Global and Stochastic Analysis
Vol. 13 No. 1 (2026)

http://dx.doi.org/10.64837/GSA.13.1.7

Received: 10th November 2025 *Revised: 19th January 2026* *Accepted: 06th March 2026*

THE APPLICATION OF MARTIN'S METHOD TO MHD FLOWS OF VISCOUS FLUID THROUGH POROUS MEDIA WITH VARIABLE INCLINATION

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Abstract. In this study, the exact solution of a steady, variably inclined MHD viscous fluid flow with infinite electrical conductivity through a porous medium has been found using Martin's approach. Martin's method is used to find the closed-form solution of the MHD viscous fluid flow for porous media. Several features of the flow geometry have been recognized, and the closed-form solutions of fluid flow have been seen for two distinct flow situations (radial flow and parallel flow) with variably inclined flows. The non-linear governing equations of fluid flows have been converted into a novel form called Martin's form using differential geometry. The curvilinear co-ordinates (ϕ, ψ) in the plane of flows have been represented in this manner by the lines of coordinates, where ψ represents the streamlines of the flow and ϕ is arbitrary. Both of these lines are constants. Finally, the values of the current density function, pressure function, vorticity, velocity vector, and magnetic field vector have been established. Additionally, a graphical depiction of the changes within the function of pressure and magnetic line patterns with varying fluid densities ρ and $\frac{\eta}{K}$ values have been created. The observed graphical representation has considerably represented the various characteristics of the porous material.

Keywords: MHD, Martin's method, exact solution, porous media, variably inclined flow, stream function.

2020 A.M.S. subject classification: Primary 76Wxx; 74Sxx; 76Sxx; 35Gxx; 35Axx, Secondary 76W05; 74S99; 6S05; 35G20; 35A25.

1. Introduction

The mathematical equations governing an electrically conductive fluid's flow have not been able to be thoroughly examined due to their mathematical complexity since Alfven's early theoretical work (1940). It became necessary to make some geometrical approximations, such as plane or axisymmetric flows, and some assumptions regarding the fluid's essential properties, such as its incompressibility, inviscid, infinite conductivity, high and low Reynolds numbers, etc., to make simple problems. Additionally, every analytical approach used to date has necessitated imposing limitations on the angle formed by the velocity vector field **v** and the magnetic vector field **H**.

Accordingly, the recent research points out flows that are constantly inclined (constant angle between **v** and **H**), orthogonal (**v** perpendicular to **H**), and associated (**v** parallel to **H**). Some geometrical properties for aligned flows have been developed by Chandna and Nath (1973). Chandna et al. (1976; 1979) employed Martin's approach to establish analytic solutions in the flow of a viscous MHD fluid and in plane compressible MHD electrical conducting fluid flows. In the investigations they conducted on the viscous flow of fluid in plane, incompressible MHD, Barton and Chandna (1981), Chandna et al. (1982; 1983; 1994) have presented several findings dealing with aligned, orthogonal, constantly inclined, and variably inclined flows. In a porous medium, the fluid motion is usually expressed by way of volume or the collective average of the motion of the individual fluid components over certain spatial areas because the fluid-rock boundary conditions make it impossible to determine the real trajectory of a single fluid particle when a fluid flows through a porous medium. Usually accomplished via the well-known Darcy's law, this causes the resistance term to take the place of the viscous component in fluid motion equations$-\frac{\eta}{K}\vec{V}$, where η is the fluid's viscosity, K is the medium's permeability, and $\vec{V}$ is the fluid's seepage velocity. Studies of fluid flows through porous media in various flow problems have been conducted by several scholars (Thakur C and Kumar 2008; Sil et al. 2012; Kumar 2014; Sil and Kumar 2015; Kumar et al. 2015; Prajapati 2018; Sil and Kumar 2019; Krishna and Chamkha 2019; Krishna et al. 2020; Sil et al. 2021). The exact solution to the unsteady transverse MHD flow rotating frame was found by Vishwakarma et al. (2022) using Hodograph Transformation. Singh et al. (2024) described a method for applying the Inverse method to effectively solve the flow of an analytic solution with the Hall effect via porous media. Sil (2024) investigated an analytical solution for MHD micropolar fluid using the Hodograph approach in porous formations.

Several transformation techniques are used to turn the nonlinear partial differential equations into linear differential equations. There is no universal solution according to the Navier-Stokes equations or the basic formulas governing the flow of fluids that are magnetohydrodynamic (MHD), as it is, by nature, nonlinear partial differential equations. To transform equations into a solvable form, Martin (1971) developed a new technique that uses differential geometry. The curvilinear coordinates (ϕ, ψ) are utilized in the flow plane, and the coordinate lines ψ = constant are assumed to become the streamlines of flow, while the coordinate lines ϕ = constant are left arbitrary. In this study, we apply Martin's methodology. The Von Mises coordinates (x, ψ) that have been made use of need to make ϕ = x in Martin's coordinates (ϕ, ψ). A methodology created by Martin has been extended to address MHD problems by studying the planar incompressible flow of an electrically conducting viscous fluid by several scholars (Naeem and Nadeem 1996; Naeem and Ali 2001; Ali et al. 2007; Thakur et al. 2008; Kumar et al. 2009; Naeem et al. 2009; Naeem et al. 2009; Sil S, and Kumar 2016; Sil et al. 2020; Vishwakarma et al. 2024). Manglesh and Gorla (2013) study free-convective MHD flow across a porous surface with an angled magnetic field and Hall currents, expressly assuming this. They demonstrate that by setting the induced-field term to zero for $R_m \geq 1$, the leading-order velocity and heat-transfer profiles remain unchanged. Using a horizontal flat plate covered with a fluid-saturated porous medium, Postelnicu (2007) investigated the effects of thermophoretic particle deposition on the flow of free convection boundary layers. After converting the governing transport equations into coupled nonlinear differential equations allowing for thermophoresis effects, the finite difference technique was used to solve the equations analytically. Using an analytical approach, Tripathi et.al. (2020) examined the electro-osmotic transport in a microchannel embedded with a porous formation and encircled by geometrically complex wavy surfaces.

The novelty of this study is that it extends Martin's differential-geometric technique to provide accurate analytical solutions for viscous MHD flows through porous media that are stable and variably inclined. This is an area that has not been explored in theoretical MHD research before. Supported by graphical analysis of pressure and magnetic line variations, the study offers new insights into the geometrical and physical properties of porous MHD systems by converting the nonlinear governing equations into Martin's form in curvilinear coordinates and taking into account variable inclination between velocity and magnetic field vectors.

The more general problem of taking into account a variable angle between $\vec{V}$ and $\vec{H}$, in porous formations, has not been investigated in any theoretical investigations in MHD so far. In this work, we used Martin's method to determine certain geometrical characteristics and derive an exact solution for viscous, variably inclined, plane, incompressible, via porous medium, MHD flows. By determining the equations of flow and providing a few important differential geometry concepts, the findings in part 2. The conversion of the flow equations to a system similar to Martin's system is covered in part 3. In part 4, the geometrical characteristics of these radial and parallel flows with variable inclinations are fully investigated, and solutions to these problems are obtained. The obtained results are significantly observed for the physical phenomenon of MHD flow along with porous media.

2. Equation of Flow

When the magnetic field is in the plane of flow, the general equations governing the steady plane, the incompressible, viscous fluid flow with infinite electrical conductivity are

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0, \tag{1}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial p}{\partial x} = \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \mu H_2\left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\right) - \frac{\eta}{\kappa}u, \tag{2}$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial p}{\partial y} = \eta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \mu H_1\left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\right) - \frac{\eta}{\kappa}v \tag{3}$$

$$u\,H_2 - v\,H_1 = \sqrt{u^2+v^2}\,,\quad \sqrt{{H_1}^2+{H_2}^2}\ sin\theta\ = k\,, \tag{4}$$

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0, \tag{5}$$

where u, v are the velocity components, p is the pressure function, and H_1, H_2 are the magnetic vector field components, and the angle at any given position (x, y) between the magnetic (H) and velocity (v) vectors is θ(x, y).In this system, the terms ρ, η, and μ stand for constant fluid density, constant viscosity coefficient, and constant magnetic permeability, respectively, and k is an arbitrary constant.

Introducing the functions

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},\ j = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y},\qquad h = \frac{1}{2}\rho q^2 + p\,, \tag{6}$$

where q=$\sqrt{u^2+v^2}$.

The set of equations given has been replaced to the following system-

(Continuity), $$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{7}$$

(Linear momentum), $$\rho v\omega - \eta\frac{\partial \omega}{\partial y} - \mu j H_2 - \frac{\eta}{\kappa}u = \frac{\partial h}{\partial x}, \tag{8}$$

$$-\rho u\omega + \eta\frac{\partial \omega}{\partial x} + \mu j H_1 - \frac{\eta}{\kappa}v = \frac{\partial h}{\partial y}, \tag{9}$$

(Diffusion equation), $$u\,H_2 - v\,H_1 = qH sin\theta = k, \tag{10}$$

(Solenoidal), $\frac{\partial H_1}{\partial x}+\frac{\partial H_2}{\partial y}=0$ (11)

(vorticity), $\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\omega$ (12)

(Current density), $j=\frac{\partial H_2}{\partial x}-\frac{\partial H_1}{\partial y}$ (13)

seven unknowns (u, v, H_1, H_2, ω, j) and h as x and y functions are involved in seven non-linear partial differential equations. Martin has successfully used a decrease in order from two to one to study viscous non-MHD flows.

According to the equation of continuity (7), there must be a stream function ψ(x, y) such that

$$\frac{\partial\psi}{\partial x}=-\rho v\,,\quad \frac{\partial\psi}{\partial y}=\rho u\,, \tag{14}$$

In the physical plane, we consider ϕ(x, y) = const. to be an arbitrary family of curves that produces a curvilinear net (ϕ,ψ) with the streamlines ψ(x, y) = const.

Let x = x (ϕ,ψ), y = y (ϕ,ψ), (15)

Use the squared element of arc length to define the curvilinear net along any curve provided by

$$ds^2=E(\phi,\psi)d\phi^2+2F(\phi,\psi)d\phi\,d\psi+G(\phi,\psi)d\psi^2, \tag{16}$$

Here,

$$E=\left(\frac{\partial x}{\partial\phi}\right)^2+\left(\frac{\partial y}{\partial\phi}\right)^2,\quad F=\frac{\partial x}{\partial\phi}\frac{\partial x}{\partial\psi}+\frac{\partial y}{\partial\phi}\frac{\partial y}{\partial\psi},\quad G=\left(\frac{\partial x}{\partial\psi}\right)^2+\left(\frac{\partial y}{\partial\psi}\right)^2. \tag{17}$$

Solving transformation equations (15) yields ϕ,ψ as functions of x, y such that

$$\frac{\partial x}{\partial\phi}=J\frac{\partial\psi}{\partial y}\,,\frac{\partial x}{\partial\psi}=-J\frac{\partial\phi}{\partial y}\,,\frac{\partial y}{\partial\phi}=-J\frac{\partial\psi}{\partial x}\,,\ \frac{\partial y}{\partial\psi}=J\frac{\partial\phi}{\partial x}\,, \tag{18}$$

given $0<|J|<\infty$, where J is the Jacobian transformation. With (17), we have

$$J=\frac{\partial(x,y)}{\partial(\phi,\psi)}=\pm\sqrt{EG-F^2}=\pm M \tag{19}$$

The local angle of inclination of the tangent to the coordinate line ψ = constant, indicated by β and directed in the direction of increasing ϕ.

Based on differential geometry, we obtain

$$\frac{\partial x}{\partial\phi}=\sqrt{E}cos\beta\,,\quad \frac{\partial y}{\partial\phi}=\sqrt{E}\sin\beta, \tag{20}$$

$$\frac{\partial x}{\partial\psi}=\frac{F}{\sqrt{E}}\sin\beta-\frac{J}{\sqrt{E}}\cos\beta,\quad \frac{\partial y}{\partial\psi}=\frac{F}{\sqrt{E}}\cos\beta-\frac{J}{\sqrt{E}}\sin\beta, \tag{21}$$

$$\frac{\partial\beta}{\partial\phi}=\frac{J}{E}\Gamma_{11}^2,\qquad \frac{\partial\beta}{\partial\psi}=\frac{J}{E}\Gamma_{12}^2, \tag{22}$$

$$K=\frac{1}{M}\left[\frac{\partial}{\partial\psi}\left(\frac{M}{E}\Gamma_{11}^2\right)-\frac{\partial}{\partial\phi}\left(\frac{M}{E}\Gamma_{12}^2\right)\right], \tag{23}$$

Where,

$$\Gamma_{11}^2=\frac{1}{2M^2}\left[-F\frac{\partial E}{\partial\phi}+2E\frac{\partial F}{\partial\phi}-E\frac{\partial E}{\partial\psi}\right],\quad \Gamma_{12}^2=\frac{1}{2M^2}\left[E\frac{\partial G}{\partial\phi}-F\frac{\partial E}{\partial\psi}\right]. \tag{24}$$

3. Methodology

3.1 Equations of flow in Martin's form. This section involves rewriting the system (7)-(13) equations in terms of the new independent variables ϕ, ψ. It is assumed that, without losing generality, the variable angle θ between the velocity vector v and the magnetic field H is such that $0 < \theta < \pi$.

3.2 Equation of continuity. Martin [21] has determined the necessary and sufficient conditions for a fluid to flow along the coordinate lines ψ = constant of a curvilinear coordinate system (15) with ds^2 provided by (16), to satisfy the principle of conservation of mass as

$$\rho M q = \sqrt{E}\,, \qquad u + iv = \frac{\sqrt{\mathrm{E}}}{\rho \mathrm{J}} \exp(i\beta), \tag{25}$$

Additionally, it has demonstrated that the fluid moves toward greater or lesser values for the parameters ϕ in accordance with J is positive or negative.

3.3 Vorticity equation. Martin [21] has established that

$$\omega = \frac{1}{\rho M}\left[\frac{\partial}{\partial \phi}\left(\frac{F}{M}\right) - \frac{\partial}{\partial \psi}\left(\frac{E}{M}\right)\right]. \tag{26}$$

3.4 Solenoidal condition on H. In either an upward or downward direction, ϕ parameter values, the fluid moves along the streamlines, and H forms an angle with the x-axis of $\theta + \beta$ or $\theta + \beta - \pi$.

$$H_1 = \pm H\cos(\theta+\beta),\ H_2 = \pm H\sin(\theta+\beta) \tag{27}$$

where J is either positive or negative, and the positive or negative sign is considered, and $H = \sqrt{{H_1}^2 + {H_2}^2}$.

As a result of using (18) in the solenoidal equation (11),

$$\left(\frac{\partial H_1}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial H_1}{\partial \psi}\frac{\partial y}{\partial \phi}\right) - \left(\frac{\partial H_2}{\partial \phi}\frac{\partial x}{\partial \psi} - \frac{\partial H_2}{\partial \psi}\frac{\partial x}{\partial \phi}\right).$$

Using these equations (20), (21), and (27), we obtain

$$\frac{\partial H}{\partial \phi}(-F\sin\theta + J\cos\theta) + \frac{\partial H}{\partial \psi}E\sin\theta - H(F\cos\theta + J\sin\theta)\frac{\partial \theta}{\partial \phi} + HEcos\theta\frac{\partial \theta}{\partial \psi} -$$
$$H\frac{J}{E}\Gamma_{11}^2(Fcos\theta + Jsin\theta) + HJ\Gamma_{12}^2 cos\theta = 0\,.$$
(28)

3.5 The current density. Based on (13) and (18), we obtain

$$Jj = \left(\frac{\partial H_2}{\partial \phi}\frac{\partial y}{\partial \psi} - \frac{\partial H_2}{\partial \psi}\frac{\partial y}{\partial \phi}\right) - \left(\frac{\partial H_1}{\partial \phi}\frac{\partial x}{\partial \psi} - \frac{\partial H_1}{\partial \psi}\frac{\partial x}{\partial \phi}\right).$$

Using (20), (21), and (27), this equation gives

$$\sqrt{E}Mj = \frac{\partial H}{\partial \phi}(F\cos\theta + J\sin\theta) - \frac{\partial H}{\partial \psi}E\cos\theta - H(F\sin\theta - J\cos\theta)\frac{\partial \theta}{\partial \phi} + HEsin\theta\frac{\partial \theta}{\partial \psi} -$$
$$H\frac{J}{E}\Gamma_{11}^2(Fsin\theta - Jcos\theta) + HJ\Gamma_{12}^2 sin\theta. \tag{29}$$

3.6 Momentum equations

When we apply (14) and (27) to the momentum equations (8) and (9), we obtain

$\eta\left(\frac{\partial\omega}{\partial\phi}\frac{\partial\phi}{\partial y}-\frac{\partial\omega}{\partial\psi}\frac{\partial\psi}{\partial y}\right)+\omega\frac{\partial\psi}{\partial x}\pm\mu jHsin(\theta+\beta)+\frac{\eta}{\kappa}\frac{\partial\psi}{\partial y}=-\left(\frac{\partial h}{\partial\phi}\frac{\partial\phi}{\partial x}+\frac{\partial h}{\partial\psi}\frac{\partial\psi}{\partial x}\right)$,

and

$\eta\left(\frac{\partial\omega}{\partial\phi}\frac{\partial\phi}{\partial x}-\frac{\partial\omega}{\partial\psi}\frac{\partial\psi}{\partial x}\right)+\omega\frac{\partial\psi}{\partial y}\pm\mu jHcos(\theta+\beta)-\frac{\eta}{\kappa}\frac{\partial\psi}{\partial x}=\left(\frac{\partial h}{\partial\phi}\frac{\partial\phi}{\partial y}+\frac{\partial h}{\partial\psi}\frac{\partial\psi}{\partial y}\right)$.

Applying equations (18) to these two equations yields

$$\eta\left(-\frac{\partial\omega}{\partial\phi}\frac{\partial x}{\partial\psi}+\frac{\partial\omega}{\partial\psi}\frac{\partial x}{\partial\phi}\right)-\omega\frac{\partial y}{\partial\phi}+\mu MjHsin(\theta+\beta)+\frac{\eta}{\kappa}\frac{\partial x}{\partial\phi}=-\left(\frac{\partial h}{\partial\phi}\frac{\partial y}{\partial\psi}-\frac{\partial h}{\partial\psi}\frac{\partial y}{\partial\phi}\right),$$

$$\eta\left(-\frac{\partial\omega}{\partial\phi}\frac{\partial y}{\partial\psi}+\frac{\partial\omega}{\partial\psi}\frac{\partial y}{\partial\phi}\right)-\omega\frac{\partial x}{\partial\phi}+\mu MjHcos(\theta+\beta)-\frac{\eta}{\kappa}\frac{\partial y}{\partial\phi}=\left(-\frac{\partial h}{\partial\phi}\frac{\partial x}{\partial\psi}+\frac{\partial h}{\partial\psi}\frac{\partial x}{\partial\phi}\right).$$

One equation is obtained by multiplying these two equations by $\frac{\partial y}{\partial\phi}$ and $\frac{\partial x}{\partial\phi}$, respectively, and then adding them. In the subsequent set of the linear momentum equations' new equivalent form, the second equation is obtained by multiplying by $\frac{\partial\psi}{\partial y}$ and $\frac{\partial\psi}{\partial x}$, respectively, and then adding. The following are the two linear momentum equations:

$$\eta J\frac{\partial\omega}{\partial\phi}-\omega E+\mu MjH\sqrt{E}\cos\theta=-F\frac{\partial h}{\partial\phi}+E\frac{\partial h}{\partial\psi},$$

$$\eta J\frac{\partial\omega}{\partial\psi}-\omega F+\mu MjH\left(\frac{F}{\sqrt{E}}cos\theta+\frac{J}{\sqrt{E}}sin\theta\right)+\frac{\eta}{\kappa}J=-G\frac{\partial h}{\partial\phi}+F\frac{\partial h}{\partial\psi},$$

The diffusion equation is $\sqrt{E}Hsin\theta=\rho kM$. (30)

To sum up, we have:

Theorem 1

If the streamlines ψ (x, y) = const. and an arbitrary family of curves ϕ (x, y) = const. generate a curvilinear net in the physical plane of a steady, incompressible viscous MHD fluid through porous media, then the flow in the independent variables ϕ,ψ is governed by the system.

(Linear momentum)

$$\eta J\frac{\partial\omega}{\partial\phi}-\omega E+\mu MjH\sqrt{E}\cos\theta=-F\frac{\partial h}{\partial\phi}+E\frac{\partial h}{\partial\psi}, \tag{31}$$

$$\eta J\frac{\partial\omega}{\partial\psi}-\omega F+\mu MjH\left(\frac{F}{\sqrt{E}}cos\theta+\frac{J}{\sqrt{E}}sin\theta\right)+\frac{\eta}{\kappa}J=-G\frac{\partial h}{\partial\phi}+F\frac{\partial h}{\partial\psi}, \tag{32}$$

(Gauss) $\frac{\partial}{\partial\psi}\left(\frac{M}{E}\Gamma_{11}^2\right)-\frac{\partial}{\partial\phi}\left(\frac{M}{E}\Gamma_{12}^2\right)=0$, (33)

(Vorticity) $\omega=\frac{1}{\rho M}\left[\frac{\partial}{\partial\phi}\left(\frac{F}{M}\right)-\frac{\partial}{\partial\psi}\left(\frac{E}{M}\right)\right]$, (34)

(Solenoidal)

$$\frac{\partial H}{\partial\phi}(-F\sin\theta+J\cos\theta)+\frac{\partial H}{\partial\psi}E\sin\theta-H(F\cos\theta+J\sin\theta)\frac{\partial\theta}{\partial\phi}+HEcos\theta\frac{\partial\theta}{\partial\psi}-H\frac{J}{E}\Gamma_{11}^2(Fcos\theta+Jsin\theta)+HJ\Gamma_{12}^2cos\theta=0, \tag{35}$$

(Current density)

$$\sqrt{E}Mj=\frac{\partial H}{\partial\phi}(F\cos\theta+J\sin\theta)-\frac{\partial H}{\partial\psi}E\cos\theta-H(F\sin\theta-J\cos\theta)\frac{\partial\theta}{\partial\phi}+HEsin\theta\frac{\partial\theta}{\partial\psi}-H\frac{J}{E}\Gamma_{11}^2(Fsin\theta-Jcos\theta)+HJ\Gamma_{12}^2sin\theta, \tag{36}$$

(Diffusion)

$\sqrt{E}Hsin\theta = \rho kM,$ (37)

with eight unknowns and seven equations E, F, G, h, ω, j, H and θ as a function of ϕ, ψ. When this system's solution is found, the pressure component is calculated from $P = h - \frac{E}{2pM^2}$.

Additionally, the flow in both the hodograph and physical planes is explained by

$$z = \int \frac{e^{i\beta}}{\sqrt{E}}\{Ed\phi + (F + iJ)d\psi\},$$

$$\beta = \int \frac{J}{E}\{\Gamma_{11}^2 d\phi + \Gamma_{12}^2 d\psi\} \quad (38)$$

$$u + iv = \frac{\sqrt{E}}{\rho J}\exp(i\beta). \quad (39)$$

Furthermore, the magnetic field is derived from

$$H_1 + H_2 = \left(\frac{HM}{J}\right) exp[i(\theta + \beta)].$$

Due to the intrinsic arbitrariness in choosing the set of coordinate lines ϕ = constant, we observe that the system (31)-(37) is under-determined. Several methods may be used to make it determinate, but one reasonable approach is to consider coordinate lines to be ϕ = constant are the orthogonal trajectories of streamlines ψ = constant. That is, set

F = 0.

If we set F = 0 and apply the integrability condition $\frac{\partial^2 h}{\partial\phi\partial\psi} = \frac{\partial^2 h}{\partial\psi\partial\phi}$ to the linear momentum equations (31) and (32), we get

$$\eta\left[\frac{\partial}{\partial\phi}\left(\frac{J}{E}\frac{\partial\omega}{\partial\phi}\right) + \frac{\partial}{\partial\psi}\left(\frac{J}{E}\frac{\partial\omega}{\partial\psi}\right)\right] - \frac{\partial\omega}{\partial\phi} + \mu\left[\frac{\partial}{\partial\phi}\left(jH\sqrt{G}\right)cos\theta + \frac{\partial}{\partial\psi}\left(jH\frac{J}{\sqrt{G}}\right)sin\theta\right] + \frac{\eta}{\kappa}\frac{\partial}{\partial\psi}(J) = 0.$$

Corollary 1.

When the streamlines ψ = const. and the coordinate lines ϕ = constant are taken for orthogonal curvilinear coordinates, the fluid flow under study is governed by the system-

(Linear momentum)

$$\eta\left[\frac{\partial}{\partial\phi}\left(\frac{J}{E}\frac{\partial\omega}{\partial\phi}\right) + \frac{\partial}{\partial\psi}\left(\frac{J}{E}\frac{\partial\omega}{\partial\psi}\right)\right] - \frac{\partial\omega}{\partial\phi} + \mu\left[\frac{\partial}{\partial\phi}\left(jH\sqrt{G}\right)cos\theta + \frac{\partial}{\partial\psi}\left(jH\frac{J}{\sqrt{G}}\right)sin\theta\right] + \frac{\eta}{\kappa}\frac{\partial}{\partial\psi}(J) = 0, \quad (40)$$

(Gauss) $\frac{\partial}{\partial\psi}\left(\frac{1}{\sqrt{EG}}\frac{\partial E}{\partial\psi}\right) - \frac{\partial}{\partial\phi}\left(\frac{1}{\sqrt{EG}}\frac{\partial G}{\partial\phi\phi}\right) = 0$ (41)

(Vorticity) $\omega = -\frac{1}{\rho M}\frac{\partial}{\partial\psi}\left(\frac{E}{M}\right)$ (42)

(Solenoidal condition)

$$J\frac{\partial H}{\partial\phi}\cos\theta + E\frac{\partial H}{\partial\psi}\sin\theta - JHsin\theta\frac{\partial\theta}{\partial\phi} + EHcos\theta\frac{\partial\theta}{\partial\psi} - \frac{1}{2}\frac{\partial E}{\partial\psi}Hsin\theta + \frac{1}{2}\frac{J}{G}\frac{\partial E}{\partial\phi}Hcos\theta = 0, \quad (43)$$

(Current density)

$$\sqrt{E}Jj = J\frac{\partial H}{\partial\phi}\sin\theta - E\frac{\partial H}{\partial\psi}\cos\theta - \frac{1}{2}\frac{\partial E}{\partial\psi}Hcos\theta + \frac{1}{2}\frac{J}{G}\frac{\partial E}{\partial\phi}H\sin\theta + JHcos\theta\frac{\partial\theta}{\partial\phi} + EHsin\theta\frac{\partial\theta}{\partial\psi} \quad (44)$$

(Diffusion) $Hsin\theta = \rho k\sqrt{G}$ (45)

Having six unknowns (E, G,ω, H, j, and θ).

4. Results and Discussion

Using the conclusions above, we will examine flows for which the coefficients E and G of the first fundamental form are subject to certain a priori requirements in this section.

$$ds^2 = Ed\phi^2 + Gd\psi^2 \tag{46}$$

4.1 Radial flows

In this work, we study source flow solutions.

Since the point of polar positions inside the flow plane is (r, p), we use

$\psi = \psi(\alpha),\ \ \phi = (r)$, examine these flows. For these flows, the arc length squared element is $ds^2 = E\phi'^2 dr^2 + G\psi'^2 d\alpha^2$. (47)

Comparing (46) and (47) gives

$$E = \frac{1}{\phi'^2},\ G = \frac{r^2}{\psi'^2},\ J = \frac{r}{\phi'\psi'}. \tag{48}$$

Applying (48) to (45) yields

$$H\sin\theta = \frac{\rho k r}{\psi'(\alpha)} \quad , \tag{49}$$

by using (48) equation (43) and differentiating (49) with respect to r and α, it becomes

$$r\frac{\partial}{\partial r}(\cot\theta) + 2\cot\theta = \frac{\psi''(\alpha)}{\psi'(\alpha)}.$$

This has a solution

$$\cot\theta = \frac{\psi''(\alpha)}{\psi'(\alpha)} + \frac{g(\alpha)}{r^2}, \tag{50}$$

where f (α) is an arbitrary function of α. Using (48) and (49) in (44), the current density equation becomes into

$$j = \frac{1}{r\cos\theta}\frac{\partial H}{\partial \alpha} + \frac{H}{\cos\theta}\frac{\partial\theta}{\partial r}. \tag{51}$$

Similarly, the vorticity seen in (42) is

$$\omega = -\frac{1}{\rho r^2}\psi''. \tag{52}$$

Equation (41) is automatically satisfied, and (40) employing (48) and (52) implies

$$\frac{4\eta\psi''}{\rho r^3\psi'} + \frac{\eta\psi^{iv}}{\rho r^3\psi'} + \frac{2\psi''}{\rho r^3} - \frac{\mu}{\psi'}\frac{\partial}{\partial r}(\mathrm{rjHcos}\,\theta) - \frac{\mu}{\psi'}\frac{\partial}{\partial \alpha}(\mathrm{rjHsin}\,\theta) + \frac{\eta}{\mathrm{K}} r\ \frac{\psi''}{\psi'^3} = 0.$$

Using (51), in place of j, the equation gives

$$\psi'^2\sin^2\theta\{\eta\psi^{iv} + 4\eta\psi'' + 2\psi'\psi''\} + \mu\rho^4k^2r^4\left\{\frac{2\psi''}{\psi'} + \frac{2\psi''}{\psi'}\frac{\partial\theta}{\partial\alpha} + 2\cot\theta\frac{\partial\theta}{\partial\alpha} - 3r\frac{\partial\theta}{\partial r} - r^2\frac{\partial^2\theta}{\partial r^2} + 2r^2\cot\theta\left(\frac{\partial\theta}{\partial r}\right)^2 - \frac{\partial^2\theta}{\partial\alpha^2} + 2\cot\theta\left(\frac{\partial\theta}{\partial\alpha}\right)^2\right\} + \frac{\eta}{\mathrm{K}}r^4\rho\,\psi'' sin^2\theta = 0. \tag{53}$$

Finally, from (50) expressing θ and its derivatives in terms of g(α) and $\tau(\alpha) = \frac{\psi''}{\psi'}$ and assuming $\tau \neq 0$, this equation reduces to

$$\frac{1}{2}\{\tau'' - 3\tau\tau' + \tau(\tau^2 + 4)\}r^4 + \{g'' - 3\tau g' - (\tau' - 2\tau^2)g\}r^2 + \frac{4\eta}{\mu\rho^3k^2}\psi''\psi'^2 + \frac{\eta}{\mu\rho^3k^2}\psi^{iv}\psi'^2 + \frac{2}{\mu\rho^3k^2}\psi''\psi'^3 + 2\tau g^2 - 2gg' + \frac{\eta}{\mathrm{K}}r^4\rho\,\psi'' = 0 \tag{54}$$

Each power of r must have a coefficient of zero, as this must be true for every r. So, the functions $\psi'$and g (α) must be satisfied the following equations

$$\tau'' - 3\tau\tau' + \tau(\tau^2 + 4) = 0 \quad (55)$$

$$g'' - 3\tau g' - (\tau' - 2\tau^2)g = 0 \quad (56)$$

$$\eta\psi^{iv}\psi'^2 + 4\eta\psi''\psi'^2 + 2\,\psi''\psi'^3 + 2\mu\rho^3k^2\tau g^2 - 2\mu\rho^3k^2gg' = 0' \quad (57)$$

$$\psi'' = 0 \quad (58)$$

If we assuming $\psi'' \neq 0$ then Equation (57) can be written as

$$(g^2)' - 2(ln\psi')' g\ ' = \frac{\eta}{\mu\rho^3k^2}\{\eta\psi^{iv}\psi'^2 + 4\eta\psi''\psi'^2 + 2\,\psi''\psi'^3\}$$

This can be integrated to give-

$$g^2(\alpha) = \frac{\psi'^2}{\mu\rho^3k^2}\{\eta\psi''' + 4\eta\psi' + 2\psi'^2 + c_0\} \quad (59)$$

where c_0 is an arbitrary integration constant.

It can be shown the $\psi'(\alpha)$ has to satisfy by substituting (59) into (56).

$$\frac{\{\tau(3\eta\tau'+\psi')\}'}{\{\tau(3\eta\tau'+\psi')\}^2} = \frac{\psi'}{\eta\psi'(\tau'+\tau^2+4)+\psi'^2+c_0}. \quad (60)$$

The next step is to solve equation (55) for $\tau(\alpha)$ # 0. Assuming $\gamma = \gamma(\tau)$ and letting$= \tau'$, such that $\tau'' = \gamma\left(\frac{d\gamma}{d\tau}\right)$ than

$$\gamma\left(\frac{d\gamma}{d\tau}\right) - 3\tau\gamma + \tau(\tau^2 + 4) = 0 \quad (61)$$

A general solution of the form is assumed, taking into account that $\gamma = \tau^2 + 4$ is a particular solution of this equation.

$$\gamma(\tau) = \tau^2 + 4 + t(\tau) \quad (62)$$

After that, it can be demonstrated that $t(\tau)$ must satisfy

$$(\tau^2 + 4 + t)dt - \tau t d\tau = 0.$$

As $\tau \neq 0$, the equation's solution is

$$t(\tau) = \frac{1}{c_1} \pm \frac{1}{c_2}\sqrt{\left(1 + c_1(\tau^2 + 4)\right)}, \quad (63)$$

where the integration constant$c_1$is such that $c_1 \in \left(-\frac{1}{4}, 0\right) \cup (0, \infty)$. When c_1= 0, the differential equation $\frac{d\tau}{d\alpha} = \left(1/2(\tau^2 + 4)\right)$ is obtained.

To make things easier, define$C = 17 + 4c_1$. (64)

Therefore, using equations (62)–(64), the differential equation for $\tau(\alpha)$is reduced to

$$c_1\frac{d\tau}{d\alpha} = C + c_1\tau^2 \pm \sqrt{C + c_1\tau^2} \quad (65)$$

for both 0 < C < 1 and 1 < C < ∞. When C=1, the $\tau(\alpha)$equation is

$$\frac{d\tau}{d\alpha} = \frac{1}{2(\tau^2+4)}. \quad (66)$$

Case 1.

C = 0. Equation (66) gives

$$\tau(\alpha) = 2tan(\alpha + \alpha_0), \quad (67)$$

again, on integrating to find, as $\tau(\alpha) = \frac{\psi''}{\psi'}$ and α_0 is an arbitrary constant.

$$\psi' = D_0sec^2(\alpha + \alpha_0)\quad , \quad (68)$$

D_0 is an arbitrary positive constant in this case. When we enter (67) and (68) into (60), we obtain that D_0= -6η and c_0 = 0. However, D_0 > 0 and the viscosity coefficient is (positively) η. Therefore, (68) is not a suitable solution.

Case 2.

0 < C< 1, we define $C^{*2} = -c_1$ in this instance, since c_1< 0, write equation (65) as follows:

$-C^{*2}\frac{d\tau}{d\alpha} = C - C^{*2}\tau^2 \pm \sqrt{C - C^{*2}\tau^2}$.

This equation turns to $C^*\tau = \sqrt{Csin\, v}$., $\frac{dv}{\sqrt{C\, cosv}\pm 1} = -\frac{1}{C^*}d\alpha$,

On integrating $\pm\frac{2}{\sqrt{1-C}}\tan^{-1}\left(\frac{s}{A_\pm}\right) = -\frac{1}{C^*}(\alpha+\alpha_0)$, (69)

where α_0 is an arbitrary constant ${A_\pm}^2 = \frac{1\pm\sqrt{C}}{1\pm\sqrt{C}}$, and $s = tan\frac{v}{2} = \frac{C^*\tau}{\sqrt{C+\sqrt{C-C^{*2}\tau^2}}}$.

In (69), substituting t and calculating τ yields

$$\tau(\alpha) = \pm\frac{2\sqrt{CA_\pm}}{C^*}\left\{\frac{tan(\alpha+\alpha_0)}{1+{A_\pm}^2tan^2(\alpha+\alpha_0)}\right\}. \quad (70)$$

The result of solving for $\psi'(\alpha)$ is $\psi'(\alpha) = D_0\left\{\frac{tan(\alpha+\alpha_0)}{1+{A_\pm}^2tan^2(\alpha+\alpha_0)}\right\}$, (71)

where the constant D_0 > 0 is arbitrary. By substituting (70) and (71) into (60), it is once more possible to demonstrate that no option of c_0,D_0, or C will meet (5.13). Therefore, for $\psi'(\alpha)$), 0 < C < 1 results in an inadmissible result.

Case 3.

1 < C< ∞ , Equation (65), taking $\tau = \sqrt{C/c_1}$ tan v, simplifies to $\frac{dv}{\sqrt{C}\pm cosv} = \frac{1}{\sqrt{c_1}}d\alpha$,

It possesses the solution $s = R_\pm\tan(\alpha+\alpha_0)$, α_0 is an arbitrary constant.

$${R_\pm}^2 = \frac{\sqrt{C\pm1}}{\sqrt{C\pm1}}, = tan\frac{v}{2} = \frac{\sqrt{c_1}\ \tau}{\sqrt{C+\sqrt{C-c_1\tau^2}}},\ \tau(\alpha) = \pm\frac{2R_\pm\sqrt{C}}{\sqrt{C}}\left\{\frac{tan(\alpha+\alpha_0)}{1-{R_\pm}^2tan^2(\alpha+\alpha_0)}\right\}, \quad (72)$$

$$\psi'(\alpha) = D_0\left\{\frac{1+tan(\alpha+\alpha_0)}{1+{R_\pm}^2tan^2(\alpha+\alpha_0)}\right\}, \quad (73)$$

where the constant D_0 > 0 is arbitrary. Equation (60), once more, cannot be satisfied by expressions (72) and (73); hence 1 < C< ∞ is not possible.

Regarding the case t = 0, g $\neq 0, \tau \neq 0$ (62) becomes

$\frac{d\tau}{d\alpha} = \tau^2 + 4$,

The solution is

$$\tau(\alpha) = 2tan2(\alpha+\alpha_0), \quad (74)$$

Consequently, we determine

$$\psi'(\alpha) = D_0 sec2(\alpha+\alpha_0). \quad (75)$$

In this case, D_0 > 0. When we put (74) and (75) into (60), we see that G'(p), as provided by (75), does not satisfy (60). As a result, there is no ideal flow solution for z = 0.

Following the same analysis as before, while taking into account g = 0, t $\neq 0$, $\tau \neq 0$, and g $\neq 0$ likewise, results in a contradiction.

The above analysis shows that our assumption that $\psi'' \neq 0$ is wrong. So, the flow must be irrotational, i.e. $\psi'' = 0$ or $\tau = 0$, to exhibit a pattern of radial flow.

i.e. $\psi'(\alpha) = N: constant.$

We find out $g = constant = \frac{1}{D}$, $\cot\theta = \frac{1}{Dr^2}$,

The flow in the physical plane can be identified by using Theorem 1, (51), and (52)

$$\omega = 0, \quad j = \frac{2\rho k}{N}, \quad u = \frac{N}{\rho}\frac{x}{x^2+y^2}, \quad v = \frac{N}{\rho}\frac{y}{x^2+y^2}, \tag{76}$$

$$H_1 = \frac{\rho K}{DN}\left(\frac{x}{x^2+y^2} - Dy\right), \quad H_2 = \frac{\rho K}{DN}\left(\frac{y}{x^2+y^2} + Dx\right),$$

$$P = \frac{2\mu\rho^2 k^2}{DN^2}\left\{-\frac{D(x^2+y^2}{2} + \tan^{-1}\left(\frac{y}{x}\right)\right\} - \frac{N^2}{2\rho(x^2+y^2)} - \frac{\eta}{\mathrm{K}}\ln(x^2+y^2) + P_0, \tag{77}$$

where N, P_0, D, and k are all arbitrary constants.

The magnetic lines in the radial flow are determined by $4\tan^{-1}(x/y) + Dx^2 + Dy^2 = constant$.

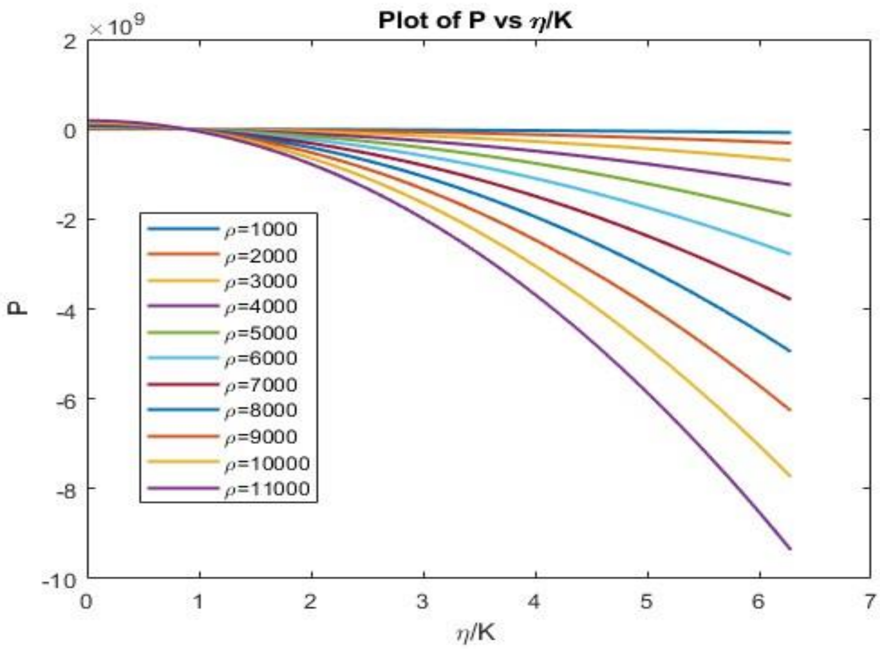

Figure 1: Pressure P variation with $\frac{\eta}{\mathrm{K}}$ for various fluid density ρ values.

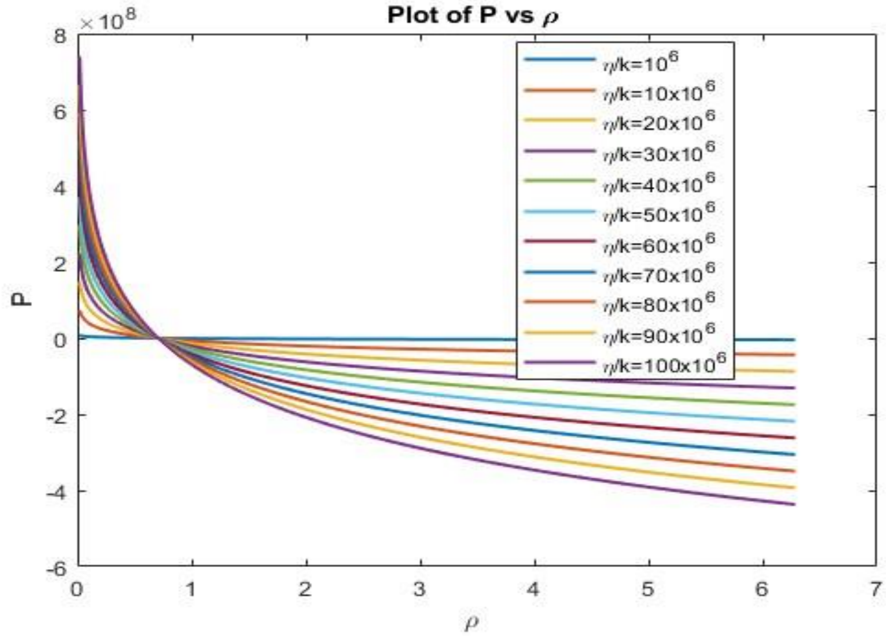

Figure 2: Pressure P variation with ρ the fluid density for various $\frac{\eta}{\kappa}$ values.

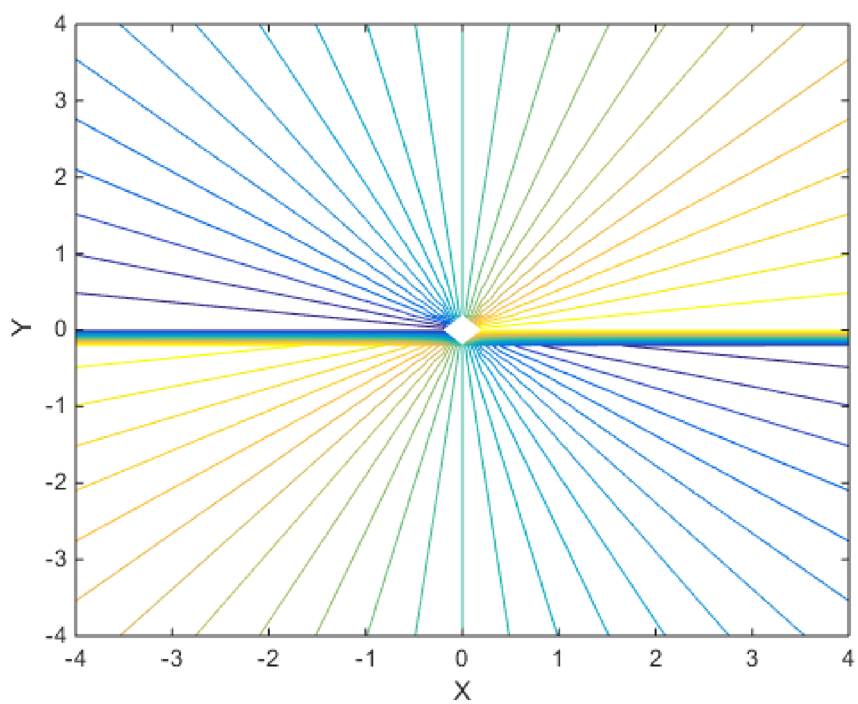

Figure 3: magnetic lines provided by $4\ \tan^{-1}(x/y) + Dx^2 + Dy^2 = constant$

In Radial flow:

- The current density is constant, and the vorticity is zero, as given by equations (76). Accordingly, equations (76) and (77) give the pressure, velocity components and also components of magnetic fields. The patterns of the magnetic lines are plotted in figure 3.
- Pressure decreases with constant $\frac{\eta}{K}$ and varying density ρ in figure 2, according to the plot of pressure against $\frac{\eta}{K}$. This shows that as the density ρ changes, the pressure variation rises, and the rate of drop also increases.
- Figure 3, evident from the pressure vs. fluid density ρ graphs for distinct values of $\frac{\eta}{K}$ that the pressure function lowers as density ρ increases. The pattern exhibits a consistent similarity across varying values of $\frac{\eta}{K}$.

4.2 Parallel Flows

This illustration shows flows in which the velocity vector is consistently parallel to the x-axis, which are taken into consideration. Using rectangular coordinates, we write

$$ds^2 = dx^2 + dy^2 = Ed\phi^2 + G\,\psi^2, \tag{78}$$

then, examine those flows where $\phi = \phi(x),\ \psi = (y)$. Therefore, (76) provides

$$E = \frac{1}{\phi'^2},\quad G = \frac{1}{\psi'^2},\quad J = \frac{1}{\phi'(x)\psi'(y)}\,. \tag{79}$$

Equation (41) is automatically satisfied, and the remaining equations can be simplified to

$$\omega = -\frac{\psi''}{\rho}\ , \tag{80}$$

$$j = \frac{\rho k}{\psi'}\left\{\left(\frac{\psi''}{\psi'}\right)^2 x - \left(\frac{\psi''}{\psi'}\right)' x - \frac{\psi''}{\psi'} f - f'\right\}, \tag{81}$$

$$\cot\theta = \frac{\psi''}{\psi'} x + f(y)\ , \tag{82}$$

$$\cot\theta\frac{\partial j}{\partial x} + \frac{\partial j}{\partial y} - \frac{\eta}{K}\frac{\psi''}{\mu\rho k\psi'^2} = \frac{\eta\psi'\psi^{iv}}{\mu\rho^2 k}, \tag{83}$$

where f(y) an unknown function of y, has been located. Using (82) in (83) and comparing the coefficients of powers of x we have,

$$\frac{2\psi''}{\psi'}\left(\frac{\psi''}{\psi'}\right)' - \left(\frac{\psi''}{\psi'}\right)'' = 0, \tag{84}$$

$$\frac{\eta}{K}\frac{\psi''}{\mu\rho k\psi'^2} - \frac{\eta\psi'\psi^{iv}}{\mu\rho^2 k^2} + \frac{1}{\psi'}\left\{\left(\frac{\psi''}{\psi'}\right)^2 - \left(\frac{\psi''}{\psi'}\right)'\right\} f + \left(\frac{\psi''}{\psi'} f\right)' - \left(\frac{f'}{\psi'}\right)' = 0, \tag{85}$$

Various distinct cases result from solving (84) as

Case I

$\psi'' = 0,\ \textit{This is}\ \psi' = N; constant,\ \mathrm{Cot}\,\theta = Ay + A_{0,}$

$$u = \frac{N}{\rho},\quad v = 0,\quad \omega = 0, \tag{86}$$

$$j = -\frac{\rho k A}{N},\ \ H_1 = \frac{\rho k}{N}(Ay + A_1),\ \ H_2 = \frac{\rho k}{N}, \tag{87}$$

$$P = \frac{\mu A\rho^2 k^2}{N^2}x - \frac{\mu A\rho^2 k^2}{N^2}\left(\frac{Ay^2}{2} + A_1 y\right) + \frac{\eta}{K}\frac{N}{\rho}x + A_2 \ , \quad (88)$$

Where A, $A_{0,}, A_{1,}, A_{2\,,} N > 0$ are random constants. The magnetic lines in the uniform parallel flow are determined by $\left(\frac{A}{2}\right)y^2 + A_1 y - x = constant.$

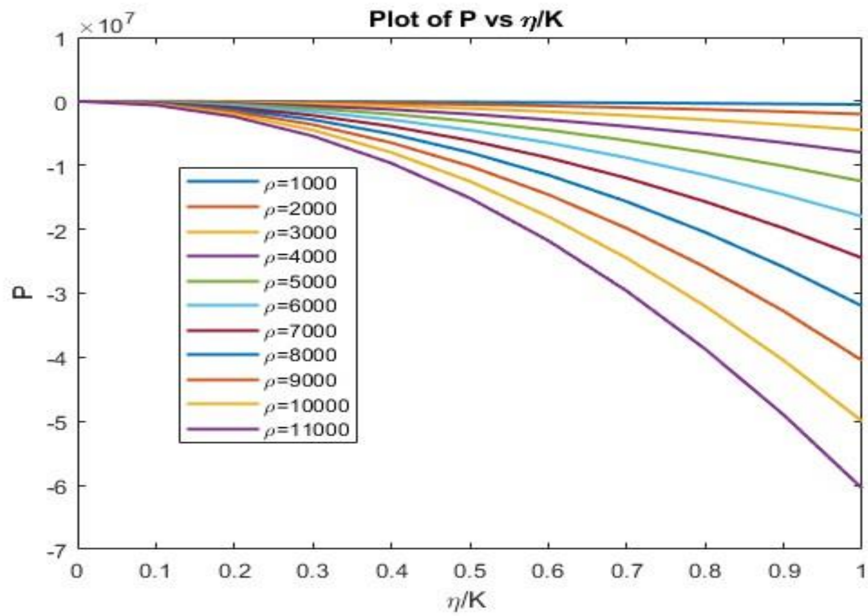

Figure 4: Pressure P variation with $\frac{\eta}{K}$ for various fluid density ρ values.

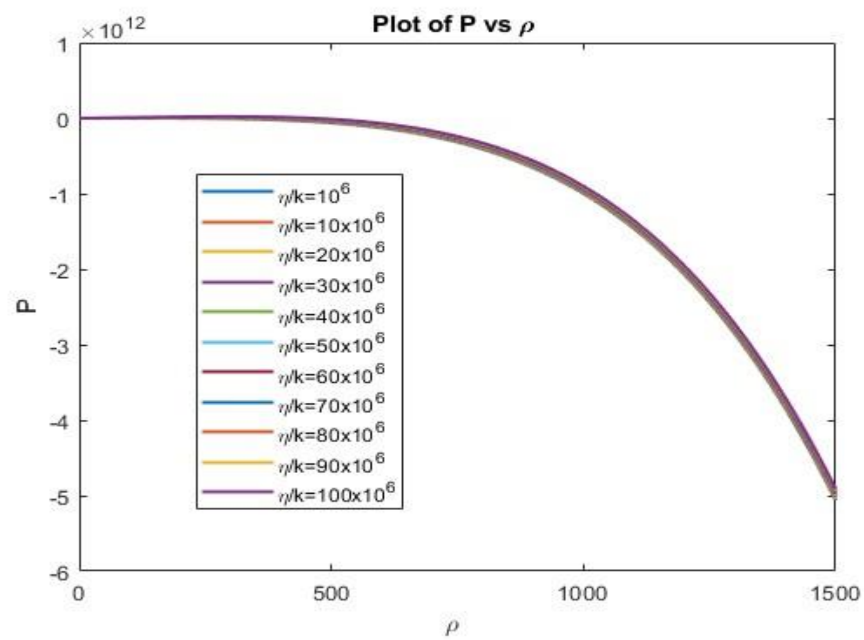

Figure 5: Pressure P variation with ρ, the fluid density for various $\frac{\eta}{K}$ values.

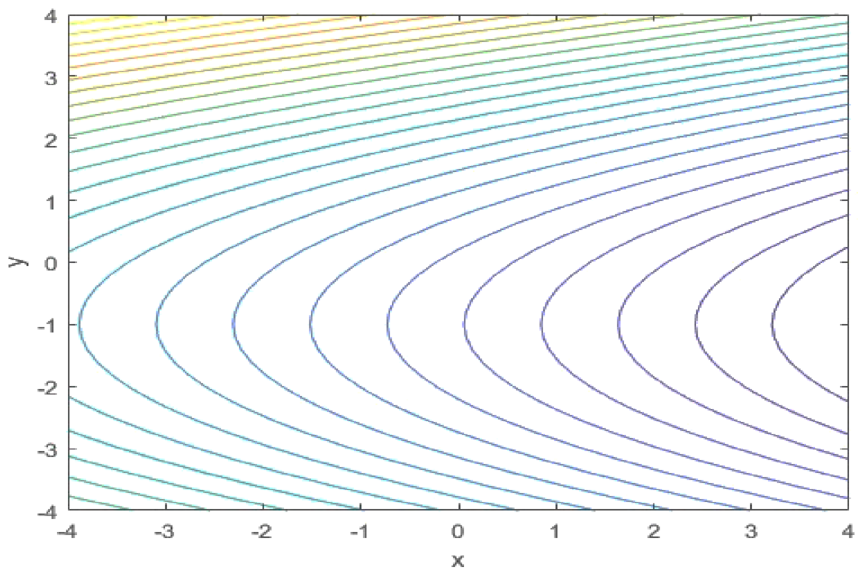

Figure 6: magnetic lines provided by $\left(\frac{A}{2}\right)y^2 + A_1 y - x = constant$

Case II.

$\left(\frac{\psi''}{\psi'}\right)' = 0,\ \psi'' \neq 0.$ In the present situation, the flow is described as $\psi' = B_0 \exp(B_1 y)$,

$\cot\theta = B_1 - \frac{\eta B_1 {B_0}^2}{3\mu\rho^3 k^2}\exp(3B_1 y) + \frac{B_2}{2B_1}\exp(2B_1 y) + B_3$,

$$u = \frac{B_0}{\rho}\exp(B_1 y),\quad v = 0,\quad \omega = -\frac{B_1 B_0}{\rho}\exp(B_1 y), \tag{89}$$

$$j = \frac{\rho k}{B_0}\left\{\left({B_1}^2 x + B_1 B_3\right)\exp(-B_1 y) - \frac{B_2}{2}\exp(2B_1 y) + \frac{2\eta {B_1}^2 {B_0}^2}{3\mu\rho^3 k^2}\exp(2B_1 y)\right\}, \tag{90}$$

$$H_1 = \frac{\rho k}{B_0}\left\{(B_1 x + B_3)\exp(-B_1 y) + \frac{B_2}{2B_1}\exp(B_1 y) - \frac{\eta {B_1}^2 {B_0}^2}{3\mu\rho^3 k^2}\exp(2B_1 y)\right\}, \tag{91}$$

$H_2 = \frac{\rho K}{B_0} exp(-B_1 y)$,

$$P = -\frac{\mu\rho^2 k^2}{{B_0}^2}\left\{\frac{{B_1}^2 x^2}{2} + B_1 B_3 x + \frac{{B_3}^2}{2}\right\}\exp(-2B_1 y) + \frac{\mu\rho^2 k^2 B_2}{2{B_0}^2}x - \frac{\mu\rho^2 k^2 {B_2}^2}{8{B_1}^2 {B_0}^2}\exp(2B_1 y) + \frac{\eta B_0 B_2}{6\rho}\exp(3B_1 y) + \frac{\eta B_1 B_0}{3\rho}\{B_1 x + B_3\}\exp(B_1 y) - \frac{\eta^2 {B_1}^2 {B_0}^4}{18\mu\rho^4 k^2}\exp(4B_1 y) + \frac{\eta B_2}{K\rho}\exp(2B_1 y)\,x + B_4 \tag{92}$$

where $B_1 \neq 0,\ B_0 \neq 0, B_{2,}\ B_3,\ B_4\ \ are$ arbitrary constants.

This illustrates a non-uniform parallel flow using magnetic lines that are provided as

$\{B_1 x + B_3\}\exp(-B_1 y) - \frac{B_2}{2B_1}\exp(B_1 y) - \frac{\eta B_1 {B_0}^3}{6\mu\rho^3 k^2}\exp(2B_1 y) = constant.$

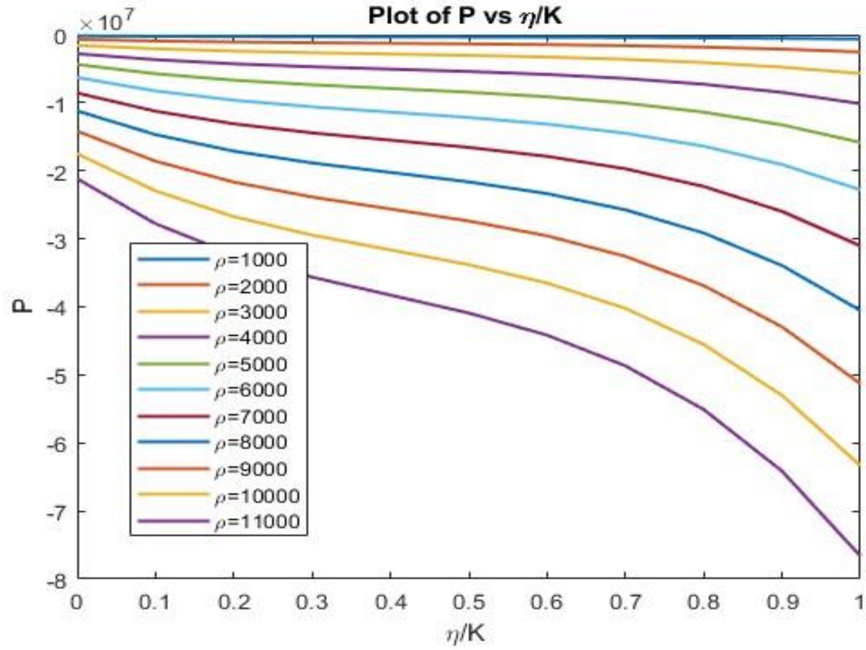

Figure 7: Pressure P variation with $\frac{\eta}{\kappa}$ for different fluid density levels.

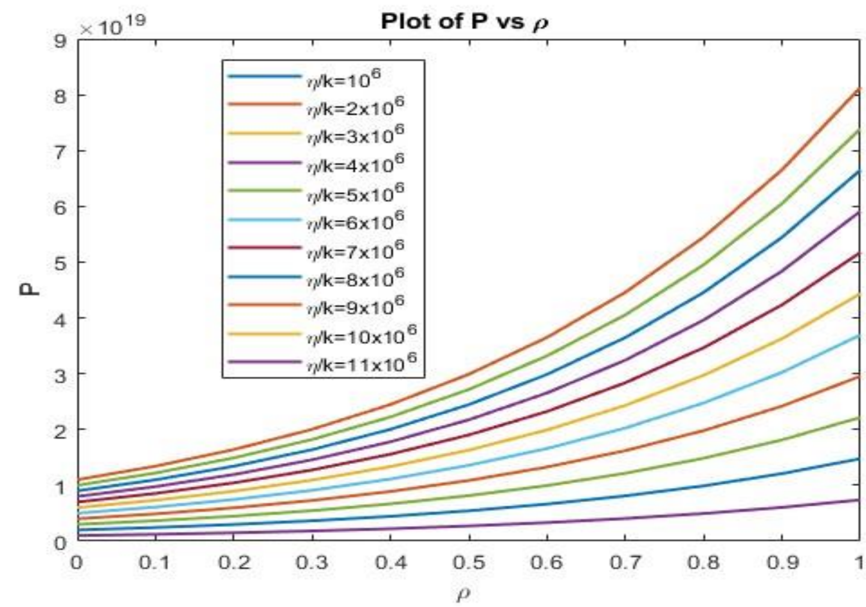

Figure 8: Pressure P variation with ρ for various $\frac{\eta}{\kappa}$ values.

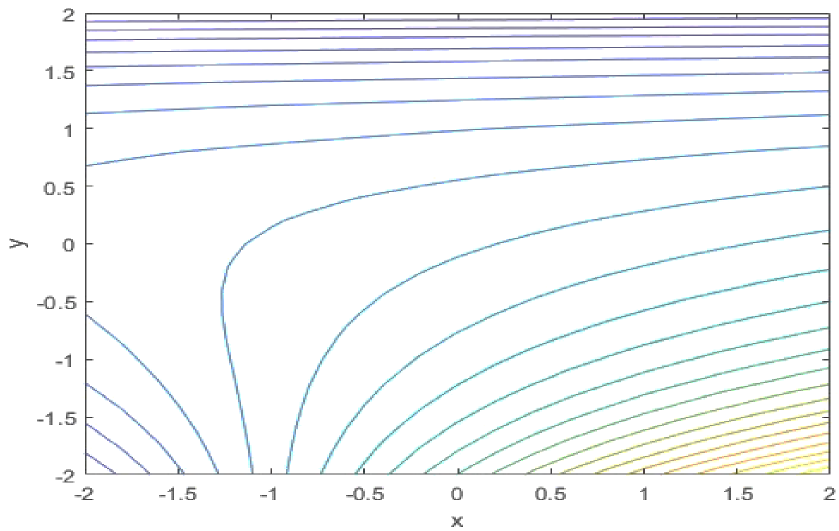

Figure 9: Magnetic lines provided by $\{B_1 x + B_3\}\exp(-B_1 y) - \frac{B_2}{2B_1}\exp(B_1 y) - \frac{\eta B_1 B_0{}^3}{6\mu\rho^3 11k^2}\exp(2B_1 y) = constant.$

In Parallel flow:

- The magnetic lines in parallel flow for Case I can be seen in figures 6, and for Case II, it can be seen in figures 9. We also observe that the current density is constant and the vorticity is zero, as given by equations (86) for case I and from equations (86) to (92) accordingly provide the velocity components, components of magnetic fields and pressure, respectively, for both case I and II.
- The pressure function changes as the ratio $\frac{\eta}{\mathrm{K}}$ varies for different values of density ρ, as seen in figure 4 for case I. The graphs indicate that the pressure variation P decreases progressively at higher values of density ρ, and also, the nonlinearity of the curves suggests that P and $\frac{\eta}{\mathrm{K}}$ have a nonlinear relationship.
- Figure 5 for case I plot of pressure against density ρ at constant $\frac{\eta}{\mathrm{K}}$ shows that for various $\frac{\eta}{\mathrm{K}}$ values nearly overlap, it can be concluded that changes in $\frac{\eta}{\mathrm{K}}$ have little effect on the pressure variation with respect to ρ.
- The plot of pressure against density $\frac{\eta}{\mathrm{K}}$ at fixed density ρ in figure 7 for case II indicates that the decrease in pressure is nonlinear and that greater densities lead to lower pressures for the same $\frac{\eta}{\mathrm{K}}$.
- Figure 8 for Case II shows how pressure variation changes with density ρ for different $\frac{\eta}{\mathrm{K}}$ values . As $\frac{\eta}{\mathrm{K}}$ increases, the pressure P also increases for the same density value, and higher viscous resistance (or lower permeability) leads to higher pressures. The relationship between pressure P and density ρ is nonlinear for all values of $\frac{\eta}{\mathrm{K}}$.

5. Conclusions

The steady, variably inclined viscous MHD fluid flow with infinite electrical conductivity across porous formation has been examined in this paper. A method for determining the exact solution has been implemented. Expressions for current density, pressure distribution, vorticity, velocity, and magnetic field are determined. Additionally, variations in the pressure function and magnetic line patterns with different fluid density ρ and $\frac{\eta}{\mathrm{K}}$ values have also been sketched in the graphs. This study presents a more generalized framework, as the absence of porous media leads to the retrieval of the findings of Chandna, Barron, and Chew (1983). The main findings from the present work are given below:

- Both radial flow and parallel flow, the current density function j, the vorticity function ω, the velocity component of u and v, the component of the magnetic field H, and the function of pressure P are determined.
- Plotting of the pressure variation is done for each case, for both changing the density ρ of various fluids at constant $\frac{\eta}{\kappa}$ value and changing the porous media parameter $\frac{\eta}{\kappa}$ (the permeability of the medium is varied.)

Acknowledgement: The Authors are thankful to BIT Sindri, Dhanbad, and P. K. Roy Memorial College, Dhanbad, for providing research facilities.

Conflict of Interest: The Authors declare that there is no conflict of interest.

Ethics approval: This study was conducted following ethical standards, with approval and consent obtained from all individual participants included in the study.

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