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ENERGY OF AN ENHANCED HYPERCUBE NETWORKS

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Abstract. Enhanced hypercube networks, a class of hypercube-based interconnection networks, are widely recognized for their regular structure and scalability. These networks extend traditional hypercube topologies and find applications in various fields, including network theory, condensed matter physics, and quantum field theory. In this work, we explore the application of graph energy to enhanced hypercube structures, focusing on their energy, Laplacian energy, distance energy, and Randic energy. These energy measures provide insights into the spectral properties of the networks, which are crucial for understanding their behavior in physical systems. In condensed matter physics, graph energy helps model the electronic and vibrational properties of materials, such as graphene and carbon nanotubes. In quantum field theory, the eigenvalues of Feynman diagrams, represented as graphs, determine the energy of particle interactions and scattering amplitudes. We also present standard algorithms and numerical examples to compute the energy of enhanced hypercube structures, demonstrating their utility in analyzing complex networks and quantum systems. This study bridges the gap between graph theory, condensed matter physics, and quantum field theory, offering a unified framework for understanding energy-related phenomena in diverse scientific domains.

1. Introduction

In 1978, Ivan Gutman made the first description of the energy of graph G. This idea emerged since by earlier results related to $\pi-$ electron energy [13] in the field of chemical terms, where expected numerical variables, such as the temperature of production of a hydrocarbon, are connected to the overall -electron energy, which can be computed as the energy of the relevant ”molecular” structure [5]. There are number of uses of graph energy in the chemistry of unsaturated conjugated compounds. Applications in radiography, macro molecule theory, as well as analysis and comparison of protein sequences, are all somewhat relevant. Besides that, graph energies were applied to networking, which covered issues with satellite technology, biology, computer engineering, and process analysis. One of the most common formulations of graph energy arises from spectral graph theory, which associates the energy of a graph with the eigenvalues of its adjacency matrix. The spectrum of a graph, represented by its eigenvalues, encapsulates essential information about the graph’s global topology and local connectivity, offering insights

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into its stability, resilience, and information propagation dynamics, the readers should consult the references [9].

The motivation of study Hypercube network graphs are versatile tools in mathematical physics, providing insights into the electronic, vibrational, and thermodynamic properties of materials, as well as the behavior of quantum systems. Their regular structure and scalability make them ideal for modeling complex systems in condensed matter physics, quantum field theory, and statistical mechanics. By analyzing the eigenvalues of hypercube graphs, researchers can predict and understand a wide range of physical phenomena.

Hypercubes have been extensively studied for their applications in parallel processing, distributed computing, and interconnection networks. They serve as fundamental building blocks for designing efficient and scalable parallel algorithms and architectures, making them an essential concept in the field of computer science. In [7] gives the results related to Hypercube and its complements. This leads to study the Enhanced hypercube. In [14, 11] listed the results importent and application of extra links in hypercube. In features hypercube becomes the first choice for the topological structure of parallel processing and computing systems. Since the impression of results related to topological index of different graph stucher listed in [4, 12].

Let G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For every vertex $v \in V$, the open neighborhood $N_G(v) = N(v)$ is the set $\{u \in V/uv \in E\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup v$. The degree of a vertex $v \in V$ is $d_G(v) = d(v) = |N(v)|$. The minimum and maximum degree of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively.

Let $A(G) = (a_{ij})$ be the adjacency matrix for G and $det(\Lambda I - A(G))$ denote the characteristic polynomial of the adjacency matrix. Zero of this characteristic polynomial is called the eigen value of the $A(G)$. Let $\Lambda_1, \Lambda_2, \dots, \Lambda_m$ denote the eigen value of the $A(G)$. Hence, the energy of the graph

$$\varepsilon(G) = \sum_{i=1}^{m} |\Lambda_i|$$

The distance matrix of graph G is defined as $D = [d_{ij}]$ where d_{ij} is distance between two vertices v_i and v_j. The distance energy of graph G is $\varepsilon_D(G)$ defined as sum of absolute eignen value of distance matrix of graph G. Also Randic matrix $R(G) = [r_{ij}]$ is defined as $r_{ij} = \frac{1}{\sqrt{d_i d_j}}$, if the vertices v_i and v_j are adjacent, otherwise $r_{ij} = 0$, where d_i is degree of the vertex v_i. The Randic energy of graph G is $RE(G)$ defined as sum of absolute eigenvalue of Randic matrix of graph G.

2. Fundamentals

Let n be a positive integer then n-dimensional hypercube Q_n is defined with 2^n vertices and each vertex represented by n-tuples $(v_1, v_2, \dots, v_n)$ whose entries are 0 and 1. Also, the adjacency relation is defined by two vertices are adjacent if and only if they are differ exactly one place.

The enhanced hypercube $Q_{n,k}$, $0 \le k \le n$, is a graph with vertex set $V(Q_{n,k}) = V(Q_n)$ and edge set

$E(Q_{n,k}) = E(Q_n) \cup \{v_1, v_2, \dots v_{k-1}, v_k \dots v_n, v_1, v_2, \dots v_{k-1}, \bar{v_k} \dots \bar{v_n}\}$. The edges of Q_n in $Q_{n,k}$ are hypercube edges and the remaining edges of $Q_{n,k}$ are called complementary edges. Here, $k = 0$ then enhanced hypercube $Q_{n,k}$ is same as hypercube Q_n.

Theorem 2.1. ([11, Lemma 4.1]) *Let G be an $k-$regular graph. Then Laplacian energy and energy of G are equal.*

Theorem 2.2. ([11, Theorem 2.3]) *Let $E(Q_n$ be the (ordinary) energy of $n-$dimensional hypercube Q_n. Then*

$$E(Q_n) = 2\lceil \frac{n}{2} \rceil \binom{n}{\lceil \frac{n}{2} \rceil}$$

where $n \geq 1$, $\lceil a \rceil$ is the ceiling of number of a.

3. main Results

Let $Q_{n,n}$ be enhanced hypercube then, the following algorithm to find the adjacency matrix and Laplacian matrix of $Q_{n,n}$ respectively.

Algorithm 1 Adjacency matrix of $Q_{n,n}$

```
$n \leftarrow$ Order of $Q_{n,n}$
$A \leftarrow zeros(n)$;
for $i \leftarrow 1$ to $n$ do
    for $j \leftarrow 1$ to $n$ do
        if $i$ and $j$ differ exactly one and $n$ place then
            $a(i,j) \leftarrow 1$;                ▷ This is Adjacency matrix of $Q_{n,n}$
        end if
    end for
end for
```

Algorithm 2 Laplacian matrix of $Q_{n,n}$

```
$n \leftarrow$ Order of the group
$A \leftarrow$ Adjacency matrix of $Q_{n,n}$;                ▷ 1
$Deg \leftarrow sum(A)$;
$D \leftarrow zeros(n)$;
for $i \leftarrow 1$ to $n$ do
    for $j \leftarrow 1$ to $n$ do
        if $i == j$ then
            $D(i,j) \leftarrow deg(1,i)$;                ▷ Degree matrix of $Q_{n,n}$
        end if
    end for
end for
$L \leftarrow D - A$;                ▷ Laplacian matrix of $Q_{n,n}$
```

Theorem 3.1. *Let n be non zero positive integer and $Q_{n,n}$ is enhanced hypercube. Then energy*

$$\varepsilon(Q_{n,n}) = \begin{cases} (n-2)(n-1)(n+1), & \textit{If } n \textit{ is odd} \\ \frac{n+2}{2}\binom{n+1}{\frac{n}{2}+1}, & \textit{If } n \textit{ is even.} \end{cases}$$

Proof. **Case i** If $Q_{n,n}$ be the enhanced hypercube graph with n is odd, then the eigen values of $Q_{n,n}$ are $\Lambda_i = 4i-(n+1)$ where $i = 0, 1, \ldots, \frac{n+1}{2}$ and multiplicity of the eigen values Λ_i are $\binom{n+1}{2i}$. Hence the energy

$$\begin{aligned}
\varepsilon(Q_{n,n}) &= \sum_{i=0}^{\frac{n+1}{2}} |\Lambda_i| \\
&= \sum_{i=0}^{\frac{n-3}{2}} (n+1-4i)\binom{n+1}{2i} + \sum_{i=\frac{n-1}{2}}^{\frac{n+1}{2}} 4i-(n+1)\binom{n+1}{2i} \\
&= \sum_{i=0}^{\frac{n-3}{2}} (n+1-4i)\binom{n+1}{2i} + \sum_{i=0}^{\frac{n+1}{2}} (4i-(n+1))\binom{n+1}{2i} \\
&\quad - \sum_{i=0}^{\frac{n-3}{2}} \frac{n-1}{2} 4i-(n+1)\binom{n+1}{2i} \\
&= 2\sum_{i=0}^{\frac{n-3}{2}} (n+1-4i)\binom{n+1}{2i} + 4(2^{n-2}(n+1)) - 2^n(n) - 2^n \\
&= 2\sum_{i=0}^{\frac{n-3}{2}} (n+1-4i)\binom{n+1}{2i} + 2^n(n+1-n-1) \\
&= 2\sum_{i=0}^{\frac{n-3}{2}} (n+1-4i)\binom{n+1}{2i} + 0 \\
&= 2\Big[(n+1)\sum_{i=0}^{\frac{n-3}{2}} \binom{n+1}{2i} - 4\sum_{i=0}^{\frac{n-3}{2}} i\binom{n+1}{2i}\Big] \\
&= 2\Big[(n+1)(-\frac{n^2}{2} - \frac{n}{2} + 2^n - 1) - 4(\frac{n+1}{4}(-n^2+n+2^n-2))\Big] \\
&= 2\Big[\frac{(n-2)(n-1)(n+1)}{2}\Big] \\
\varepsilon(Q_{n,n}) &= (n-2)(n-1)(n+1)
\end{aligned}$$

Case ii If n is even then eigen values of $Q_{n,n-1}$ are $\Lambda_i = (-1)^i\Big[n-2i+1\Big]$ where $i = 0, 1, \ldots, \frac{n}{2}$ and multiplicity of the eigen values Λ_i is $\binom{n+1}{i}$. Hence the energy

$$\begin{aligned}
\varepsilon(Q_{n,n}) &= \sum_{i=0}^{\frac{n}{2}} |\Lambda_i| \\
&= \sum_{i=0}^{\frac{n}{2}} |(-1)^i (n-2i+1)| \\
&= \sum_{i=0}^{\frac{n}{2}} (n-2i+1) \\
\varepsilon(Q_{n,n}) &= \frac{n+2}{2}\binom{n+1}{\frac{n}{2}+1}
\end{aligned}$$

□

Consider the following figure to know the flow of energy of enhanced hypercube $Q_{n,n}$

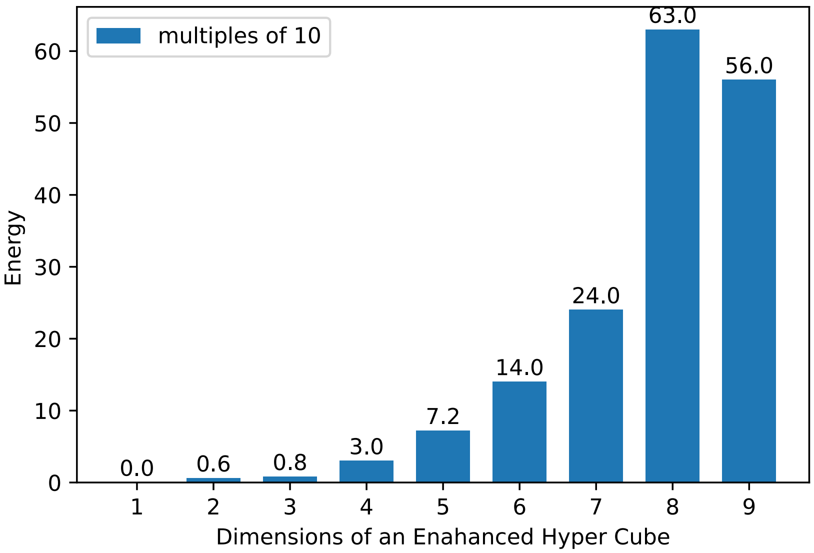

FIGURE 1. $\varepsilon(Q_{n,n})$

By lemma 2.1 proves the following theorem.

Theorem 3.2. *Let n be non zero positive integer and $Q_{n,n}$ is enhanced hypercube. Then Laplacian energy*

$$L\varepsilon(Q_{n,n}) = \begin{cases} (n-2)(n-1)(n+1), & \textit{If } n \textit{ is odd} \\ \frac{n+2}{2}\binom{n+1}{\frac{n}{2}+1}, & \textit{If } n \textit{ is even.} \end{cases}$$

Let $Q_{n,n}$ be enhanced hypercube then, the following algorithm to find the distance matrix of $Q_{n,n}$. From distance matrix we can find the distance spectra by Matlab tool.

Algorithm 3 Distance matrix of $Q_{n,n}$

```
$n \leftarrow$ Order of $Q_{n,n}$
$D \leftarrow zeros(n)$;
for $i \leftarrow 1$ to $n$ do
    for $j \leftarrow 1$ to $n$ do
        if $i$ and $j$ differ exactly one and $n$ place then
            $a(i,j) \leftarrow d_{ij}$;        ▷ This is distance matrix of $Q_{n,n}$
        end if
    end for
end for
```

Theorem 3.3. *Let n be non zero positive integer and $Q_{n,n}$ is enhanced hypercube. The distance energy*

$$\varepsilon_D(Q_{n,n}) = 2(n2^n - 2^{n-1} - n^2 + 1)$$

Proof. Let n be non zero positive integer, then distance eigen values of $Q_{n,n}$ are $n(2^{n-1}-1)+1, n-1, 1-n, 2^{n-1}-n+1$ with multiplicity $1, 2^{n-1}-n, 2^{n-1}-1, n$ respectively. Hence the distance energy is given by

$$\begin{aligned}
\varepsilon_D(Q_{n,n}) &= \sum |\Lambda_i| \\
&= |n(2^{n-1}-1)+1| + (2^{n-1}-n)|n-1| + (2^{n-1}-1)|1-n| \\
&\quad +n|2^{n-1}-n+1| \\
&= n(2^{n-1}-1)+1+(n-1)(2^{n-1}-n)+(n-1)(2^{n-1}-1) \\
&\quad +n(n-1)+n2^{n-1} \\
&= (2^n-1)(2n-1)-2n(n-1)+1 \\
\varepsilon_D(Q_{n,n}) &= 2(n2^n - 2^{n-1} - n^2 + 1)
\end{aligned}$$

□

Consider the following figure to know the flow of distance energy of enhanced hypercube $Q_{n,n}$

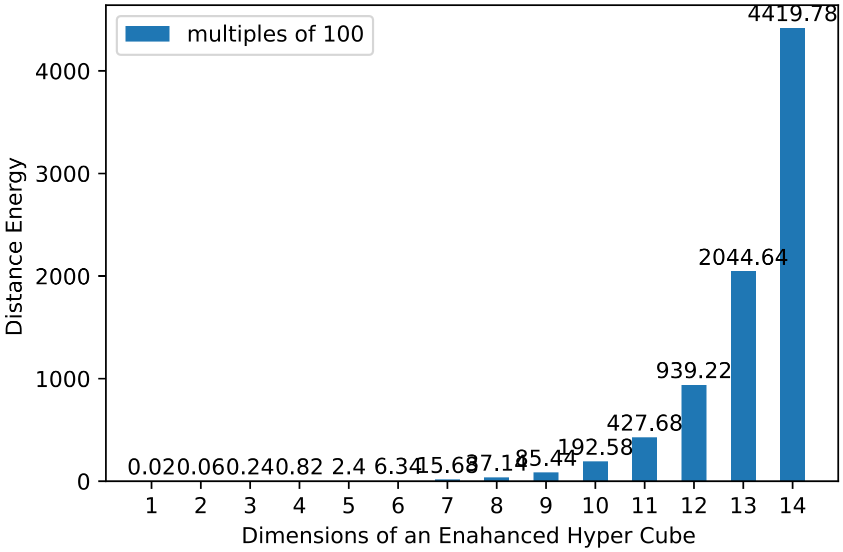

FIGURE 2. $\varepsilon_D(Q_{n,n})$

Let $Q_{n,n}$ be enhanced hypercube then, the following algorithm to find the randic matrix of $Q_{n,n}$. From randic matrix we can find the randic spectra by Matlab tool.

Algorithm 4 Randic matrix of $Q_{n,n}$

```
$n \leftarrow$ Order of $Q_{n,n}$
$R \leftarrow zeros(n)$;
for $i \leftarrow 1$ to $n$ do
    for $j \leftarrow 1$ to $n$ do
        if $i$ and $j$ differ exactly one and $n$ place then
            $a(i,j) \leftarrow \frac{1}{d_i d_j}$;          ▷ This is randic matrix of $Q_{n,n}$
        end if
    end for
end for
```

Theorem 3.4. *Let n be non zero positive even integer and $Q_{n,n}$ is enhanced hypercube. The randic energy*

$$RE(Q_{n,n}) = 0$$

Proof. Let n be non zero positive even integer, then Randic eigen values of $Q_{n,n}$ are $\Lambda_i = \frac{n+1-4i}{n+1}$ with multiplicity $\binom{n+1}{2i}$ respectively, where $0 \leq i \leq \frac{n}{2}$. Hence the randic energy is given by

$$\begin{aligned} RE(Q_{n,n}) &= \sum |\Lambda_i| \\ &= \sum_{i=0}^{\frac{n}{2}} \frac{n+1-4i}{n+1} \binom{n+1}{2i} \\ RE(Q_{n,n}) &= 0 \end{aligned}$$

□

Observation: Let n be non zero positive integer and $Q_{n,n-1}$ is enhanced hypercube. The spectrum of $Q_{n,n-1}$ is given by the following, If n is odd then
$$\begin{pmatrix} n+1 & n-1 & n-3 & n-5 & . & . & . & n-1 & -n+1 \\ \tbinom{n}{n} & \tbinom{n}{n} & \tbinom{n}{n-2} & \tbinom{n}{n-2} & . & . & . & \tbinom{n}{1} & \tbinom{n}{1} \end{pmatrix}$$

If n is even, $\begin{pmatrix} n+1 & n-1 & n-3 & n-5 & . & . & . & -(n-1) & -(n+1) \\ \tbinom{n}{n} & \tbinom{n}{n} & \tbinom{n}{n-2} & \tbinom{n}{n-2} & . & . & . & \tbinom{n}{0} & \tbinom{n}{0} \end{pmatrix}$

Let n be non zero positive integer and $Q_{n,n-1}$ is enhanced hypercube. The Laplacian spectrum of $Q_{n,n-1}$ is given by the following,

If n is odd then $\begin{pmatrix} 0 & 2 & 4 & 6 & . & . & . & 2n-2 & 2n \\ \tbinom{n}{n} & \tbinom{n}{n} & \tbinom{n}{n-2} & \tbinom{n}{n-2} & . & . & . & \tbinom{n}{1} & \tbinom{n}{1} \end{pmatrix}$

If n is even, $\begin{pmatrix} 0 & 2 & 4 & 6 & . & . & . & 2n & 2n+2 \\ \tbinom{n}{n} & \tbinom{n}{n} & \tbinom{n}{n-2} & \tbinom{n}{n-2} & . & . & . & \tbinom{n}{0} & \tbinom{n}{0} \end{pmatrix}$

Theorem 3.5. *Let n be non zero positive integer and $Q_{n,n-1}$ is enhanced hypercube. Then distance energy*

$$\varepsilon_D(Q_{n,n-1}) = (4n-2)2^{n-2}$$

Proof. Let n be non zero positive integer, then distance eigen values of Q_{n,n_1} are $(2n-1)(2^{n-2}), 0, -2^{n-2}, -2^{n-1}$ with multiplicity $1, 2^n-n-2, 3, n-2$ respectively. Hence the distance energy is given by

$$\begin{aligned} \varepsilon_D(Q_{n,n-1}) &= \sum |\Lambda_i| \\ &= |(2n-1)(2^{n-2})| + (2^n-n-2)|0| + 3|-2^{n-2}| + (n-2)|-2^{n-1}| \\ &= (2n-1)(2^{n-2}) + 0 + 3(2^{n-2}) + (n-2)2^{n-1} \\ &= 2(n+1)2^{n-2} + (n-2)2^{n-1} \\ \varepsilon_D(Q_{n,n-1}) &= (4n-2)2^{n-2} \end{aligned}$$

□

Consider the following figure to know the flow of distance energy of enhanced hypercube $Q_{n,n-1}$

4. conclusion

In this article, we mainly focused on to find the energy, Laplacian energy, distance energy and Randic energy of enhanced hypercube. Consider the following

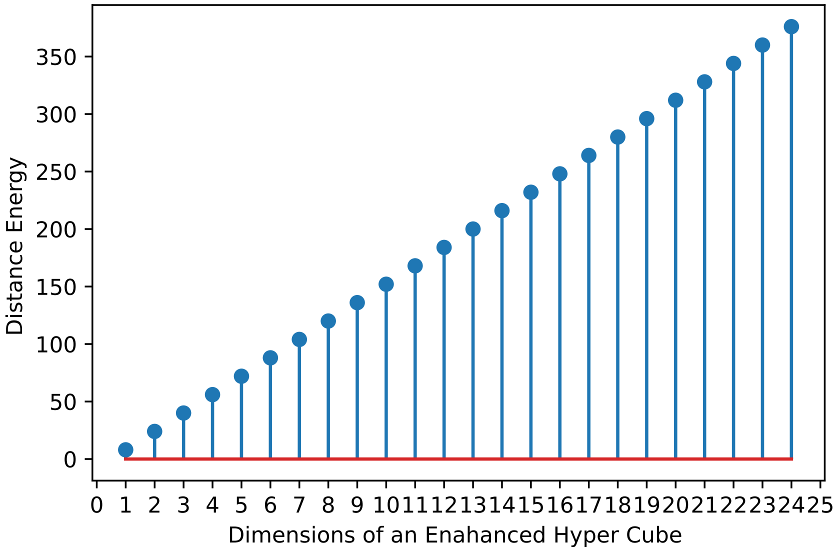

FIGURE 3. $\varepsilon_D(Q_{n,n-1})$

table1 and figure to analyse the variety of energy of enhanced hypercube $Q_{n,k}$, where k is any positive integer less than n.

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Dimension	$\varepsilon(Q_{n,n})$	$\varepsilon_D(Q_{n,n})$	$\varepsilon_D(Q_{n,n-1})$
1	0	2	8
2	6	6	24
3	8	24	40
4	30	82	56
5	72	240	72
6	140	634	88
7	240	1568	104
8	630	3714	120
9	560	8544	136
10	2772	19258	152
11	1080	42768	168
12	12012	93922	184
13	1848	204464	200
14	51480	441978	216
15	2912	949824	232
16	218790	2031106	248
17	4320	4324800	264
18	923780	9174394	280
19	6120	19397936	296
20	3879876	40893666	312

TABLE 1

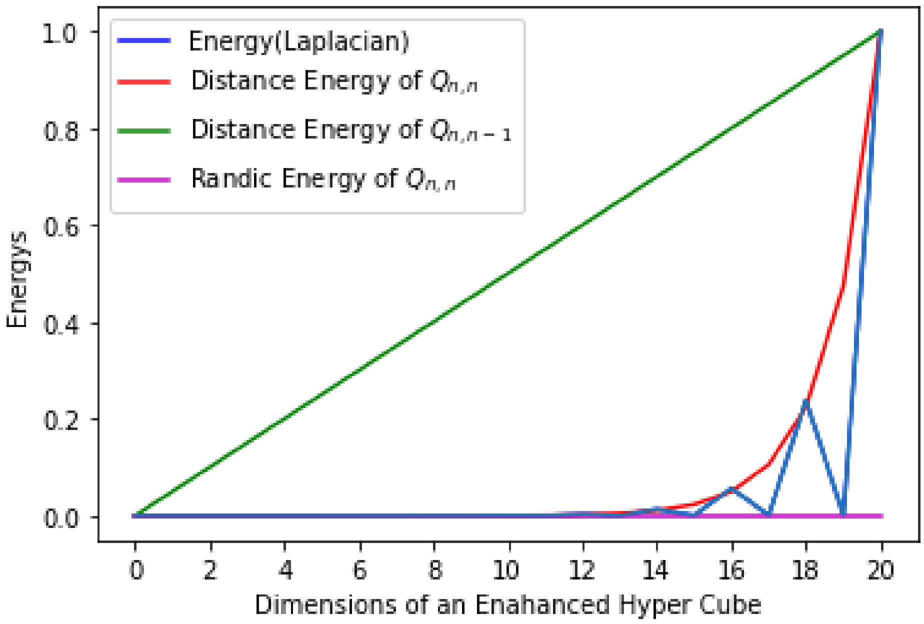

FIGURE 4

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