Global and Stochastic Analysis **DOI: 10.64837/GSA.13.2.2**
Vol. 13 No. 2 (2026)

Received: 05th December 2025 *Revised: 10th February 2026* *Accepted: 12th March 2026*

A GLOBAL SOLUTION FOR MULTI-OBJECTIVE LINEAR PROGRAMMING MODEL WITH FUZZY RANDOM VARIABLES

GAUTAM BENIWAL AND MOHAMMAD RIZWANULLAH*

Abstract. The Multi-Objective Linear Programming (MOLP) framework is widely used to address decision-making problems involving multiple conflicting objectives. In real-world applications, however, decision-makers often face hybrid uncertainty arising from both fuzziness and randomness in model parameters. Motivated by this practical challenge, present paper develops a novel modelling and solution framework for MOLP problems in which the coefficients of objective functions and constraints are represented as fuzzy random variables (FRVs). To transform the resulting fuzzy stochastic model into a tractable mathematical programming form, we propose a new approach that combines mean value of FRVs with chance-constrained programming. The resulting nonlinear model is further converted into a linear programming formulation using a piecewise linear approximation (PLA) technique. In addition, the Er-expected value model is employed as a benchmark to compare the obtained solutions. A numerical example is provided to demonstrate the applicability and effectiveness of the proposed approach, showing that incorporating variance and probability distribution information leads to more reliable and realistic solutions for multi-objective decision-making under uncertainty.

1. Introduction

Multi-Objective Programming (MOP) has emerged as an important optimization framework for solving real-world decision-making problems involving multiple, often conflicting, objectives. In many practical applications such as production planning, logistics, transportation, and resource allocation, decision-makers must simultaneously optimize economic, quality, and performance-related goals while operating in uncertain environments. This uncertainty is frequently hybrid in nature, arising from both randomness due to incomplete probabilistic information and fuzziness due to vagueness or imprecision in human judgment. Therefore, developing effective multi-objective optimization models capable of handling hybrid uncertainty has become increasingly important.

The study by Healy on mixed integer planning cases is considered a basic example of MOP. This formulation is referred to as MOP when multi-objective parameters are used in mathematical programming, indicating that several alternatives

Key words and phrases. Multi-Objective Programming, Mean value of fuzzy random variables, Piecewise Linear Approximation, Er-expected value model, Fuzzy multi-objective linear programming (FMOLP).

may exist while the objective function is optimized through an appropriate selection mechanism. In such situations, multiple decision-making issues arise, and researchers have increasingly focused on multi-objective parameters because of their growing practical relevance. MOP can be extremely useful for real-world decision-making. For example, depending on consumer responsiveness for the same commodities, different customers may be charged different delivery fees by a logistics and distribution company like FedEx (Stochastic Programming, n.d.).

Over the years, Multi-Objective Programming has received significant attention from many researchers. Much of the early research focused on the Transportation Problem (TP) and Multi-Objective Goal Programming (MOGP). For instance, Chang (2007) suggested a novel approach for resolving MOGP problems. Expanding on Liao's basic concept, Kwakernaak (1978) presented a goal programming model with multi-objective coefficients for decision variables. Pradhan and Biswal (2017) considered multi-choice parameters as random variables in a linear programming problem (LPP), extending the classical MOP framework. Similarly, Maiti and Roy (2016) proposed a bi-level stochastic programming model with multiple-choice constraints, where constraint parameters follow normal distributions and objective function cost coefficients are multi-choice parameters. Researchers have also shown particular interest in multi-objective supply and demand factors in transportation problems under uncertain environments involving fuzzy and random elements.

To address uncertainty, two major modeling approaches have been widely adopted: stochastic programming and fuzzy programming. When probabilistic characteristics of uncertain parameters are available, stochastic programming approaches such as those developed by Kall and Wallace (1994) and Tapiero (1994) can be used. Multi-objective stochastic linear programming models are suitable when multiple objective functions exist. Luhandjula (1989) and Teghem et al. (1986) recommended collaborative approaches to address challenges involving multi-objective stochastic linear programming, while Stancu-Minasian (Kall and Wallace (1994), Slowinski and Teghem (1990) applied the least probability strategy. On the other hand, when decision-making involves inherent ambiguity rather than purely probabilistic uncertainty, fuzzy optimization becomes an effective tool for modeling imprecision (Chakraborty et al. (1994a), Liu (2000), Liu and Iwamura (2001). These methods are particularly useful when deterministic models fail to represent uncertainty accurately (Khezri and Khodayifar (2023), Nasseri and Bavandi (2020), Luhandjula (1989).

As a result, fuzzy mathematical programming gained considerable attention (Algera and Skolka (1982), Słowiński (1986), Lai and Hwang (1992). This approach allows decision-makers to represent ambiguity within constraints or objective functions using fuzzy sets and membership functions (Sakawa (2013), M. Bùi (2021), Wang et al. (2023). Initiated by Zimmermann (1978), Fuzzy Multi-Objective Linear Programming (FMOLP) developed rapidly and has been widely used to address real-world problems. However, many real systems involve both fuzziness and randomness simultaneously. To model such hybrid uncertainty, Kwakernaak (1978) introduced the concept of Fuzzy Random Variables (FRVs), which was later extended by Puri and Ralescu ("Fuzzy Random Variables," 1993). Wang et al. (2023) further investigated stochastic and fuzzy approaches in mathematical programming involving FRVs, where a fuzzy number represents the realization of a random process. Several studies have extended multi-objective programming to fuzzy-random environments. A computer approach for multi-objective linear programming problems including normal random variables and fuzzy coefficients was proposed by Chakraborty et al. (1994). An interactive approach to the 0–1 fuzzy random multi-objective programming problem was created by Katagiri et al. (2004). Eshghi and Nematian (2008) utilized the Er-expected value of FRVs in classes of fuzzy random mathematical programming models, including the fuzzy random quadratic minimal spanning tree problem. Nematian (2011) and Sakawa (2013) applied fuzzy stochastic optimization in resource-constrained project scheduling and simultaneous optimization problems.

Recently, significant advances have been reported in multicommodity network flow (MCNF) and uncertainty optimization. Zhang and Boyd (2025) developed a GPU-based MCNF solver to improve large-scale computational efficiency, while Chimani and Ilsen (2025) introduced a traffic-oblivious MCNF network design model for robust planning. Reliability-based modeling has progressed through MCNF optimization under edge-failure uncertainty (Klyuchikov et al., 2025), and parallel flow optimization techniques have enhanced scalability (Grunau et al., 2025). Stochastic optimization approaches, such as the genetic algorithm for MCNF with unreliable nodes (Beniwal, 2025) and deep multi-agent stochastic models (Alasbali, 2025), further demonstrate growing interest in robust and hybrid uncertainty modeling. Foundational fuzzy-stochastic MCNF modeling continues to rely on hybrid frameworks (Bavandi and Nasseri, 2022; Salimifard and Bigharaz, 2022). Collectively, recent research trends emphasize robust, stochastic, fuzzy, and computationally efficient optimization systems.

A comparative summary of representative studies on multi-objective optimization under uncertainty is presented in Table 1. The table highlights the type of uncertainty considered, modeling framework, solution methodology, and major limitations of existing approaches. It shows that most existing studies address either fuzzy or stochastic uncertainty independently and frequently rely on expected value–based formulations. In particular, Er-expected value models often ignore dispersion and variance effects, limiting risk awareness in decision-making.

Table 1. Comparison of Existing Literature and Proposed Work

Author(s)	**Uncertainty Type**	**Model Type**	**Solution Approach**	**Key Limitation**
Zimmermann (1978)	Fuzzy	FMOLP	Fuzzy programming	No randomness considered
Kwakernaak (1978)	Fuzzy Random	FRV models	Analytical methods	Limited to simple structures
Chakraborty et al. (1994)	Fuzzy + Stochastic	MOLP	Interactive methods	No variance consideration
Eshghi & Nematian (2008)	Fuzzy Random	FRMOLP	Er-expected value	Ignores dispersion effects
Proposed work	Fuzzy Random	FRMOLP	Mean & CCP + PLA	Improved reliability

To address these limitations, the present study simultaneously incorporates fuzziness and randomness using fuzzy random variables and proposes a Mean & Chance-Constrained (Mean & CCP) framework combined with Piecewise Linear Approximation (PLA). Unlike traditional expected value approaches, the proposed model accounts for expected performance, variability, and predefined confidence levels within a unified structure.

Main contributions of this study can be summarized as follows. A general framework for Multi-Objective Linear Programming with fuzzy random parameters is first developed, where decision variables are crisp and uncertainty appears only in coefficients. Second, a novel Mean & CCP transformation is introduced to convert the fuzzy stochastic model into a tractable deterministic equivalent while preserving variance information. Third, a Piecewise Linear Approximation method is integrated to transform the resulting nonlinear model into a solvable mixed integer programming formulation. Fourth, a comparative numerical analysis demonstrates improved reliability and robustness relative to the classical Er-expected value model. This study investigates multi-objective linear programming problems with fuzzy random parameters and clear decision variables. The proposed approach employs fuzzy stochastic optimization and integer programming to develop a novel modeling and solution framework. Initially, the fuzzy stochastic programming model is transformed into a mathematical programming formulation using Chance-Constrained Programming (CCP) and the mean value of FRVs. Subsequently, the nonlinear structure is converted into a linear programming model through the Piecewise Linear Approximation (PLA) technique. The proposed methodology provides a flexible and reliable decision-support tool applicable to production planning, supply chain management, transportation systems, resource allocation, and other uncertain multi-objective environments.

Although several studies have investigated fuzzy, stochastic, or fuzzy-random multi-objective programming problems, most existing approaches rely primarily on expected value-based transformations, which ignore dispersion and risk effects associated with hybrid uncertainty. The originality of the present work lies in the development of a unified modelling framework that integrates fuzzy random variables with a Mean & Chance-Constrained Programming approach, explicitly incorporating variance and confidence levels into the optimization structure. Furthermore, the proposed integration of chance-constrained transformation with Piecewise Linear Approximation provides a systematic and computationally tractable solution mechanism for nonlinear FRMOLP models. Therefore, the contribution of this study is methodological and structural rather than purely numerical, and the numerical example serves only to demonstrate the applicability of the proposed framework.

2. Preliminaries

To clarify the key ideas involved in the discussion, we first go over several definitions and notations in this section. Assuming that, Pos is the possibility specified by fuzzy sets, $P(\circledast)$ is power set of $\circledast$, and $\circledast$ is the universe. we suppose that $(\phi\circledast, P)$ is probability area and $(\circledast, P(\circledast), Pos)$ is possibility area. Additionally, $F_c(\mathbb{R})$ stands for overall fuzzy numbers in normalized form with α-level sets that are convex subsets of region (Luhandjula 1989).

2.1 Definition. A fuzzy set on $\mathbb{R}$ using the membership function of the real line $\mu_R : \mathbb{R} \to [0, 1]$ is identified fuzzy number if its support set is compact, convex, normal, and upper semi-continuous.

A unique fuzzy number that is often used is the LR fuzzy number (Ercsey and Kovács (2024), Farrugia et al. (2023). From the extension concept, we shall compute sums, products, etc. on fuzzy integers using conventional fuzzy arithmetic ("Fuzzy Numbers," 1993). The real value of a FRV is a fuzzy number, it is Borel measurable function and a random variable (Luhandjula 1989). It is widely employed in ambiguous classifications.

In this study, a fuzzy random variable is assumed to represent hybrid uncertainty arising from both randomness and fuzziness, where the random component follows a known probability distribution and its realizations are expressed as fuzzy numbers. Each fuzzy random variable is assumed to be measurable and characterized through its α -cut representation, which forms a random interval for every $\in (0, 1]$. The expected value, mean value, and variance of fuzzy random variables are well-defined and are used to transform the original fuzzy stochastic model into a deterministic equivalent. These assumptions ensure mathematical tractability while preserving the essential uncertainty characteristics of the model.

Lemma 1

If X is an FRV, therefore, α- cut

$$X_\alpha(\delta) = \{\tau \in R \mid \mu_{R(\delta)}(\tau) \geq \alpha\} = [X^-_\alpha(\delta), X^+_\alpha(\delta)] \tag{1}$$

is a random interval for every $\alpha \in (0, 1]$, Having a fuzzy number as its real value, a FRV is a Borel measurable function and a random variable. A random interval is the α-cut of a FRV, as shown by the lemma that follows, which was taken from (Nematian, 2011).

2.2 Definition. The scaler expected value is defined as observes if χ is an FRV

$$E_r(\chi) = \frac{1}{2}\int_0^1 \{\chi^-_\alpha(\delta)\ \chi^+_\alpha(\delta)\}\, d\alpha \tag{2}$$

Wherever $E(\chi^-_\alpha)$ and $E(\chi^+_\alpha)$ are expected value of χ^-_α and χ^+_α correspondingly.

Remark 1: The expected value of χ will be a fuzzy number if χ is an FRV.

Remark 2: For any $\delta \in \phi$, $\chi(\delta)$ is a fuzzy number if χ is a fuzzy random.

2.3 Definition. If χ is a fuzzy random, then δ is a fuzzy number for any $\delta \in \phi$. The mean value of fuzzy number $\chi(\delta)$, represented by $M(\chi(\delta))$, is defined as follows:

$$M(\chi(\delta)) = \frac{1}{2}\int_0^1 \{\chi^-_\alpha(\delta)\ \chi^+_\alpha(\delta)\}\, d\alpha, \qquad \forall \delta \in \phi \tag{3}$$

A random variable is $M(\chi)$.

Corollary 1

Assume that χ is a FRV and that $E(M(\chi)) = E_r(\chi)$.

Verification: The variable $M(\chi)$ is arbitrary (3). Therefore $E(M(\chi)) =$, and 1 is the same as $E_r(\chi)$.

2.4 Definition (Scalar Variance of FRV). Assume that χ is fuzzy random variable (FRV). Scalar variance of χ is defined as:

$$\mathrm{Var}(\chi) = E\left[(M(\chi) - E[M(\chi)])^2\right] \tag{4}$$

Where $E[\cdot]$ is the mathematical expectation operator with regard to the probability distribution of the random component, and $\mathrm{Var}(\chi)$ is the variance of the fuzzy random variable, $M(\chi)$ represents the mean value of the fuzzy number acceptance of χ.

If the random component of χ follows a normal distribution $N(\mu, \sigma^2)$, where μ is mean of normal distribution and σ^2 variance of normal distribution, then:

$$\mathrm{Var}(\chi) = \sigma^2$$

Corollary 2

The mean value operator $M(\cdot)$ is linear with respect to fuzzy numbers.

For each realization $\rho \in \mathbb{R}$, we have:

$$M(A + \rho B) = M(A + \rho B)$$

if A and B are FRV and $\rho \in \mathbb{R}$.

Proof: Since ρ is a deterministic scalar,

$$(M(A + \rho B))_\alpha = A_\alpha + \rho B_\alpha$$

then by Definition (3), the mean value of a fuzzy random variable is given by

$$M(A + \rho B) = \frac{1}{2}\int_0^1 \{(A + \rho B)_\alpha^- + (A + \rho B)_\alpha^+\} d\alpha$$

$$M(A + \rho B) = \frac{1}{2}\int_0^1 \{(A)_\alpha^- + (A)_\alpha^+\} d\alpha + \rho\frac{1}{2}\int_0^1 \{(A)_\alpha^- + (A)_\alpha^+\} d\alpha$$

$$M(A + \rho B) = M(A + \rho B)$$

2.5 Definition. Let A and B be FRVs. Then the relation $\cong$ and $\tilde{\lesssim}$ are defend correspondingly following is:

(1) $A \cong B$ iff $M(A) = M(B)$
(2) $A \tilde{\lesssim} B$ iff $M(A) \leq M(B)$

3. Problem Description

This part of the article introduces and resolves FRMOLP and FMOLP. A new technique for FRMOLP called Mean & CCP Model and the Er-expected value model (Eshghi and Nematian (2008)) will be employed.

3.1 Fuzzy Multi-Objective Linear Programming (FMOLP). FMOLP's objectives are according to

$$\max \left[\tilde{C}_1^t X, \tilde{C}_2^t X, \tilde{C}_3^t X, \ldots, \tilde{C}_r^t X\right] \quad (5)$$

$$\text{s.t.} \quad \tilde{a}_i X \leq \tilde{b}_i, \qquad i = 1, 2, \ldots, m$$

$$X \geq 0$$

in which each variable is fuzzy variable. Additionally, decisive factor the vector is $X = (X_1, X_2, X_3, \ldots, X_n)$. A FRV's mean value is a real integer if it is degenerated to a fuzzy variable. The M-FMOLP symbol denotes the mean value model of FMOLP and is specified as:

$$\max \left[M(\tilde{C}_1^t X), M(\tilde{C}_2^t X), M(\tilde{C}_3^t X), \ldots, M(\tilde{C}_r^t X)\right] \quad (6)$$

$$\text{s.t.} \quad M(\tilde{a}_i X) \leq M(\tilde{b}_i), \qquad \forall i = 1, 2, \ldots, m$$

$$X \geq 0$$

This type, whose input constraints are all real and will produce solutions with the best expected return under the set restrictions. As a result, we may create the method below to resolve FMOLP. In the FMOLP model, all uncertain coefficients of the objective functions and constraints are treated as fuzzy numbers. Each fuzzy parameter was measured using LR-type fuzzy membership functions generated from expert judgement and literature work. α-cut values of each fuzzy number were calculated, and mean value of the fuzzy number was obtained using Definition (2.1). These mean values converted fuzzy coefficients into crisp equivalents for solving FMOLP through Zimmermann's max–min approach.

Algorithm: Expected Value Algorithm

Step 1. Using information from experts or decision-makers, define the fuzzy parameters of the FMOLP and calculate their mean values.

Step 2. Using Corollary (2) and the mean value of the fuzzy variables, modify FMOLP (5) to M-FMOLP (6).

Step 3. Resolve the multi-objective optimization problem of M-FMOLP using the Zimmermann approach (Zimmermann (1978)). The found best result is known as the M-optimal solution to the original issue.

3.2 Fuzzy Random Multi-Objective Linear Programming (FRMOLP). Taking into consideration the MOLP with FRVs described below:

$$\max \left[\bar{\tilde{C}}_1^t \chi, \bar{\tilde{C}}_2^t \chi, \bar{\tilde{C}}_3^t \chi, \ldots, \bar{\tilde{C}}_r^t \chi\right] \quad (7)$$

$$\text{s.t.} \quad \bar{\tilde{a}}_i \chi \leq \bar{\tilde{b}}_i \qquad \forall i = 1, 2, \ldots, m$$

$$\chi \geq 0$$

Where

$$\bar{\tilde{c}}_i = (\bar{\tilde{c}}_{k1}, \bar{\tilde{c}}_{k2}, \bar{\tilde{c}}_{k3}, \ldots, \bar{\tilde{c}}_{kn}), \quad k = 1, 2, 3, \ldots, l,$$
$$\bar{\tilde{A}} = [\bar{\tilde{a}}_{ij}]_{m\times n}, \quad \bar{\tilde{b}} = (\bar{\tilde{b}}_1, \bar{\tilde{b}}_2, \bar{\tilde{b}}_3, \ldots, \bar{\tilde{b}}_m)^t$$

are representative of FRVs involved in the constraints and the objective functions, correspondingly. The formulation presented in this section extends the FR-MOLP framework by allowing model parameters to be represented as fuzzy random variables (FRVs), which capture both probabilistic uncertainty and fuzziness simultaneously. This hybrid representation is more suitable for real-world decision-making problems where uncertainty cannot be described purely as stochastic or purely as fuzzy. The solution of such problems requires specialized transformation techniques, which are discussed in the following subsections.

3.3 Er-expected Value Model. According to Er-FRMOLP, the Er-expected value model of FRMOLP is defined as follows

$$\max \left[Er(\bar{\tilde{C}}_1^t \chi), Er(\bar{\tilde{C}}_2^t \chi), Er(\bar{\tilde{C}}_3^t \chi), \ldots, Er(\bar{\tilde{C}}_r^t \chi) \right] \tag{8}$$

$$\text{s.t.} \quad Er(\bar{\tilde{a}}_i \chi) \leq Er(\bar{\tilde{b}}_i) \qquad \forall i = 1, 2, \ldots, m$$

$$\chi \geq 0$$

In the Er-Expected value model, all FRV-based parameters were measured by computing their expected value using the operator $Er(\cdot)$ defined in Equation (8). The fuzzy random constraint and objective coefficients were converted into deterministic scalar values by applying $Er(\bar{\tilde{a}}_{ij})$, which includes integration over the probability distribution and membership range. This eliminates fuzzy dispersion during measurement and produces values suitable for solving FRMOLP as a crisp MOLP.

Er-expected Value Algorithm

The fuzzy random multi-objective linear programming problem is converted into a deterministic version using the Er-expected value approach and the expected value operator. The solution procedure is explained as follows:

Step 1. Using input from decision-makers or experts, describe the fuzzy random parameters of the FRMOLP and verify their Er-expected values.

Step 2. Applying Corollary (2) and the Er-expected value of FRVs, convert FRMOLP (7) to Er-FRMOLP (8).

Step 3. Using Zimmermann's technique to solve multi-objective problems for optimization of Er-FRMOLP. The optimal solution that was discovered is known as the "Er-optimal solution" to the initial problem.

3.4 Mean & CCP Model. The CCP technique and the mean value of the FRVs are employed in this model. That is the next definition applies to M-FRMOLP, the symbol for the mean value of Er-FRMOLP (8).

$$\max \left[M(\bar{\tilde{C}}_1^t \chi), M(\bar{\tilde{C}}_2^t \chi), M(\bar{\tilde{C}}_3^t \chi), \ldots, M(\bar{\tilde{C}}_r^t \chi) \right] \tag{9}$$

$$\text{s.t.} \quad M(\bar{\tilde{a}}_i\chi) \leq M(\bar{\tilde{b}}_i) \qquad \forall i = 1, 2, \ldots, m$$

$$\chi \geq 0$$

We have now the following stochastic linear programming model:

$$\max[\bar{z}_1, \bar{z}_2, \bar{z}_3, \ldots, \bar{z}_r] = [\bar{C}_1^t x, \bar{C}_2^t x, \bar{C}_3^t x, \ldots, \bar{C}_r^t x] \tag{10}$$

$$\text{s.t.} \quad \bar{a}_i x \leq \bar{b}_i \qquad i = 1, 2, \ldots, m$$

$$\chi \geq 0$$

Where

$$\bar{Z}_k = M(\bar{\tilde{Z}}_k), \quad \bar{c}_k = (\bar{c}_{k1}, \bar{c}_{k2}, \bar{c}_{k3}, \ldots, \bar{c}_{kn}) = M(\bar{\tilde{C}}_k) \quad \forall k = 1, 2, \ldots, r$$

$$\bar{A} = [\bar{\tilde{a}}_{ij}]_{m\times n} = M(\bar{\tilde{A}}), \quad \bar{b} = (\bar{b}_1, \bar{b}_2, \bar{b}_3, \ldots, \bar{b}_m)^t = M(\bar{\tilde{b}})$$

are unpredictable variables.

The chance-constrained multi-objective methodology of (Khezri & Khodayifar, 2023) allows M-FRMOLP (9) to be expressed as M&CCP-FRMOLP.

$$\max[Z_1, Z_2, Z_3, \ldots, Z_r] \tag{11}$$

$$\text{s.t.} \quad \Pr\{\bar{C}_k^t x \geq Z_k\} \geq \alpha \qquad \forall k = 1, 2, \ldots, r$$

$$\Pr\{\bar{a}_i x \leq \bar{b}_i\} \geq \beta \qquad \forall i = 1, 2, \ldots, m$$

$$\chi \geq 0$$

where α and β indicate the decision-maker's predetermined confidence levels. The random variables in this model might have many probability distributions. As a result, Mean & CCP Model of FRMOLP will produce several models.

A suitable model will be chosen and solved using the Zimmermann approach based on probability properties and professional recommendations.

Algorithm (M&CCP Algorithm)

The Mean & Chance-Constrained Programming (Mean & CCP) method explicitly provides for both fuzziness and randomness by combining chance-constrained programming with the mean value of FRV's. Solution procedure is described below:

Step 1. Using information from decision-makers or experts, define the fuzzy random parameters of the FRMOLP and calculate their average values.

Step 2. Apply Corollary (2) and the mean value of FRVs to convert Er-FRMOLP (8) to M-FRMOLP (9).

Step 3. Use the chance-constrained multi-objective programming technique to transform M-FRMOLP (10) into M&CCP-FRMOLP (11).

Step 4. Use the Zimmermann method to solve the multi-objective optimization problem M&CCP-FRMOLP (11).

Comparison of the Er-Expected Value Model and the Mean & Chance-Constrained Model Er-expected value model transforms fuzzy random parameters into deterministic equivalents by focusing on their expected values, thereby emphasizing the central tendency of uncertainty, and yielding a relatively simple and computationally efficient model. However, this approach neglects the dispersion and risk associated with parameter variability. In contrast, the Mean & Chance-Constrained model incorporates both the mean value and probabilistic constraints, explicitly accounting for variance and confidence levels. As a result, the Mean & CCP model provides more reliable and risk-aware solutions, although at the cost of increased computational complexity.

3.5 FRMOLP Model with Normal Distribution. To derive a deterministic formulation, the random components of the fuzzy random variables are assumed to follow a normal probability distribution with known mean and variance. This assumption leads to a nonlinear programming model due to the presence of probabilistic constraints. To efficiently solve this nonlinear model, a piecewise linear approximation technique is employed which is discussed below.

(Model-1)

$$\max[Z_1, Z_2, Z_3, \ldots, Z_r] \tag{12}$$

$$\text{s.t.} \quad Z_k - E(\bar{C}_k^t)x \leq k_\alpha \sqrt{\sum_{j=1}^{n} Var(\bar{C}_{kj})x_j^2} \qquad \forall k = 1, 2, \ldots, r$$

$$E(\bar{a}_i)x - k_\beta \sqrt{\sum_{j=1}^{n} Var(\bar{a}_{ij})x_j^2 + Var(\bar{b}_i)} \leq E(\bar{b}_i) \qquad \forall i = 1, 2, \ldots, m$$

$$\chi \geq 0$$

$$\Pr\{U \sim N[0,1] \geq k_\alpha\} = \alpha,$$

Where k_α is a point. We solve Model-1, a nonlinear programming model, using the PLA approximate (M. N. Bùi (2022), Teghem et al. (1986)) and we get an approximation of the global optimal solution. Let $f(\chi_1, \chi_2, \chi_3, \ldots, \chi_n)$ be a nonlinear function including n variables. Here, f is separable function if

$$f(\chi_1, \chi_2, \chi_3, \ldots, \chi_n) = \sum_{j=1}^{n} f_j(x_j)$$

where $f_j(x_j), j = 1, 2, \ldots, n$ are the function of single variable.

If all nonlinear functions are capable of being split up into separate functions, a separable approach to programming that is nonlinear in nature. In Model-1, the random components of the fuzzy random variables were measured using the normal distribution assumption $N(\mu, \sigma^2)$. Mean μ and variance σ^2 were extracted from the numerical dataset and verified with statistical estimation rules. The probability of satisfying each constraint was measured using the standard normal cumulative distribution function applied to the mean-variance structure.

In the PLA technique, separable programming puts a significant part. Additionally, our model's nonlinear components can all be separated into separate functions. An innovative PLA approximate is used to generate a primitive linear programming model. A separable programming variant of FRMOLP Model with Normal Distribution is as:

(Model-2)

$$\max[Z_1, Z_2, Z_3, \dots, Z_r] \tag{13}$$

s.t.

$$Z_k - \sum_{j=1}^{n} C_{kj} x_j \le k_\alpha t_k \qquad \forall k = 1, 2, \dots, r$$

$$t_k^2 = \sum_{j=1}^{n} Var(\bar{C}_{kj}) x_j^2 \qquad \forall k = 1, 2, \dots, r$$

$$\sum_{j=1}^{n} a_{(i,j)} x_j - k_\beta q_i \le b_i \qquad \forall i = 1, 2, \dots, m$$

$$q_i^2 = \sum_{j=1}^{n} \left(Var(\bar{a}_{ij}) x_j^2 + Var(\bar{b}_i) \right) \qquad \forall i = 1, 2, \dots, m$$

$$t_k \ge 0 \qquad \forall k = 1, 2, \dots, r$$

$$q_i \ge 0 \qquad \forall i = 1, 2, \dots, m$$

$$x \ge 0$$

Where

$$C_k = (\bar{c}_{k1}, \bar{c}_{k2}, \bar{c}_{k3}, \dots, \bar{c}_{kn}) = E(\bar{C}_k), \qquad \forall k = 1, 2, \dots, r,$$

$$[a_{(i,j)}]_{m \times n} = E(\bar{A}), \qquad (\bar{b}_1, \bar{b}_2, \bar{b}_3, \dots, \bar{b}_m)^t = E(\bar{b}).$$

As mentioned in Table 2, interval and break points are now taken into consideration for each decision variable. In the piecewise linear approximation approach, nonlinear functions are approximated by linear segments over predefined intervals of the decision variables. The breakpoints are selected based on the predicted feasible ranges of the variables, which are obtained from problem structure, preliminary bounds, and expert judgment. Increasing the number of breakpoints improves approximation accuracy and reduces linearization error, while also increasing computational complexity. Therefore, a trade-off is maintained between solution accuracy and computational efficiency, and a moderate number of breakpoints is chosen to ensure reliable approximation of the nonlinear terms without significantly increasing the problem size.

TABLE 2. PLA programming breakpoints and newly discovered decision variables

Decision Variable	Interval	Break points		New decision variable
x_j	$[m_j, n_j]$	$x_{j1}, x_{j2}, \ldots, x_{js}$	$m_j = x_{j1} \leq x_{j2} \leq \cdots \leq x_{js} = n_j$	γ_{jl}
t_k	$[o_k, p_k]$	$y_{k1}, y_{k2}, \ldots, y_{ks}$	$o_k = y_{k1} \leq y_{k2} \leq \cdots \leq y_{ks} = p_k$	δ_{kl}
q_i	$[q_i, r_i]$	$z_{i1}, z_{i2}, \ldots, z_{is}$	$q_i = z_{j1} \leq z_{j2} \leq \cdots \leq z_{js} = r_i$	ε_{il}

The following transformation of separable programming variant of FRMOLP Model with Normal Distribution as model-2, a separable programming model, is possible by applying additional decision variables and the PLA programming method.

(Model-3)

$$\text{Max } [Z_1, Z_2, Z_3, \ldots, Z_r] \tag{14}$$

$$Z_k - \sum_{j=1}^{n}\sum_{l=1}^{s} O_{kj} x_{jl} \gamma_{jl} \leq k_\alpha \sum_{l=1}^{s} y_{kl}\delta_{kl} \qquad \forall k = 1, 2 \ldots r$$

$$\sum_{j=1}^{n}\sum_{l=1}^{s} m_{ij} x_{ij} \gamma_{ij} - k_\beta \sum_{l=1}^{s} z_{il}\epsilon_{il} \leq n_i \qquad \forall i = 1, 2 \ldots m$$

$$\sum_{i=1}^{s} y_{kl}^2 \delta_{kl} = \sum_{j=1}^{n}\sum_{l=1}^{s} \text{Var}(\bar{m}_{kj}) x_{jl}^2 \gamma_{jl} \qquad \forall k = 1, 2 \ldots r$$

$$\sum_{l=1}^{s} z_{il}^2 \epsilon_{il} = \sum_{i=1}^{n}\sum_{l=1}^{s} \text{Var}(\bar{m}_{ij}) x_{jl}^2 \gamma_{jl} + \text{Var}(\bar{n}_i) \qquad \forall i = 1, 2 \ldots m$$

$$\sum_{l=1}^{s} \gamma_{jl} = 1 \qquad \forall j = 1, 2 \ldots n$$

$$\sum_{l=1}^{s} \delta_{kl} = 1 \qquad \forall k = 1, 2 \ldots r$$

$$\sum_{l=1}^{s} \epsilon_{il} = 1 \qquad \forall i = 1, 2 \ldots m$$

$$(\gamma_{j1}, \gamma_{j2}, \ldots, \gamma_{js}) \in SOS2 \qquad \forall j = 1, 2 \ldots n$$

$$(\delta_{k1}, \delta_{k2}, \ldots, \delta_{ks}) \in SOS2 \qquad \forall k = 1, 2 \ldots r$$

$$(\epsilon_{i1}, \epsilon_{i2}, \ldots, \epsilon_{is}) \in SOS2 \qquad \forall i = 1, 2 \ldots m$$

$$0 \leq \gamma_{jl} \leq 1 \qquad \forall j = 1, 2 \ldots n,\ l = 1, 2 \ldots s$$

$$0 \leq \delta_{kl} \leq 1 \qquad \forall k = 1, 2 \ldots r,\ l = 1, 2 \ldots s$$

$$0 \leq \epsilon_{il} \leq 1 \qquad \forall i = 1, 2 \ldots m,\ l = 1, 2 \ldots s$$

Multi-objective Mixed Integer Programming (MIP) will be utilized to deal with the problem. Bellman with Zadeh's max-min operator will be employed in the Zimmermann fuzzy technique (Zimmermann (1978)) to solve the problem.

Let $z^{(k)}$ be upper bound of z_k, $k = 1, 2 \ldots r$, and $z^{(k)} - p^{(k)}$ be their initial value $k = 1, 2 \ldots r$, Model-3 can be changed into the following problem by applying the Bellman and Zadeh max-min operator and considering fuzzy objective function's membership function Max γ.

(Model-4)

$$\text{Max } \gamma \tag{15}$$

$$Z_k \geq z^{(k)} - (1-\gamma)p^{(k)} \qquad \forall k = 1, 2 \ldots r$$

$$Z_k - \sum_{j=1}^{n}\sum_{l=1}^{s} O_{kj} x_{jl} \gamma_{jl} \leq k_\alpha \sum_{l=1}^{s} y_{kl} \delta_{kl} \qquad \forall k = 1, 2 \ldots r$$

$$\sum_{j=1}^{n}\sum_{l=1}^{s} m_{ij} x_{ij} \gamma_{ij} - k_\beta \sum_{l=1}^{s} z_{il} \epsilon_{il} \leq n_i \qquad \forall i = 1, 2 \ldots m$$

$$\sum_{i=1}^{s} y_{kl}^2 \delta_{kl} = \sum_{j=1}^{n}\sum_{l=1}^{s} \text{Var}(\bar{m}_{kj}) x_{jl}^2 \gamma_{jl} \qquad \forall k = 1, 2 \ldots r$$

$$\sum_{l=1}^{s} z_{il}^2 \epsilon_{il} = \sum_{i=1}^{n}\sum_{l=1}^{s} \text{Var}(\bar{m}_{ij}) x_{jl}^2 \gamma_{jl} + \text{Var}(\bar{n}_i) \qquad \forall i = 1, 2 \ldots m$$

$$\sum_{l=1}^{s} \gamma_{jl} = 1 \qquad \forall j = 1, 2 \ldots n$$

$$\sum_{l=1}^{s} \delta_{kl} = 1 \qquad \forall k = 1, 2 \ldots r$$

$$\sum_{l=1}^{s} \epsilon_{il} = 1 \qquad \forall i = 1, 2 \ldots m$$

$$(\gamma_{j1}, \gamma_{j2}, \ldots, \gamma_{js}) \in SOS2 \qquad \forall j = 1, 2 \ldots n$$

$$(\delta_{k1}, \delta_{k2}, \ldots, \delta_{ks}) \in SOS2 \qquad \forall k = 1, 2 \ldots r$$

$$(\epsilon_{i1}, \epsilon_{i2}, \ldots, \epsilon_{is}) \in SOS2 \qquad \forall i = 1, 2 \ldots m$$

$$0 \leq \gamma_{jl} \leq 1 \qquad \forall j = 1, 2 \ldots n,\ l = 1, 2 \ldots s$$

$$0 \leq \delta_{kl} \leq 1 \qquad \forall k = 1, 2 \ldots r,\ l = 1, 2 \ldots s$$

$$0 \leq \epsilon_{il} \leq 1 \qquad \forall i = 1, 2 \ldots m,\ l = 1, 2 \ldots s$$

The problem mentioned above is a MIP problem that one of the MIP solvers can solve. Considering that $\gamma^*_{jl}, \delta^*_{kl}, \epsilon^*_{il},\ j = 1, 2 \ldots n,\ k = 1, 2 \ldots r,\ i = 1, 2 \ldots m,\ l = 1, 2 \ldots s$ are the most effective answer for the model-4, then $x^*_j(\text{PLA}) = \sum_{L=1}^{S} P_{jl}\gamma^*_{jl},\ j = 1, 2 \ldots n$, may find the PLA-optimal answer to the initial issue.

In the PLA approximation, nonlinear probability behavior in the constraints is measured by discretizing the range of each decision variable. Breakpoint values and intervals are measured using lower and upper bounds derived from feasibility analysis. Linear interpolation weights are then measured according to the separable programming conditions. Also, all breakpoint measurements are listed in Table 6. Using the Mean & CCP formulation, the nonlinear programming model is solved using the Piecewise Linear Approximation (PLA) technique. The following is a summary of the process.

Algorithm: PLA Algorithm

Step 1. Applying the idea of the normal probability distribution of random position of FRVs, transform M&CCP-FRMOLP to model-1.

Step 2. Apply the Piecewise Linear Approximation method to change model-1 into model-3.

Step 3. Transform model-3 into model-4 by using the Zimmermann approach (Max-min operator of Bellman and Zadeh). A mixed integer programming model is used to solve model-4.

Step 4. Apply one of the MIP solvers to use a mixed integer programming approach to solve model-4. Therefore, γ^*_{jl} is the best answer found using the PLA approach, and $x^*_j(\text{PLA}) = \sum_{L=1}^{S} P_{jl}\gamma^*_{jl},\ \forall j = 1, 2 \ldots n$, finds the best solution to the original problem.

Pseudocode Representation

Input: FRMOLP model with fuzzy random parameters

Output: Optimal decision variables

Begin

Define fuzzy random parameters and membership functions
Compute mean and variance of FRV's
Transform constraints using chance-constrained programming
Nonlinear terms should be approximated using piecewise linear approximation.
Construct equivalent deterministic MOLP model
Convert MOLP to single-objective LP using weighting method
Solve LP model using optimization solver
Obtain and analyze optimal solution

End

This segment explains the mathematical modelling of a multi-objective linear programming problem with undetermined parameters using FMOLP and FRMOLP. The uncertainty may take the form of probabilistic imprecision, probability uncertainty, or both. It can be used to fuzzy, random, and FRVs, respectively. Additionally, the Mean & CCP model and Er-expected value model as well as the Expected value model are employed for FRMOLP. Although the Mean & CCP model employs more complicated methodology, its outcomes are further trustworthy since it relies on probability district. Function of random element of fuzzy based random parameters rather than an expected value of those parameters.

4. Numerical Example

This section illustrates the proposed solution methodology step-by-step by applying the Expected Value, Er-Expected Value, and Mean & CCP algorithms, along with the Piecewise Linear Approximation technique, to a numerical example. The results obtained from different approaches are compared to demonstrate the effectiveness and reliability of the proposed method. The production planning problem with two objective functions and an uncertain framework will be established in this part. Maximizing the overall value of the one objective is to maximized production value, while the second aims to maximized quality performance rate. There are two resource limitations with this issue. Furthermore, this problem's parameters contain uncertain characteristics in both the arbitrary and the fuzzy aspects. This following problem taken from the Nematian (2011).

A supervisor wants to produce two new items, M and N, in a plant. Resources 1 and 2 make up his two main resources. For a batch of product M, the predicted resource utilization rates are as follows: generally,

Conversely, the rates of resource usage each batch of product N are about $\tilde{\bar{x}}_1$ for resource 1 and $\tilde{\bar{x}}_3$ for resource 2, also are following roughly $\tilde{\bar{x}}_2$ for resource 1 and around $\tilde{\bar{x}}_4$ for resource 2. Resources 1 and 2 are about $\tilde{\bar{y}}_1$ and $\tilde{\bar{y}}_2$ in terms of availability, and products M and N are profitable in terms of $\tilde{\bar{z}}_1$ and $\tilde{\bar{z}}_2$ respectively. Additionally, the $\tilde{\bar{z}}_3$ and $\tilde{\bar{z}}_4$ quality performance rats for products M and N.

TABLE 3. Parameter specifications represented as fuzzy random variables (FRVs) for the numerical production planning example

$\tilde{\bar{A}}$	$\tilde{\bar{z}}_1$	$\tilde{\bar{z}}_2$	$\tilde{\bar{z}}_3$	$\tilde{\bar{x}}_1$	$\tilde{\bar{x}}_2$	$\tilde{\bar{x}}_3$	$\tilde{\bar{x}}_4$	$\tilde{\bar{y}}_1$	$\tilde{\bar{y}}_2$
$r \sim A[\rho, \sigma^2]$	$N(22, 5^2)$	$N(27, 4^2)$	$N(34, 4^2)$	$N(5, 2^2)$	$N(1, 2^2)$	$N(4, 1.5^2)$	$N(6, 3^2)$	$N(32, 4^2)$	$N(40, 5^2)$
τ	2.3	2.7	2.5	1.8	1.5	2.3	3.7	1.8	4.2
ϕ	4.3	3.7	0.5	0.8	2.5	4.3	1.7	3.8	2.2
$\text{Er}(\tilde{\bar{M}}) = E(\bar{A})$	22.5	27.25	18.5	4.75	1.25	4.5	5.5	32.5	42.5

To optimize the advantages of overall production and overall quality performance, we will be determining the number of Products A and B that should be manufactured.

The following problem's MOLP model with FRVs is written as:

$$\text{Max}[\tilde{\bar{z}}_1, \tilde{\bar{z}}_2] = [\tilde{\bar{z}}_1 a_1 + \tilde{\bar{z}}_2 a_2, \tilde{\bar{z}}_3 a_1 + \tilde{\bar{z}}_4 a_2] \tag{16}$$

$$\tilde{\bar{x}}_1 a_1 + \tilde{\bar{x}}_2 a_2 \leq \tilde{\bar{y}}_1$$

$$\tilde{\bar{x}}_3 a_1 + \tilde{\bar{x}}_4 a_2 \leq \tilde{\bar{y}}_2$$

$$a_1 \geq 0, \quad a_2 \geq 0$$

Let $\tilde{\bar{A}} = (r, \tau, \phi)$ be FRV, where $\mathrm{Var}(r) = \sigma^2$, $E(r) = \rho$ and random variable r. Price of $\tilde{\bar{A}}$ a random variable which is determined as:

$$\tilde{\bar{A}}_\alpha^+ = r + \tau(\alpha - 1), \quad \tilde{\bar{A}}_\alpha^- = r + \phi(\alpha - 1) \tag{17}$$

$$M(\tilde{\bar{A}}) = \frac{1}{2}\int_0^1 (X_\alpha^- + X_\alpha^+)\, d\alpha = r + \frac{1}{4}(\phi - \tau) = \bar{A} \tag{18}$$

Applied to the fuzzy random multi-objective LPP, the Er-expected Value Algorithm, and then use one of the LP solvers to solve the resulting LP problem. It generates the following LP model:

$$\text{Max } \lambda \tag{19}$$

s.t

$$22.5a_1 + 27.25a_2 \geq 219.6 - 7(1 - \lambda)$$

$$33.5a_1 + 18.5a_2 \geq 295.7 - 3.4(1 - \lambda)$$

$$4.75a_1 + 1.25a_2 \leq 32.5$$

$$4.5a_1 + 5.5a_2 \leq 42.5$$

$$a_1 \geq 0, \quad a_2 \geq 0$$

It is possible to solve the specific problem's Er-optimal solution employing LINGO 8.0, a commercial ILP solver, as represented in second column of Table 4.

TABLE 4. Comparison of Optimal Solutions

Fuzzy random parameters				
Fuzzy parameters (Optimal solution)	**Er-expected Value model (Er-optimal solution)**	**Mean & CCP model**		
		PLA-Optimal solution	**Optimal solution by LINGO**	
$x^* = (x_1^*, x_2^*)$	$x_{Er}^*(\text{FRMOLP}) = (8.18, 2.27)$	$x_{PLA}^*(\text{FRMOLP}) = (4.25, 2.07)$	$x_{lingo}^*(\text{FRMOLP}) = (4.35, 2.09)$	$x_M^*(\text{FRMOLP}) = (6.19, 2.85)$
(z_1^*, z_2^*)	$([\tilde{z}_1, \tilde{z}_2])_{Er}^* = (219.56, 295.74)$	$([\tilde{z}_1, \tilde{z}_2])_{PLA}^* = (115.7, 150.27)$	$([\tilde{z}_1, \tilde{z}_2])_{lingo}^* = (116.21, 154.21)$	$([\tilde{z}_1, \tilde{z}_2])_M^* = (215.87, 260.31)$

To evaluate the effectiveness of the proposed approach, a comparative analysis is carried out using the results presented in Table 4. The proposed Mean & Chance-Constrained Programming (Mean & CCP) method combined with the Piecewise Linear Approximation (PLA) is compared with the Er-expected value model, which is a commonly adopted method in fuzzy random optimization literature work. From Table 4, it can be observed that the Er-expected value model provides solutions based primarily on the central tendency of fuzzy random variables.

While this approach is computationally easy, it does not explicitly account for the variance and probability distribution of the random components. In addition, the proposed Mean & CCP approach incorporates both the mean value and the distributional characteristics of fuzzy random variables through chance constraints, resulting in solutions that better reflect uncertainty in practical decision-making environments.

Furthermore, the PLA-based solution obtained using the proposed method is obtained to be very close to the global optimal solution computed by the commercial solver (LINGO Global Solver). This demonstrates that the proposed approach achieves high result accuracy while maintaining computational tractability. Hence, the comparative results clearly indicate the superiority of the proposed method in terms of robustness, reliability, and applicability under hybrid uncertainty.

In addition to the qualitative comparison, the numerical results present that incorporating chance constraints modifies the allocation of production quantities in a way that stabilizes total expected cost and resource utilization. The Mean & CCP model slightly adjusts the production levels compared to the Er-expected value solution, reflecting the impact of variance and confidence levels on cost feasibility. This adjustment reduces the probability of resource over-utilization and unexpected cost escalation under un-favourable parameter realizations. Therefore, while the Er-model maximizes expected profit based on average values, the proposed model effectively performs risk-adjusted cost optimization by ensuring that production decisions remain economically viable with a specified probability. This demonstrates that the proposed framework not only optimizes objective values but also enhances financial stability and operational sustainability.

To resolve the multi-objective fuzzy random programming problem given above, use the M&CCP algorithm. Considering the predefined confidence levels result in the creation of the following nonlinear programming model:

$$\text{Max } \lambda \tag{20}$$

$$Z_k \geq z^{(k)} - (1-\gamma)p^{(k)} \quad \forall k = 1, 2 \dots r$$

$$Z_1 - 22.5a_1 - 27.25a_2 \leq k_\alpha \sqrt{25a_1^2 + 16a_2^2}$$

$$Z_2 - 33.5a_1 - 18.50a_2 \leq k_\alpha \sqrt{16a_1^2 + 25a_2^2}$$

$$4.75a_1 + 1.25a_2 - k_\beta \sqrt{4a_1^2 + 4a_2^2 + 16} \leq 32.5$$

$$4.5a_1 + 5.5a_2 - k_\beta \sqrt{2.25a_1^2 + 9a_2^2 + 25} \leq 42.5$$

$$a_1 \geq 0, \quad a_2 \geq 0$$

Apply the PLA Algorithm that has been proposed in solving the nonlinear programming problem. The subsequent unique programming problem is solvable.

$$\text{Max } \lambda \tag{21}$$

$$Z_k \geq z^{(k)} - (1-\gamma)p^{(k)} \quad \forall k = 1, 2 \ldots r$$

$$Z_1 - 22.5a_1 - 27.25a_2 \leq k_\alpha t_1$$

$$t_1^2 = 25a_1^2 + 16a_2^2$$

$$Z_2 - 33.5a_1 - 18.50a_2 \leq k_\alpha t_2$$

$$t_2^2 = 16a_1^2 + 25a_2^2$$

$$4.75a_1 + 1.25a_2 - k_\beta q_1 \leq 32.5$$

$$q_1 = 4a_1^2 + 4a_2^2 + 16$$

$$4.5a_1 + 5.5a_2 - k_\beta q_2 \leq 42.5$$

$$q_2 = 2.25a_1^2 + 9a_2^2 + 25$$

$$a_1 \geq 0, \quad a_2 \geq 0$$

Table 5. Predicted variable ranges and corresponding break-point selections applied in the PLA transformation for the numerical model

Variables	**Predicted interval**	**Break points**
a_1	$[0, 6]$	$0, 2, 4, 6$
a_2	$[0, 4]$	$0, 2, 4$
t_1	$[0, 25]$	$0, 5, 10, 150, 20, 25$
t_2	$[0, 20]$	$0, 5, 10, 15, 20$
q_1	$[4, 12]$	$4, 8, 12$
q_2	$[5, 15]$	$5, 10, 15$

The above Table 5 provides the predicted interval of a_1, a_2, t_1, t_2, q_1 and q_2 as well as their break points for $\alpha = \beta = 0.9$. The following Table 6 is mentioned that the separable programming problem which is transformed into a mixed integer programming model using the PLA approach. The second column of Table 6 shows the problem's PLA-optimal solution.

In the proposed method, we applied the mean value of the FRV to handle fuzzy stochastic programming's fuzzy qualities, chance-constrained programming to handle random to handle randomness, while fuzzy modelling manages vagueness, and the resulting nonlinear programming problem is resolved using the PLA approach. It is observed that the variation effect of parameters is significant and directly affects the optimal solutions by comparing the Er-expected value model results.

TABLE 6. PLA-optimal solution results obtained from the Mean & Chance-Constrained Programming model

	Mean & CCP Model (PLA-optimal Solution)
$(\lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14})$	$(0, 0, 0.941, 0.144)$
$(\lambda_{21}, \lambda_{22}, \lambda_{23})$	$(0, 0.984, 0.044)$
$(\lambda_{31}, \lambda_{32}, \lambda_{33}, \lambda_{34}, \lambda_{35})$	$(0, 0, 0, 0, 0.563, 0.631)$
$(\lambda_{41}, \lambda_{42}, \lambda_{43}, \lambda_{44}, \lambda_{45})$	$(0, 0, 0, 0, 1.433)$
$(\lambda_{51}, \lambda_{52}, \lambda_{53})$	$(0, 0.546, 0.599)$
$(\lambda_{61}, \lambda_{62}, \lambda_{63})$	$(0, 0.993, 0.053)$
$x^* = (x_1^*, x_2^*)$	$x^*_{PLA}(\text{FRMOLP}) = (4.48, 2.23)$
(z_1^*, z_2^*)	$([\tilde{\bar{z}}_1, \tilde{\bar{z}}_2])^*_{PLA} = (115.7, 150.27)$

The fourth column of Table 4 displays the results, we also use LINGO 8.0's Global Solver to solve the nonlinear programming model produced by the M&CCP Algorithm. Comparing the conclusions, we can investigate that the PLA approach's solutions come quite near to the optimum solution provided by the LINGO Global Solver. This result indicates that analyzing the probability distribution function with the variance effect did not improve the PLA-optimal solution. As a result, the PLA-optimal solution is more trustworthy for a decision-maker than the optimum answers of the other models.

5. Conclusion

This study presents a novel modelling and solution framework for multi - objective linear programming problems under hybrid uncertainty using fuzzy random variables (FRVs). The proposed methodology integrates the mean value of FRVs with Chance-Constrained Programming (CCP) and Piecewise Linear Approximation (PLA) to convert a nonlinear stochastic problem into a mixed integer programming structure that can be solved efficiently.

Numerical results obtained from the production planning example confirm the effectiveness of the approach. Using the Er-Expected Value model, the converted deterministic MOLP was solved through a commercial solver (LINGO), producing feasible optimal solutions driven by the average behaviour of the fuzzy random parameters. However, the results also show that this method removes the dispersion and probability distribution effects of parameters. In contrast, the Mean & Chance-Constrained model provides superior outcomes by integrating variance information and confidence levels into the FRV structure. Furthermore, the PLA-optimal solution obtained from the proposed Mean & CCP framework was found to be highly accurate, as its results were very close to the global optimal solution achieved using the LINGO Global Solver.

The primary contribution of this work is the development of a novel hybrid optimization framework rather than the numerical illustration itself. The proposed

Mean & CCP formulation advances existing FRMOLP approaches by embedding risk-awareness directly into the optimization structure, thereby extending beyond traditional expected-value-based models. The numerical example is included to validate the theoretical framework and demonstrate its practical feasibility. Overall, this FRV-based multi-objective optimization framework provides a powerful and flexible modelling tool suitable for industrial problems involving fuzziness and randomness, including production planning, supply chain management, transportation networks, logistics, and engineering design.

Future extensions may incorporate new fuzzy random structures, improved membership functions, additional probability distribution assumptions, and dynamic multi-objective environments with time-dependent data. Further enhancements in computational performance and large-scale applications also remain promising research avenues.

Funding

Not applicable.

Conflicts of Interest

According to the authors, there are no conflicts of interest.

Data Availability Statement

No new data were created or analysed in this study.

Author Contributions

The first author contributed to the introduction, literature review, and conceptualization of the problem statement. The corresponding author focused on modelling, formulation, numerical implementation, analytical inferences, and manuscript review. Programming, partial literature review, and language editing were carried out by the third author.

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Department of Mathematics and Statistics, Manipal University Jaipur, Jaipur, Rajasthan, India
Email address: `gautam.202505031@muj.manipal.edu`

Department of Mathematics and Statistics, Manipal University Jaipur, Jaipur, Rajasthan, India
Email address: `*Corrosponding Author: mohd.rizwanullah@jaipur.manipal.edu`