the initial fuzzy datum and the perturbation. A trajectory-wise Yosida approximation result is also obtained. A two-dimensional example with closed formulas for the α-Cuts illustrate the theory and give geometric visualizations of the induced fuzzy reachable tube.

Keywords: fuzzy differential inclusion, m-accretive operator, Banach space, Hausdorff stability, Yosida approximation, reachable fuzzy tube, process innovation.

1. Introduction

Since Zadeh's seminal paper introduced fuzzy sets as a mathematical language for uncertainty, fuzzy-valued differential models have matured into a branch of nonlinear analysis with connections to dynamical systems, control, and uncertain evolution problems [34]. Early differential constructions were formulated through Hukuhara-type differences and embeddings of fuzzy-number spaces into normed spaces [21, 15]. This led to foundational theories of existence for fuzzy differential equations and their Cauchy problems [32, 27, 5]. Subsequent work clarified generalized differentiability and the role of interval and fuzzy arithmetic in differential models [2,29,28,3,1,16]. Recent developments include linearly correlated fuzzy-number calculi and Banach-space embeddings tailored to more refined fuzzy dynamics [24, 25, 26, 33].

A second, conceptually robust, line of research interprets fuzzy differential equations through differential inclusions acting on the α-cuts. This viewpoint is especially useful when generalized Hukuhara differentiability is too restrictive, when nonuniqueness is intrinsic, or when the uncertain forcing is more naturally set-valued [8,9,10,19,20,11]. In recent years, this inclusion perspective has been combined with fractional and hemivariational structures in Banach spaces [14, 17, 31, 22], showing that Banach-space geometry is indispensable once one moves beyond finitedimensional fuzzy arithmetic.

The operator-theoretic side of the subject is equally rich. The nonlinear semigroup framework of Crandall and Liggett remains the canonical tool for autonomous and nonautonomous evolution equations driven by m-accretive operators in Banach

spaces [12]. Multivalued perturbations of such problems were studied systematically by Bothe [4], and the modern evolution-inclusion literature now contains refined solvability and stability results for time-dependent maximal monotone or accretive operators [13, 30, 6, 23]. At the algorithmic level, monotone inclusion methods based on resolvents and forward-backward splitting continue to shape both theory and computation [7]. The survey [18] shows how these operator-theoretic ideas are increasingly influential in current fuzzy differential research.

The purpose of the present paper is to develop a monotone-operator framework for fuzzy differential inclusions in Banach spaces. Instead of working directly with fuzzy derivatives, we encode the evolution through nested α-level differential inclusions

$$x'(t) + Ax(t) \in F_\alpha(t, x(t)), x(0) \in [u_0]^\alpha$$

where A is m-accretive and the family $\{F_\alpha\}_{\alpha\in(0,1]}$ represents the α-cuts of a fuzzy perturbation. This viewpoint allows us to combine the stability of resolvent methods with the geometry of fuzzy sets. The main contributions of the paper are as follows.

(i) We establish level-wise existence of integral solutions under measurable upper semicontinuous perturbations with bounded convex values.
(ii) We construct a reachable fuzzy solution tube from the family of terminal sets of the α-level problems and prove a priori and time-continuity estimates for its cuts.
(iii) We derive quantitative stability bounds in the Hausdorff metric with respect to both the initial fuzzy datum and the perturbation.
(iv) We record a trajectory wise Yosida approximation principle that makes the operator-theoretic content explicit.
(v) We provide an explicit planar example whose α-cuts can be written in closed form and visualized exactly.

The analysis is intended to fit naturally into nonlinear and stochastic/global analysis themes, while remaining close to current fuzzy-differential literature. Figure 1 summarizes the conceptual flow of the paper.

2. Preliminaries

Throughout the paper, X denotes a real separable reflexive Banach space with norm $\|\cdot\|$ and closed unit ball B_X. For nonempty closed bounded sets $A, B \subset X$, we use the Hausdorff distance

$$d_H(A,B) = \max\left\{\sup_{a\in A}\text{dist}(a,B), \sup_{b\in B}\text{dist}(b,A)\right\}$$

Definition 2.1. A fuzzy set on X is a function $u: X \to [0,1]$. For $\alpha \in (0,1]$ its α-cut is

$$[u]^\alpha = \{x \in X: u(x) \geq \alpha\}$$

and its 0-cut is defined by

$$[u]^0 = \text{cl}\{x \in X: u(x) > 0\}.$$

We write $\mathcal{F}_b(X)$ for the class of normal upper semicontinuous fuzzy sets whose α-cuts are nonempty closed bounded sets for all $\alpha \in [0,1]$.

For $u, v \in \mathcal{F}_b(X)$ we consider the metric

$$D(u,v) = \sup_{\alpha\in[0,1]} d_H([u]^\alpha, [v]^\alpha)$$

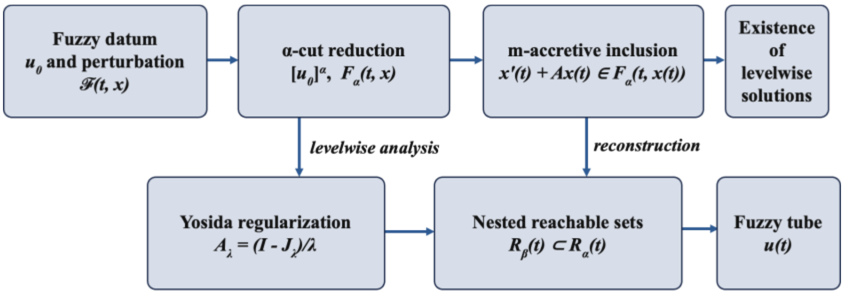

Figure 1. Monotone-operator construction of the fuzzy solution tube. The diagram summarizes the passage from fuzzy input data and α-cuts to levelwise m-accretive inclusions, Yosida regularization, and reconstruction of the fuzzy reachable tube.

This metric is standard in fuzzy differential analysis because it reduces the geometry of fuzzy uncertainty to nested families of level sets; see [21, 15, 33].

Definition 2.2. A set-valued operator $A: \mathrm{dom}(A) \subset X \to 2^X$ is called accretive if for every $x_i \in \mathrm{dom}(A)$ and $y_i \in Ax_i, i = 1,2$, one has

$$\|x_1 - x_2 + \lambda(y_1 - y_2)\| \geq \|x_1 - x_2\| \text{ for all } \lambda > 0.$$

It is called m-accretive if, in addition, $I + \lambda A$ is onto X for every $\lambda > 0$.

In the Banach-space literature, accretive operators play the role maximal monotone operators play in Hilbert spaces. If A is m-accretive, its resolvent and Yosida approximation are given by

$$J_\lambda = (I + \lambda A)^{-1}, A_\lambda = \frac{1}{\lambda}(I - J_\lambda), \lambda > 0.$$

Then J_λ is nonexpansive, A_λ is single-valued, Lipschitz, and accretive, and $A_\lambda x \in AJ_\lambda x$ whenever $x \in X$; see [12,4].

Remark 2.3. Because the Banach-space setting is essential for our main results, the recent embedding results of Wu [33] are especially relevant: they show that families of fuzzy sets with suitable support-function regularity can be viewed inside Banach spaces in a structurally efficient way. Our paper does not require an explicit embedding theorem, but the α-cut formalism employed here is compatible with that viewpoint.

3. Problem formulation and hypotheses

Let $T > 0$ be fixed. We consider a fuzzy perturbation

$$\mathfrak{F}: [0, T] \times X \to \mathcal{F}_b(X),$$

and denote by $F_\alpha(t, x)$ the α-cut of $\mathfrak{F}(t, x)$ for $\alpha \in (0,1]$. The initial state is a fuzzy set $u_0 \in \mathcal{F}_b(X)$. Our fuzzy differential inclusion is interpreted levelwise as

$$x'(t) + Ax(t) \in F_\alpha(t, x(t)) \text{ for a.e. } t \in [0, T], x(0) = x_0 \in [u_0]^\alpha$$

Definition 3.1. Fix $\alpha \in (0,1]$. A function $x \in C([0, T]; X)$ is called an α-level integral solution of the above problem if there exists $f \in L^1(0, T; X)$ such that

$$f(t) \in F_\alpha(t, x(t)) \text{ for a.e. } t \in [0, T]$$

and x is an integral solution in the Crandall-Liggett sense of

$$x'(t) + Ax(t) \ni f(t), x(0) = x_0$$

For each $\alpha \in (0,1]$ and $t \in [0, T]$ we define the reachable terminal set

$$R_\alpha(t) = \mathrm{cl}\{x(t): x \text{ is an } \alpha\text{-level integral solution with } x(0) \in [u_0]^\alpha\}.$$

The basic hypotheses are collected next and summarized in Table 1.

Assumption 3.2.

(A1) $A: \mathrm{dom}(A) \subset X \to 2^X$ is m-accretive and $0 \in A0$.

(A2) $u_0 \in \mathcal{F}_b(X)$.

(A3) For every $\alpha \in (0,1]$, the map $F_\alpha: [0,T] \times X \to 2^X$ has nonempty closed convex bounded values; for each $x \in X, t \mapsto F_\alpha(t,x)$ is measurable; and for a.e. $t \in [0,T], x \mapsto F_\alpha(t,x)$ is upper semicontinuous.

(A4) There exist nonnegative functions $m, \ell \in L^1(0,T)$ such that for all $\alpha \in (0,1]$, a.e. $t \in [0,T]$, and all $x, y \in X$,

$$\sup\{\|z\|: z \in F_\alpha(t,x)\} \leq m(t) + \ell(t)\|x\|$$
$$d_H(F_\alpha(t,x), F_\alpha(t,y)) \leq \ell(t)\|x - y\|$$

(A5) The family is nested with respect to α : if $0 < \alpha \leq \beta \leq 1$, then

$$[u_0]^\beta \subset [u_0]^\alpha, F_\beta(t,x) \subset F_\alpha(t,x) \text{ for all } (t,x) \in [0,T] \times X$$

(A6) The cut family of the perturbation is left continuous in the sense that for every $\alpha \in (0,1]$,

$$F_\alpha(t,x) = \bigcap_{0<\beta<\alpha} F_\beta(t,x) \text{ for all } (t,x) \in [0,T] \times X$$

Remark 3.3. Assumptions 3.2(A3)-(A4) are consistent with the existence frameworks used for fuzzy differential inclusions in Banach spaces in [14, 17, 31] and with the perturbation theory for m-accretive evolution inclusions developed in [4, 30, 6]. The α-cut interpretation also agrees with the differential-inclusion philosophy emphasized in [10].

4. Existence and structure of fuzzy reachable solution tubes

We begin with the levelwise problem.

Theorem 4.1 (*Levelwise existence*). Assume 3.2. For every $\alpha \in (0,1]$ and every $x_0 \in [u_0]^\alpha$, the levelwise problem admits at least one integral solution on $[0,T]$.

Table 1. Hypotheses used throughout the paper and their role in the main results.

Hypothesis	Content	Main use
(A1)	m-accretive generator A with $0 \in A0$	nonlinear semigroup framework, resolvents, comparison estimates
(A2)	fuzzy initial datum with bounded cuts	initial geometry and α-level reachability
(A3)	measurable, upper semicontinuous perturbation with bounded convex values	levelwise solvability and measurable selections
(A4)	linear growth and Hausdorff-Lipschitz control	a priori bounds, time regularity, stability
(A5)	nesting of initial and perturbation cuts	monotonicity of reachable families
(A6)	left continuity of perturbation cuts	faithful reconstruction of a fuzzy tube

Proof. Fix $\alpha \in (0,1]$. By 3.2(A3), the multifunction F_α has nonempty closed convex bounded values, is measurable in time, and is upper semicontinuous in the state variable. Since X is reflexive, bounded closed convex subsets are weakly compact. Together with the growth bound in 3.2(A4), these hypotheses place the perturbation F_α within the standard class of multivalued perturbations of m-accretive evolution inclusions treated by Bothe [4]. Therefore, for every $x_0 \in [u_0]^\alpha$ there exists an integral solution of

$$x'(t) + Ax(t) \in F_\alpha(t, x(t)), x(0) = x_0$$

This proves the claim.

The next estimate is the basic bound from which the fuzzy-level estimates follow.

Theorem 4.2 (A *priori bound*). Assume 3.2. Fix $\alpha \in (0,1]$ and let

$$r_\alpha = \sup\{\|x\|: x \in [u_0]^\alpha\}, L(t) = \int_0^t \ell(s)ds$$

If x is an α-level integral solution, then for all $t \in [0, T]$,

$$\|x(t)\| \leq M_\alpha(t)$$

where

$$M_\alpha(t) = e^{L(t)}\left(r_\alpha + \int_0^t e^{-L(s)}m(s)ds\right)$$

Consequently,

$$R_\alpha(t) \subset M_\alpha(t)B_X \text{ for all } t \in [0, T]$$

Proof. Let x be an α-level integral solution and let $f \in L^1(0, T; X)$ be such that $f(t) \in F_\alpha(t, x(t))$ a.e. Since $0 \in A0$, the zero function is the integral solution of the unforced problem $z'(t) + Az(t) \ni 0, z(0) = 0$. The comparison inequality for m-accretive evolution equations yields

$$\|x(t)\| \leq \|x(0)\| + \int_0^t \|f(s)\|ds$$

Using the growth assumption in 3.2(A4) we obtain

$$\|x(t)\| \leq r_\alpha + \int_0^t (m(s) + \ell(s)\|x(s)\|)ds$$

An application of Gronwall's inequality gives

$$\|x(t)\| \leq e^{L(t)}\left(r_\alpha + \int_0^t e^{-L(s)}m(s)ds\right) = M_\alpha(t)$$

Taking closures in the definition of $R_\alpha(t)$ proves the second claim.

Proposition 4.3 (*Nestedness of reachable sets*). Assume 3.2. If $0 < \alpha \leq \beta \leq 1$, then

$$R_\beta(t) \subset R_\alpha(t) \text{ for all } t \in [0, T].$$

Proof. Take $\xi \in R_\beta(t)$. By definition, there exists a sequence of β-level solutions $\{x_n\}$ such that $x_n(t) \to \xi$. By 3.2(A5), every admissible initial value for the β problem belongs to $[u_0]^\alpha$, and every perturbation value in $F_\beta(s, x_n(s))$ also belongs to $F_\alpha(s, x_n(s))$. Hence each x_n is also an α-level solution. Therefore $\xi \in R_\alpha(t)$.

Proposition 4.4 (*Time regularity of the reachable cuts*). Assume 3.2. Fix $\alpha \in (0,1]$ and set

$$q_\alpha(t) = m(t) + \ell(t)M_\alpha(t)$$

Then for all $0 \leq s \leq t \leq T$,

$$d_H(R_\alpha(t), R_\alpha(s)) \leq \int_s^t q_\alpha(\tau)d\tau$$

In particular, $t \mapsto R_\alpha(t)$ is continuous from $[0, T]$ into the metric space of nonempty closed bounded subsets of X endowed with d_H.

Proof. Let $\xi \in R_\alpha(t)$ and $\varepsilon > 0$. Choose an α-level solution x such that

$$\|x(t) - \xi\| < \varepsilon$$

For a.e. τ, the derivative bound

$$\|x'(\tau)\| \leq \|f(\tau)\| \leq m(\tau) + \ell(\tau)\|x(\tau)\| \leq q_\alpha(\tau)$$

holds for some measurable selection $f(\tau) \in F_\alpha(\tau, x(\tau))$. Hence

$$\|x(t) - x(s)\| \leq \int_s^t q_\alpha(\tau)d\tau$$

Because $x(s) \in R_\alpha(s)$, it follows that

$$\text{dist}(\xi, R_\alpha(s)) \leq \varepsilon + \int_s^t q_\alpha(\tau)d\tau$$

Letting $\varepsilon \downarrow 0$ yields

$$\sup_{\xi \in R_\alpha(t)} \text{dist}(\xi, R_\alpha(s)) \leq \int_s^t q_\alpha(\tau)d\tau$$

By interchanging s and t the symmetric estimate follows, and therefore so does the Hausdorff bound.

The reachable family $\{R_\alpha(t)\}_{\alpha \in (0,1]}$ is nested but need not be left continuous in α. The standard regularization is therefore built into the definition of the fuzzy tube.

Definition 4.5. For each $t \in [0, T]$ define

$$[\mathcal{U}(t)]^0 = \text{cl}\left(\bigcup_{\gamma \in (0,1]} R_\gamma(t)\right),$$

and for $\alpha \in (0,1]$ set

$$[\mathcal{U}(t)]^\alpha = \bigcap_{0<\beta<\alpha} R_\beta(t).$$

The family $\mathcal{U} = \{\mathcal{U}(t)\}_{t \in [0,T]}$ will be called the reachable fuzzy solution tube.

Theorem 4.6 (*Reconstruction of a fuzzy solution tube*). Assume 3.2. For every $t \in [0, T]$ the family $\{[\mathcal{U}(t)]^\alpha\}_{\alpha \in [0,1]}$ defines a unique fuzzy set $\mathcal{U}(t) \in \mathcal{F}_b(X)$. Moreover,

$$[\mathcal{U}(t)]^\beta \subset [\mathcal{U}(t)]^\alpha \text{ whenever } 0 \le \alpha \le \beta \le 1,$$

and

$$D(\mathcal{U}(t), \mathcal{U}(s)) \le \sup_{\alpha\in(0,1]} \int_s^t q_\alpha(\tau)d\tau \text{ for } 0 \le s \le t \le T.$$

If the initial cut family is left continuous in α, then $\mathcal{U}(0) = u_0$.

Proof. By Theorem 4.2, each $R_\alpha(t)$ is nonempty and bounded; by definition it is closed. Proposition 4.3 gives the nesting in α, and the regularized family is automatically left continuous by construction. The standard representation theorem for upper semicontinuous fuzzy sets with closed bounded cuts therefore yields a unique fuzzy set $\mathcal{U}(t) \in \mathcal{F}_b(X)$ with the stated cuts.

To estimate the time variation, fix $\alpha \in (0,1]$. Since

$$[\mathcal{U}(t)]^\alpha \subset R_\beta(t) \text{ for every } \beta < \alpha,$$

Proposition 4.4 implies

$$\operatorname{dist}\big([\mathcal{U}(t)]^\alpha, R_\beta(s)\big) \le \int_s^t q_\beta(\tau)d\tau.$$

Now pass to the intersection over $0 < \beta < \alpha$, using the monotonicity of the right-hand side with respect to the bound M_β. This gives

$$d_H([\mathcal{U}(t)]^\alpha, [\mathcal{U}(s)]^\alpha) \le \sup_{\beta<\alpha} \int_s^t q_\beta(\tau)d\tau \le \sup_{\gamma\in(0,1]} \int_s^t q_\gamma(\tau)d\tau$$

Taking the supremum over $\alpha \in [0,1]$ proves the announced estimate in the fuzzy metric.

Finally, at $t = 0$ we have $R_\alpha(0) = [u_0]^\alpha$ by definition. If the cut family of u_0 is left continuous, then the regularization reproduces u_0 exactly.

Remark 4.7. The object $\mathcal{U}(t)$ is best viewed as a reachable fuzzy tube rather than a single path. This is precisely the right analogue of the solution set of a differential inclusion. In the presence of uniqueness, discussed below, the tube collapses to the image of the initial fuzzy set under a deterministic evolution operator.

5. Stability and Yosida regularization

We now compare two fuzzy differential inclusions driven by the same m-accretive operator A but different fuzzy perturbations. Let $\mathfrak{F}$ and $\mathfrak{G}$ be fuzzy perturbations with α-cuts F_α and G_α, and let $u_0, v_0 \in \mathcal{F}_b(X)$ be their initial data. Denote the corresponding reachable sets by $R_\alpha^F(t)$ and $R_\alpha^G(t)$ and the reconstructed fuzzy tubes by $\mathcal{U}_F(t)$ and $\mathcal{U}_G(t)$.

Assumption 5.1. In addition to 3.2, there exists a nonnegative $\eta \in L^1(0,T)$ such that for all $\alpha \in (0,1]$, a.e. $t \in [0,T]$, and all $x, y \in X$,

$$d_H(F_\alpha(t,x), G_\alpha(t,y)) \le \ell(t)\|x-y\| + \eta(t)$$

Theorem 5.2 (*Hausdorff stability of reachable cuts*). Assume 3.2 for both systems and assume 5.1. Then for every $\alpha \in (0,1]$ and every $t \in [0,T]$,

$$d_H(R_\alpha^F(t), R_\alpha^G(t)) \le e^{L(t)}\left(d_H([u_0]^\alpha, [v_0]^\alpha) + \int_0^t \eta(s)ds\right)$$

where $L(t) = \int_0^t \ell(s)ds$.

Proof. Fix $\alpha \in (0,1], t \in [0,T]$, and $\varepsilon > 0$. Let $\xi \in R_\alpha^F(t)$. By definition there exists an α-level integral solution x of the F-system with initial value $x_0 \in [u_0]^\alpha$ such that

$$\|x(t) - \xi\| < \varepsilon$$

Choose $y_0 \in [v_0]^\alpha$ satisfying

$$\|x_0 - y_0\| \le d_H([u_0]^\alpha, [v_0]^\alpha) + \varepsilon.$$

We approximate x by Yosida-regularized strong solutions. For $\lambda > 0$ let x_λ solve

$$x_\lambda'(s) + A_\lambda x_\lambda(s) = f_\lambda(s), x_\lambda(0) = x_0$$

where $f_\lambda(s) \in F_\alpha(s, x_\lambda(s))$ a.e. and $x_\lambda \to x$ uniformly on $[0,T]$ as $\lambda \downarrow 0$. Such approximations are standard in the accretive-operator theory of evolution inclusions; see [12,4]. For each λ define the measurable multifunction

$$M_\lambda(s,z) = G_\alpha(s,z) \cap \big(f_\lambda(s) + (\ell(s)\|x_\lambda(s) - z\| + \eta(s))\big)B_X$$

Assumption 5.1 guarantees that $M_\lambda(s,z)$ is nonempty for a.e. s and every $z \in X$. By measurable selection, we may choose $g_\lambda(s,z) \in M_\lambda(s,z)$ and solve

$$y_\lambda'(s) + A_\lambda y_\lambda(s) = g_\lambda(s, y_\lambda(s)), y_\lambda(0) = y_0$$

Since A_λ is accretive, the standard comparison estimate yields

$$\|x_\lambda(t) - y_\lambda(t)\| \le \|x_0 - y_0\| + \int_0^t \|f_\lambda(s) - g_\lambda(s, y_\lambda(s))\|ds$$

By construction of M_λ,

$$\|f_\lambda(s) - g_\lambda(s, y_\lambda(s))\| \le \ell(s)\|x_\lambda(s) - y_\lambda(s)\| + \eta(s)$$

Therefore

$$\|x_\lambda(t) - y_\lambda(t)\| \le \|x_0 - y_0\| + \int_0^t (\ell(s)\|x_\lambda(s) - y_\lambda(s)\| + \eta(s))ds$$

Gronwall's inequality gives

$$\|x_\lambda(t) - y_\lambda(t)\| \le e^{L(t)}\left(\|x_0 - y_0\| + \int_0^t \eta(s)ds\right)$$

Passing to the limit along a vanishing sequence of λ yields an α-level integral solution y of the G-system such that

$$\|x(t) - y(t)\| \le e^{L(t)}\left(\|x_0 - y_0\| + \int_0^t \eta(s)ds\right)$$

Hence

$$\mathrm{dist}(\xi, R_\alpha^G(t)) \le \varepsilon + e^{L(t)}\left(d_H([u_0]^\alpha, [v_0]^\alpha) + \varepsilon + \int_0^t \eta(s)ds\right)$$

Letting $\varepsilon \downarrow 0$ gives

$$\sup_{\xi \in R_\alpha^F(t)} \mathrm{dist}(\xi, R_\alpha^G(t)) \le e^{L(t)}\left(d_H([u_0]^\alpha, [v_0]^\alpha) + \int_0^t \eta(s)ds\right)$$

By symmetry the same estimate holds with F and G interchanged, which proves the Hausdorff bound.

Corollary 5.3 (*Stability in the fuzzy metric*). Under the assumptions of Theorem 5.2,

$$D(\mathcal{U}_F(t),\mathcal{U}_G(t)) \le e^{L(t)}\left(D(u_0,v_0)+\int_0^t \eta(s)ds\right) \text{ for all } t\in[0,T]$$

Proof. Theorem 5.2 controls the Hausdorff distance between the raw reachable sets. The same estimate then passes to the regularized cut families defining $\mathcal{U}_F(t)$ and $\mathcal{U}_G(t)$ because intersection and closure do not increase the Hausdorff distance. Taking the supremum over $\alpha\in[0,1]$ yields the result.

Corollary 5.4 (*Uniqueness in the single-valued case*). Assume 3.2, and suppose that the perturbation is independent of α and single-valued:

$$F_\alpha(t,x)=\{f(t,x)\} \text{ for all } \alpha\in(0,1]$$

where f satisfies the Carathéodory and Lipschitz assumptions of 3.2(A3)-(A4). Then for every $x_0\in X$ there exists a unique integral solution of

$$x'(t)+Ax(t)=f(t,x(t)), x(0)=x_0$$

and the reachable fuzzy tube is given by

$$[\mathcal{U}(t)]^\alpha=\Phi_t([u_0]^\alpha),$$

where Φ_t is the single-valued evolution operator.

Proof. Apply Theorem 5.2 with $F=G=f$ and $\eta\equiv 0$. Then two solutions starting from the same initial point satisfy

$$\|x(t)-y(t)\|\le e^{L(t)}\|x(0)-y(0)\|=0$$

hence coincide identically. The formula for $[\mathcal{U}(t)]^\alpha$ follows from the definition of the reachable set.

The next proposition records the operator-theoretic approximation mechanism behind the above comparison argument.

Proposition 5.5 (*Trajectorywise Yosida approximation*). Assume 3.2. Fix $\alpha\in(0,1]$, an initial value $x_0\in[u_0]^\alpha$, and a measurable function $f\in L^1(0,T;X)$ such that $f(t)\in F_\alpha(t,x(t))$ a.e. for some α-level integral solution x of

$$x'(t)+Ax(t)\ni f(t), x(0)=x_0$$

For each $\lambda>0$, let x_λ be the unique strong solution of

$$x_\lambda'(t)+A_\lambda x_\lambda(t)=f(t), x_\lambda(0)=x_0$$

Then

$$\lim_{\lambda\downarrow 0}\sup_{t\in[0,T]}\|x_\lambda(t)-x(t)\|=0$$

Proof. The conclusion is a standard consequence of the Crandall-Liggett nonlinear semigroup approximation theorem [12]. Indeed, A_λ is the Yosida approximation of the m-accretive operator A, and the forcing term f belongs to $L^1(0,T;X)$. Therefore, the regularized strong solutions converge uniformly on bounded time intervals to the integral solution of the original inhomogeneous evolution equation. The statement follows.

Remark 5.6. Proposition 5.5 is deliberately formulated at the trajectory level. It says that every admissible α-level path may be approximated by solutions of Lipschitz ordinary differential equations in Banach spaces. This is the practical bridge between fuzzy differential inclusions and computational monotone-operator methods.

6. An explicit planar example

We now present a closed-form example in $X = \mathbb{R}^2$ with the Euclidean norm. Although finite-dimensional, it captures all structural features of the general theory and leads to exact figures.

Example 6.1. Let

$$A = I_{\mathbb{R}^2}, c_0 = (1,0), \rho_0 = 0.6, r_0 = 0.25$$

Let $B_{\mathbb{R}^2}$ denote the closed unit disk in $\mathbb{R}^2$ and define the initial fuzzy datum by

$$[u_0]^\alpha = c_0 + \rho_0(1-\alpha)B_{\mathbb{R}^2}, \alpha \in [0,1].$$

Further, let

$$b(t) = (\sin t, \cos t), F_\alpha(t,x) = b(t) + r_0(1-\alpha)B_{\mathbb{R}^2}, \alpha \in (0,1].$$

Then the level wise inclusion becomes

$$x'(t) + x(t) \in b(t) + r_0(1-\alpha)B_{\mathbb{R}^2}, x(0) \in [u_0]^\alpha$$

Here $\ell \equiv 0$ and $m(t) = \|b(t)\| + r_0 \leq 1 + r_0$, so Assumption 3.2 is satisfied.

Proposition 6.2. For Example 6.1, the α-level reachable set is

$$R_\alpha(t) = m(t) + \rho_\alpha(t)B_{\mathbb{R}^2}$$

where

$$m(t) = e^{-t}c_0 + \int_0^t e^{-(t-s)}b(s)ds$$

Table 2. Center coordinates and α-level radii in Example 6.1 at selected times.

t	$m_1(t)$	$m_2(t)$	$\rho_{0.2}(t)$	$\rho_{0.5}(t)$	$\rho_{0.8}(t)$
0	1.0000	0.0000	0.4800	0.3000	0.1200
1	0.7024	0.5069	0.3030	0.1894	0.0758
2	0.8657	0.1789	0.2379	0.1487	0.0595
3	0.6402	-0.4493	0.2139	0.1337	0.0535

and

$$\rho_\alpha(t) = (1-\alpha)\big(\rho_0 e^{-t} + r_0(1-e^{-t})\big).$$

Equivalently,

$$m(t) = \left(\frac{\sin t - \cos t + 3e^{-t}}{2}, \frac{\sin t + \cos t - e^{-t}}{2}\right).$$

Proof. The variation-of-constants formula for the linear inclusion gives

$$x(t) = e^{-t}x_0 + \int_0^t e^{-(t-s)}\zeta(s)ds, \zeta(s) \in b(s) + r_0(1-\alpha)B_{\mathbb{R}^2}$$

Because the set-valued forcing is independent of x and the Euclidean disk is convex, balanced, and invariant under scalar multiplication, the set of all terminal values at time t is exactly

$$e^{-t}[u_0]^\alpha + \int_0^t e^{-(t-s)} b(s)ds + \int_0^t e^{-(t-s)} r_0(1-\alpha)B_{\mathbb{R}^2} ds$$

The first term is

$$e^{-t}c_0 + \rho_0(1-\alpha)e^{-t}B_{\mathbb{R}^2}$$

whereas the last term equals

$$r_0(1-\alpha)\left(\int_0^t e^{-(t-s)}ds\right)B_{\mathbb{R}^2} = r_0(1-\alpha)(1-e^{-t})B_{\mathbb{R}^2}$$

Combining the two disk terms yields the radius formula for $\rho_\alpha(t)$.

It remains to compute the center. Direct integration gives

$$\int_0^t e^{-(t-s)}\sin s ds = \frac{\sin t - \cos t + e^{-t}}{2}, \int_0^t e^{-(t-s)}\cos s ds = \frac{\sin t + \cos t - e^{-t}}{2}$$

Adding $e^{-t}c_0 = (e^{-t}, 0)$ proves the stated expression for $m(t)$.

Figures 2-3 visualize the exact fuzzy geometry of Example 6.1. The first shows the center trajectory together with representative α-cut disks, while the second displays the analytic decay of the radii. Table 2 records representative values.

The explicit formulas immediately verify the theoretical estimates. Since $\ell \equiv 0$, Theorem 4.2 predicts the bound

$$\|x(t)\| \le r_\alpha + \int_0^t m(s)ds$$

while the exact radius is smaller and converges to $(1-\alpha)r_0$. Moreover, Theorem 5.2 reduces in this example to a purely additive bound because there is no state-Lipschitz amplification. Thus, the example may be used as a benchmark for numerical schemes based on resolvents or implicit Euler discretizations.

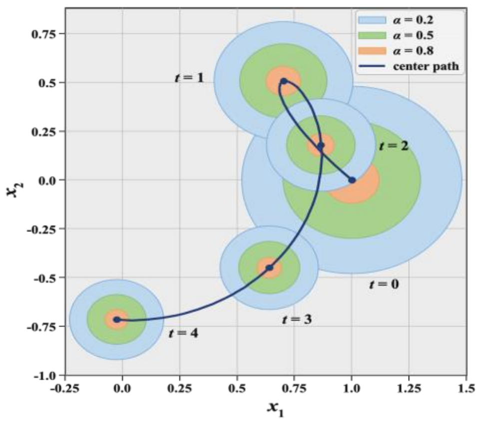

Figure 2. Planar evolution of the explicit example.

The center path $m(t)$ is shown together with α-cut disks at $t = 0,1,2,3,4$ for $\alpha \in \{0.2, 0.5, 0.8\}$. Smaller α corresponds to larger uncertainty, and the tube contracts toward the asymptotic forcing radius.

7. Concluding REMARKS

We have proposed a monotone-operator framework for fuzzy differential inclusions in Banach spaces based on three guiding principles: levelwise α-cut analysis, accretive operator theory, and fuzzy reconstruction from reachable families. This approach avoids restrictive differentiability assumptions at the fuzzy level, accommodates genuinely set-valued perturbations, and leads to quantitative estimates in the Hausdorff metric. The main results consist of levelwise existence, a priori and time-regularity bounds for reachable cuts, reconstruction of a fuzzy solution tube, stability with respect to perturbations and initial fuzzy data, and a trajectory wise Yosida approximation principle.

Several extensions are natural. One may replace the deterministic forcing by random or stochastic fuzzy perturbations, incorporate nonlocal conditions or impulses, or study numerical resolvent schemes and fractional variants in the spirit of [17, 31, 22]. Another attractive direction is to combine Banach-space embeddings

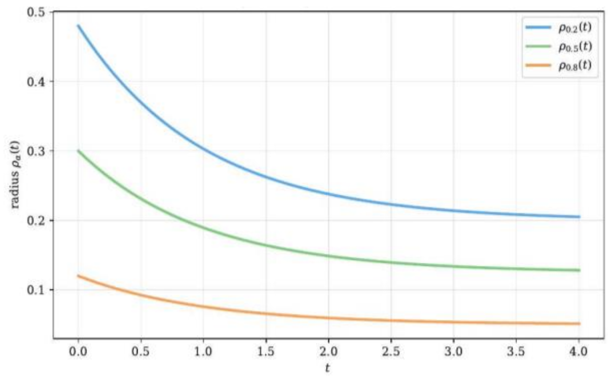

Figure 3. Decay of the α-level radii.

The exact formula $\rho_\alpha(t) = (1-\alpha)\big(0.6e^{-t} + 0.25(1-e^{-t})\big)$ shows monotone contraction from the initial radius to the limiting width $0.25(1-\alpha)$ of fuzzy sets [33] with operator splitting methods for monotone inclusions. These topics will be considered elsewhere.

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