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# PROPERTIES OF STOCK OPTION PRICES USING A MARTINGALE APPROACH IN AN AMBIGUOUS SETTING

K. MEENAKSHI[1], S. SATHISH [2], RAJASEKARAN.P [3], R. SATHESH RAAJ [4], AND PRABAKARAN N*

Abstract. Information pertaining to financial investments may not always be exact or accurate. In such situations, fuzzy environments produced better effective results. This study examines the stock price behaviour to forecast the erratic price movements in the volatile financial market. This would support investors in their analysis of the stocks that they purchase and help them to make an intelligent choice.

## 1. Introduction

The binomial approach to option pricing was first put up by Cox, Ross, and Rubinstein [2] in the year 1979. Later, Black-Scholes [1] established European option in the early 1970s and it has had tremendous impact on trading. However, conventional models are inadequate for dealing with undetermined and unclear scenario, as it frequently affects the unstable financial markets. Therefore, it is crucial to examine the fuzzy options due to its siginificant importance. In 2008, Li et al. [10] developed a stock model for European option formula and studied some its mathematical properties. Further, Liu [4] introduced fuzzy calculus to finance, where stock prices were determined through a geometric process. Peng [9] proposed a more general model to elucidate American option formula. Several scholars have used a variety of derivative options in their research [3, 8, 11, 12, 15, 13, 17, 18]. The behaviour of options in a fuzzy setting are explored in this paper. The design of the paper is as follows:
Section 2, reviews the fundamental resluts which are important to analyse the present work. Further, section 3 deals with the siginificance of martingale in a fuzzy approach to compute fuzzy prices. In section 4, some properties of the same are iilustrated. Finally, section 5 gives the conclusion.

## 2. Basic Results

The key definitions and results from [14, 16, 7, 5, 6] which are required to comprehend price movements in a fluctuating market are summarised.

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## 3. Impact of conditional expectation with fuzzy random variables on investment strategy

**Theorem 3.1.** *Suppose that* $\widetilde{X}_1$ *and* $\widetilde{X}_2$ *are integrably bounded fuzzy random variables. Then,*
*(i)* $E(\widetilde{X}_1 + \widetilde{X}_2) \approx E(\widetilde{X}_1) + E(\widetilde{X}_2)$.
*(ii) If* $\widetilde{X}_1 \preceq \widetilde{X}_2$, *then* $E(\widetilde{X}_1) \preceq E(\widetilde{X}_2)$.

*Proof.* (i) For every $x \in X$, we have
$E(\widetilde{X}_1 + \widetilde{X}_2)(x) = \sup_{\alpha\in[0,1]}\min\{\alpha, 1_{E(\widetilde{X}_1+\widetilde{X}_2)_\alpha}(x)\}$.
$\implies E(\widetilde{X}_1 + \widetilde{X}_2)(x) = \sup_{\alpha\in[0,1]}\min\{\alpha, 1_{E(\widetilde{X}_1)_\alpha}(x)\} + \sup_{\alpha\in[0,1]}\min\{\alpha, 1_{E(\widetilde{X}_2)_\alpha}(x)\}$
$\implies E(\widetilde{X}_1 + \widetilde{X}_2)(x) = E(\widetilde{X}_1)(x) + E(\widetilde{X}_2)(x)$
$\implies E(\widetilde{X}_1 + \widetilde{X}_2) \approx E(\widetilde{X}_1) + E(\widetilde{X}_2)$.
(ii) Since $\widetilde{X}_1 \preceq \widetilde{X}_2 \implies (\widetilde{X}_1)_\alpha \leq (\widetilde{X}_2)_\alpha$
i.e., $\widetilde{X}_1 \preceq \widetilde{X}_2 \implies E(\widetilde{X}_1)_\alpha \leq E(\widetilde{X}_2)_\alpha$
Hence, for every $x \in X$, we have
$E(\widetilde{X}_1)(x) = \sup_{\alpha\in[0,1]}\min\{\alpha, 1_{E(\widetilde{X}_1)_\alpha}(x)\}$.
$\implies E(\widetilde{X}_1)(x) \leq \sup_{\alpha\in[0,1]}\min\{\alpha, 1_{E(\widetilde{X}_2)_\alpha}(x)\}$
$\implies E(\widetilde{X}_1)(x) \leq E(\widetilde{X}_2)(x) \implies E(\widetilde{X}_1) \preceq E(\widetilde{X}_2)$. □

**Theorem 3.2.** *(i)* $E(\widetilde{A}\widetilde{X}_1 + \widetilde{B}\widetilde{X}_2/\mathcal{N}) \approx \widetilde{A}E(\widetilde{X}_1/\mathcal{N}) + \widetilde{B}E(\widetilde{X}_2/\mathcal{N})$, *where* $\widetilde{A}$ *and* $\widetilde{B}$ *are any two positive constant linear fuzzy numbers.*
*(ii)* $E(E(\widetilde{X}/\mathcal{N})) \approx E(\widetilde{X})$.
*(iii) If* $\widetilde{X}$ *is* $\mathcal{N}-$ *measurable, then* $E(\widetilde{X}/\mathcal{N}) \approx \widetilde{X}$.
*(iv) If* $\widetilde{X}_1 \preceq \widetilde{X}_2$, *then* $E(\widetilde{X}_1/\mathcal{N}) \preceq E(\widetilde{X}_2/\mathcal{N})$.
*(v) If* $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *are sub* $\sigma$ *- algebras of* $\mathcal{N}$ *such that* $\mathcal{N}_1 \subset \mathcal{N}_2$, *then* $E(E(\widetilde{X}/\mathcal{N}_1)/\mathcal{N}_2) \approx E(\widetilde{X}/\mathcal{N}_1) \approx E(E(\widetilde{X}/\mathcal{N}_2)/\mathcal{N}_1)$.

*Proof.* (i), (ii) and (iii) can be proved easily.
(iv) For every $\Lambda \in \mathcal{N}$, if $\widetilde{X}_1(\omega) \preceq \widetilde{X}_2(\omega)$, then we have
$\int_\Lambda (\widetilde{X}_1)^\pm_\alpha(\omega)dP(\omega) \leq \int_\Lambda (\widetilde{X}_2)^\pm_\alpha(\omega)dP(\omega)$.
$\int_\Lambda E((\widetilde{X}_1)^\pm_\alpha/\mathcal{N})(\omega)dP(\omega) = \int_\Lambda (\widetilde{X}_1)^\pm_\alpha(\omega)dP(\omega)$
$\implies \int_\Lambda E((\widetilde{X}_1)^\pm_\alpha/\mathcal{N})(\omega)dP(\omega) \leq \int_\Lambda (\widetilde{X}_2)^\pm_\alpha(\omega)dP(\omega) \implies \int_\Lambda E((\widetilde{X}_1)^\pm_\alpha/\mathcal{N})(\omega)dP(\omega) \leq \int_\Lambda E((\widetilde{X}_2)^\pm_\alpha/\mathcal{N})(\omega)dP(\omega)$, $\omega \in \Omega$ and $x \in \mathbb{R}$
$\implies E(\widetilde{X}_1/\mathcal{N}) \preceq E(\widetilde{X}_2/\mathcal{N})$.
(v) If $\mathcal{N}_1 \subset \mathcal{N}_2$, then we have $E(\widetilde{X}/\mathcal{N}_1)$ is
$\mathcal{N}_2-$ measurable.
Using (iv), $E(E(\widetilde{X}/\mathcal{N}_1)/\mathcal{N}_2) \approx E(\widetilde{X}/\mathcal{N}_1)$.
For every $\Lambda \in \mathcal{N}_1 \subset \mathcal{N}_2$, since
$\int_\Lambda E(\widetilde{X}/\mathcal{N}_2)(\omega)dP(\omega) \approx \int_\Lambda \widetilde{X}(\omega)dP(\omega)$,
we have, $E(E(\widetilde{X}/\mathcal{N}_2)/\mathcal{N}_1) \approx E(\widetilde{X}/\mathcal{N}_1)$.
Hence, $E(E(\widetilde{X}/\mathcal{N}_1)/\mathcal{N}_2) \approx E(\widetilde{X}/\mathcal{N}_1)$
$\approx E(E(\widetilde{X}/\mathcal{N}_2)/\mathcal{N}_1)$. □

**Definition 3.3.** A discrete time fuzzy stochastic process $\widetilde{Y} \approx \left\{\widetilde{Y}_n, \mathcal{M}_n\right\}_{n=0}^{N}$ is a fuzzy martingale relative to a filtration $\{\mathcal{M}_n\}_{n=0}^{N}$ for each $n = 0, 1, \ldots, N$ if
(i) $E(\widetilde{Y}_{n+1}/\mathcal{M}_n) \approx \widetilde{Y}_n$. Further if $\approx$ in (i) is replaced by ($\preccurlyeq$ or $\succcurlyeq$), we have
(ii) $E(\widetilde{Y}_{n+1}/\mathcal{M}_n) \preccurlyeq \widetilde{Y}_n$, then $\left\{\widetilde{Y}_n, \mathcal{M}_n\right\}_{n=0}^{N}$ is a fuzzy supermartingale with respect to $\{\mathcal{M}_n\}_{n=0}^{N}$
(iii) and $E(\widetilde{Y}_{n+1}/\mathcal{M}_n) \succcurlyeq \widetilde{Y}_n$, then $\left\{\widetilde{Y}_n, \mathcal{M}_n\right\}_{n=0}^{N}$ is a fuzzy submartingale equipped to a filtration $\{\mathcal{M}_n\}_{n=0}^{N}$.

In $\widetilde{\mathcal{X}}^1$ and $\widetilde{\mathcal{X}}^2$, we note that 1 and 2 refers to the index and not power.

**Theorem 3.4.** *Let $\widetilde{\mathcal{X}}^1 \approx \left\{\widetilde{X}^1_n, \mathcal{M}_n\right\}_{n=0}^{N}$ and $\widetilde{\mathcal{X}}^2 \approx \left\{\widetilde{X}^2_n, \mathcal{M}_n\right\}_{n=0}^{N}$ be two fuzzy submartingales with respect to $\{\mathcal{M}_n\}_{n=0}^{N}$. Then for any two positive constant linear fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$, $\widetilde{A}\widetilde{\mathcal{X}}^1 + \widetilde{B}\widetilde{\mathcal{X}}^2 \approx \left\{\widetilde{A}\widetilde{X}^1_n + \widetilde{B}\widetilde{X}^2_n, \mathcal{M}_n\right\}_{n=0}^{N}$ is a fuzzy submartingale with respect to $\{\mathcal{M}_n\}_{n=0}^{N}$.*

*Proof.* By Theorem 3.2, we have
$E(\widetilde{A}\widetilde{X}^1_{n+1} + \widetilde{B}\widetilde{X}^2_{n+1}/\mathcal{M}_n)$
$\approx \widetilde{A}E(\widetilde{X}^1_{n+1}/\mathcal{M}_n) + \widetilde{B}E(\widetilde{X}^2_{n+1}/\mathcal{M}_n)$.
If $\left\{\widetilde{X}^1_n, \mathcal{M}_n\right\}_{n=0}^{N}$ and $\left\{\widetilde{X}^2_n, \mathcal{M}_n\right\}_{n=0}^{N}$ are fuzzy submartingales, then we have
$E(\widetilde{X}^1_{n+1}/\mathcal{M}_n) \succeq \widetilde{X}^1_n$ and $E(\widetilde{X}^2_{n+1}/\mathcal{M}_n) \succeq \widetilde{X}^2_n$.
Since $E(\widetilde{X}^1_{n+1}/\mathcal{M}_n) \succeq \widetilde{X}^1_n$ and $\widetilde{A} \succeq \widetilde{0}$
$\implies \widetilde{A}E(\widetilde{X}^1_{n+1}/\mathcal{M}_n) \succeq \widetilde{A}\widetilde{X}^1_n$.
Similarly, if $E(\widetilde{X}^2_{n+1}/\mathcal{M}_n) \geq \widetilde{X}^2_n$ and $\widetilde{B} \succeq \widetilde{0}$
$\implies \widetilde{B}E(\widetilde{X}^2_{n+1}/\mathcal{M}_n) \succeq \widetilde{B}\widetilde{X}^2_n$.
$\implies E(\widetilde{A}\widetilde{X}^1_n + \widetilde{B}\widetilde{X}^2_n/\mathcal{M}_n)$
$\succeq \widetilde{A}E(\widetilde{X}^1_n/\mathcal{M}_n) + \widetilde{B}E(\widetilde{X}^2_n/\mathcal{M}_n)$. □

*Remark* 3.5. Similar to Theorem 3.4, the results of fuzzy martingale and fuzzy supermartingale will follow immediately.

## 4. Properties of option prices in a fuzzy set up

**Definition 4.1.** The fuzzy pay off process of American and European fuzzy put options models on fuzzy future contracts are defined as $\widetilde{P}^{\mathcal{A}P}_n(\widetilde{F}_{n,i}) \approx \widetilde{P}^{\mathcal{E}P}_n(\widetilde{F}_{n,i}) \approx \widetilde{K} - \widetilde{F}_{n,i}$, for $n = 0, 1, \ldots, N$ and $i = 0, 1, \ldots, N$.

**Definition 4.2.** The fuzzy pay off process of American and European fuzzy call options models on fuzzy future contracts are defined as $\widetilde{P}^{\mathcal{A}C}_n(\widetilde{F}_{n,i}) \approx \widetilde{P}^{\mathcal{E}C}_n(\widetilde{F}_{n,i}) \approx \widetilde{F}_{n,i} - \widetilde{K}$, for $n = 0, 1, \ldots, N$ and $i = 0, 1, \ldots, N$.

**Proposition 4.3.**
$$\widetilde{V}_n(\widetilde{F}_{n,i}) \succcurlyeq max\{\widetilde{K} - \widetilde{F}_{n,i}, \widetilde{0}\} \tag{4.1}$$
*for n=0,1,...,N and i=0,1,...,N.*

*Proof.* We prove equation (4.1) by backward induction on $n$. When $n = N$: we have
$\widetilde{V}_N(\widetilde{F}_{N,i}) \succcurlyeq \max\{\widetilde{K} - \widetilde{F}_{N,i}, \widetilde{0}\}$
Assume equation (4.1) is true for $n = N - 1$, then we have
$\widetilde{V}_{n+1}(\widetilde{F}_{n+1,i}) \succcurlyeq max\{\widetilde{K} - \widetilde{F}_{n+1,i}, \widetilde{0}\}$.
Hence, we have $\widetilde{V}_n(\widetilde{F}_{n,i}) \succcurlyeq max\{\widetilde{K} - \widetilde{F}_{n,i}, \widetilde{0}\}$. □

*Remark* 4.4. In line with Proposition 4.3, we can prove $\widetilde{V}_n(\widetilde{F}_{n,i}) \succcurlyeq max\{\widetilde{F}_{n,i} - \widetilde{K}, \widetilde{0}\}$, for$n = 0, 1, \ldots, N$
and $i = 0, 1, \ldots, N$.

**Proposition 4.5.** *Suppose that $\widetilde{V}^{\mathcal{A}P_1}, \widetilde{V}^{\mathcal{A}P_2}$ and $\widetilde{V}^{\mathcal{A}_3}$ are the fuzzy intrinsic values of American fuzzy put option model on fuzzy future contract corresponding to the constant fuzzy strike prices $\widetilde{K}_1 \succcurlyeq \widetilde{K}_2 \succcurlyeq \widetilde{K}_3$ such that $\widetilde{K}_3 - \widetilde{K}_2 \approx \widetilde{K}_2 - \widetilde{K}_1$ for the same underlying fuzzy stock and for the same expiration date $N$, then*

$$\widetilde{V}^{\mathcal{A}_1} + \widetilde{V}^{\mathcal{A}P_3} \succcurlyeq 2\widetilde{V}^{\mathcal{A}_2}. \tag{4.2}$$

*Proof.* We show equation (4.2) by reverse induction on $n$.
To prove equation (4.2) for $n = N$.
$\widetilde{V}_N^{\mathcal{A}P_1}(\widetilde{F}_N^{uu\ldots u})_{\widetilde{K}_1} + \widetilde{V}_N^{\mathcal{A}P_3}(\widetilde{F}_N^{uu\ldots u})_{\widetilde{K}_3} \approx$
$\max\{\widetilde{K}_1 - \widetilde{F}_N^{uu\ldots u}, \widetilde{0}\} + \max\{\widetilde{K}_3 - \widetilde{F}_N^{uu\ldots u}, \widetilde{0}\}$
$\Longrightarrow \widetilde{V}_N^{\mathcal{A}P_1}(\widetilde{F}_N^{uu\ldots u})_{\widetilde{K}_1} + \widetilde{V}_N^{\mathcal{A}P_3}(\widetilde{F}_N^{uu\ldots u})_{\widetilde{K}_3} \approx$
$\max\{2\widetilde{K}_2 - 2\widetilde{F}_N^{uu\ldots u}, \widetilde{0}\}$
(since $\widetilde{K}_3 - \widetilde{K}_2 \approx \widetilde{K}_2 - \widetilde{K}_1$)
$\Longrightarrow \widetilde{V}_N^{\mathcal{A}P_1}(\widetilde{F}_N^{uu\ldots u})_{\widetilde{K}_1} + \widetilde{V}_N^{\mathcal{A}P_3}(\widetilde{F}_N^{uu\ldots u})_{\widetilde{K}_3} \approx$
$2\max\{\widetilde{K}_2 - \widetilde{F}_N^{uu\ldots u}, \widetilde{0}\}$
$\Longrightarrow \widetilde{V}_N^{\mathcal{A}P_1}(\widetilde{F}_N^{uu\ldots u})_{\widetilde{K}_1} + \widetilde{V}_N^{\mathcal{A}P_3}(\widetilde{F}_N^{uu\ldots u})_{\widetilde{K}_3} \approx$
$2\widetilde{V}_N^{\mathcal{A}P_2}(\widetilde{F}_N^{uu\ldots u})_{\widetilde{K}_2}$
$\widetilde{V}_N^{\mathcal{A}P_1}(\widetilde{F}_N^{uu\ldots u})_{\widetilde{K}_1} + \widetilde{V}_N^{\mathcal{A}P_3}(\widetilde{F}_N^{uu\ldots u})_{\widetilde{K}_3} \succcurlyeq 2\widetilde{V}_N^{\mathcal{A}P_2}(\widetilde{F}_N^{uu\ldots u})_{\widetilde{K}_2}$.
Similarly, we can prove it is false for other states. Let us assume that equation (4.2) fails to hold for $n + 1$.

$$\widetilde{V}_{n+1}^{\mathcal{A}P_1}(\widetilde{F}_{n+1}^{uu\ldots u})_{\widetilde{K}_1} + \widetilde{V}_{n+1}^{\mathcal{A}P_3}(\widetilde{F}_{n+1}^{uu\ldots u})_{\widetilde{K}_3} \preceq 2\widetilde{V}_{n+1}^{\mathcal{A}P_2}(\widetilde{F}_{n+1}^{uu\ldots u})_{\widetilde{K}_2}$$

and it does not hold for other states. Now we prove equation (4.2) is false for $n$.
$\widetilde{V}_n^{\mathcal{A}P_1}(\widetilde{F}_n^{uu\ldots u})_{\widetilde{K}_1} + \widetilde{V}_n^{\mathcal{A}P_3}(\widetilde{F}_n^{uu\ldots u})_{\widetilde{K}_3} \approx$
$\max\left(\widetilde{K}_1 - \widetilde{F}_n^{uu\ldots u}, \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})}\left(\widetilde{P}_u\widetilde{V}_{n+1}^{\mathcal{A}P_1}(\widetilde{F}_{n+1}^{uu\ldots u})_{\widetilde{K}_1} + \widetilde{P}_d\widetilde{V}_{n+1}^{\mathcal{A}P_1}(\widetilde{F}_{n+1}^{uu\ldots d})_{\widetilde{K}_1}\right)\right) + \max\left(\widetilde{K}_3 - \widetilde{F}_n^{uu\ldots u}, \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})}\left(\widetilde{P}_u\widetilde{V}_{n+1}^{\mathcal{A}P_3}(\widetilde{F}_{n+1}^{uu\ldots u})_{\widetilde{K}_3} + \widetilde{P}_d\widetilde{V}_{n+1}^{\mathcal{A}P_3}(\widetilde{F}_{n+1}^{uu.}\right.\right.$
$\approx 2$
$\max\left(\widetilde{K}_2 - \widetilde{F}_n^{uu\ldots u}, \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})}\left(\widetilde{P}_u\widetilde{V}_{n+1}^{\mathcal{A}P_2}(\widetilde{F}_{n+1}^{uu\ldots u})_{\widetilde{K}_2} + \widetilde{P}_d\widetilde{V}_{n+1}^{\mathcal{A}P_2}(\widetilde{F}_{n+1}^{uu\ldots d})_{\widetilde{K}_2}\right)\right)$ (since $\widetilde{K}_3 - \widetilde{K}_2 \approx \widetilde{K}_2 - \widetilde{K}_1$)
$\Longrightarrow \widetilde{V}_n^{\mathcal{A}P_1}(\widetilde{F}_n^{uu\ldots u})_{\widetilde{K}_1} + \widetilde{V}_n^{\mathcal{A}P_3}(F_n^{uu\ldots u})_{\widetilde{K}_3} \succcurlyeq 2\widetilde{V}_n^{\mathcal{A}P_2}(\widetilde{F}_n^{uu\ldots u})_{\widetilde{K}_2}$
(by equation (**??**)
In a similar manner, we can prove it is false for other states i.e., a contradiction. Therefore equation (4.2) is true for $n$ and this implies equation (**??**) may false for $n + 1$. □

**Proposition 4.6.** *If $\widetilde{K}_1$ and $\widetilde{K}_2$ are the different constant fuzzy strike prices such that $\widetilde{K}_1 \succcurlyeq \widetilde{K}_2$ and if $\widetilde{V}_{\widetilde{K}_1}^{\mathcal{A}C}$ and $\widetilde{V}_{\widetilde{K}_2}^{\mathcal{A}C}$ are the corresponding fuzzy intrinsic values of American* *fuzzy call option model on fuzzy future contract for the same underlying fuzzy stock with the same expiration date $N$, then*

$$\widetilde{V}^{\mathcal{A}C}_{\widetilde{K}_1} \preceq \widetilde{V}^{\mathcal{A}C}_{\widetilde{K}_2}. \tag{4.3}$$

*Proof.* We prove equation (4.3) by reverse induction on $n$. First we prove for $n = N$.
$\widetilde{V}^{\mathcal{A}C}_N(\widetilde{F}^{uu\ldots u}_N)_{\widetilde{K}_1} \approx \max\{\widetilde{F}^{uu\ldots u}_N - \widetilde{K}_1, \widetilde{0}\}$
(since $\widetilde{K}_1 \succcurlyeq \widetilde{K}_2 \implies \widetilde{F}^{uu\ldots u}_N - \widetilde{K}_1 \preceq \widetilde{F}^{uu\ldots u}_N - \widetilde{K}_2$)
$\implies \widetilde{V}^{\mathcal{A}C}_N(\widetilde{F}^{uu\ldots u}_N)_{\widetilde{K}_1} \preceq \max\{\widetilde{F}^{uu\ldots u}_N - \widetilde{K}_2, \widetilde{0}\}$
$\implies \widetilde{V}^{\mathcal{A}C}_N(\widetilde{F}^{uu\ldots u}_N)_{\widetilde{K}_1} \preceq \widetilde{V}^{\mathcal{A}C}_N(\widetilde{F}^{uu\ldots u}_N)_{\widetilde{K}_2}$
Similarly, we can prove it for other states. Suppose that equation (4.3) does not hold for $n+1$, we have

$$\widetilde{V}^{\mathcal{A}C}_{n+1}(\widetilde{F}^{uu\ldots u}_{n+1})_{\widetilde{K}_1} \succeq \widetilde{V}^{\mathcal{A}C}_{n+1}(\widetilde{F}^{uu\ldots u}_{n+1})_{\widetilde{K}_2} \tag{4.4}$$

and it is false for other states. Now, we prove equation (4.3) fails to hold for $n$.
$\widetilde{V}^{\mathcal{A}C}_n(\widetilde{F}^{uu\ldots u}_n)_{\widetilde{K}_1} \approx \max$
$\left(\widetilde{F}^{uu\ldots u}_n - \widetilde{K}_1, \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})}\left(\widetilde{P}_u\widetilde{V}^{\mathcal{A}C}_{n+1}(\widetilde{F}^{uu\ldots u}_{n+1})_{\widetilde{K}_1} + \widetilde{P}_d\widetilde{V}^{\mathcal{A}C}_{n+1}(\widetilde{F}^{uu\ldots d}_{n+1})_{\widetilde{K}_1}\right)\right)$
Using equation (4.4), we have
$\widetilde{V}^{\mathcal{A}C}_n(\widetilde{S}^{uu\ldots u}_n)_{\widetilde{K}_1} \succeq$
$max\left(\widetilde{F}^{uu\ldots u}_n - \widetilde{K}_2, \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})}\left(\widetilde{P}_u\widetilde{V}^{\mathcal{A}C}_{n+1}(\widetilde{F}^{uu\ldots u}_{n+1})_{\widetilde{K}_2} + \widetilde{P}_d\widetilde{V}^{\mathcal{A}C}_{n+1}(\widetilde{F}^{uu\ldots d}_{n+1})_{\widetilde{K}_2}\right)\right)$
$\implies \widetilde{V}^{\mathcal{A}P}_n(\widetilde{F}^{uu\ldots u}_n)_{\widetilde{K}_1} \succeq \widetilde{V}^{\mathcal{A}C}_n(\widetilde{F}^{uu\ldots u}_n)_{\widetilde{K}_2}$.

Similarly, we can show it is false for other states. This is a contradiction. Since equation (4.3) holds for $n$ which imply equation (4.4) is false for $n+1$. □

**Proposition 4.7.** *Suppose that if $\widetilde{V}^{\mathcal{A}C}_{\widetilde{K}_1}$, $\widetilde{V}^{\mathcal{A}C}_{\widetilde{K}_2}$ are the fuzzy intrinsic values of American fuzzy call option model on fuzzy future contract for the same underlying fuzzy stock and having the same expiration date $N$ and if $\widetilde{K}_1 \preccurlyeq \widetilde{K}_2$, then $\widetilde{V}^{\mathcal{A}C}_{\widetilde{K}_2} - \widetilde{V}^{\mathcal{A}C}_{\widetilde{K}_1} \preccurlyeq \widetilde{K}_2 - \widetilde{K}_1$.*

*Proof.* By property (i) in Proposition 4.7, if $\widetilde{K}_1 \preccurlyeq \widetilde{K}_2$, then we have $\widetilde{V}^{\mathcal{A}}_{\widetilde{K}_1} \succeq \widetilde{V}^{\mathcal{A}P}_{\widetilde{K}_2}$.
(i.e) $\widetilde{V}^{\mathcal{A}P}_{\widetilde{K}_1} - \widetilde{V}^{\mathcal{A}P}_{\widetilde{K}_2} \succeq \widetilde{0}$. Since $\widetilde{K}_1 \preccurlyeq \widetilde{K}_2$, $\widetilde{K}_1 - \widetilde{K}_2 \preceq \widetilde{0}$.
Therefore $\widetilde{V}^{\mathcal{A}P}_{\widetilde{K}_1} - \widetilde{V}^{\mathcal{A}P}_{\widetilde{K}_2} \succeq \widetilde{K}_1 - \widetilde{K}_2$. □

*Remark* 4.8. Analogues of property proved in Proposition 4.5, we can also prove for American fuzzy call option model on fuzzy future contract.
In line with the Propositions 4.5, 4.6 and 4.7, similarly we can show them for European fuzzy put / call options models.

**Proposition 4.9.** *Suppose that the discounted fuzzy intrinsic values of the buyers of fuzzy put option model on fuzzy future contract and the discounted fuzzy intrinsic values of fuzzy call option sellers model on fuzzy future contract processes $\left\{\frac{\widetilde{V}^{\mathcal{A}P/\mathcal{A}C}_n}{(\widetilde{1}+\widetilde{R})^n}\right\}_{n=0}^N$ are $Q-$ fuzzy supermartingale processes and that $\left\{\frac{\widetilde{X}^{\mathcal{A}P/\mathcal{A}C}_n}{(\widetilde{1}+\widetilde{R})^n}\right\}_{n=0}^N$ is another $Q-$ fuzzy supermartingale process satisfying $\widetilde{X}^{\mathcal{A}P/\mathcal{A}C}_n(\widetilde{F}^{uuu\ldots u}_n) \succcurlyeq \widetilde{P}^{\mathcal{A}P/\mathcal{A}C}_n(\widetilde{F}^{uuu\ldots u}_n)$, then*

$$\widetilde{X}^{\mathcal{A}P/\mathcal{A}C}_n(\widetilde{F}^{uuu\ldots u}_n) \succcurlyeq \widetilde{V}^{\mathcal{A}P/\mathcal{A}C}_n(\widetilde{F}^{uuu\ldots u}_n), \tag{4.5}$$

*for $n = 0, 1, \ldots, N$ and it is true for other states.*

*Proof.* Since $\left\{\frac{\widetilde{V}_n^{\mathcal{A}P/\mathcal{A}C}}{(\widetilde{1}+\widetilde{R})^n}\right\}_{n=0}^{N}$ is a $Q-$ fuzzy supermartingale process and
$\widetilde{V}_n^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_n^{uuu\ldots u}) \succcurlyeq \widetilde{P}_n^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_n^{uuu\ldots u})$, then we have
$\widetilde{V}_n^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_n^{uuu\ldots u}) \succcurlyeq \frac{\widetilde{1}}{\widetilde{1}+\widetilde{R}}\widetilde{E}_n^Q(\widetilde{V}_{n+1}^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_{n+1}^{uuu\ldots u}))$, for
$n = 0, 1, \ldots, N$.
If $\left\{\frac{\widetilde{X}_n^{\mathcal{A}P/\mathcal{A}C}}{(\widetilde{1}+\widetilde{R})^n}\right\}_{n=0}^{N}$ is another $Q-$ fuzzy supermartingale process dominating $\widetilde{P}_n^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_n^{uuu\ldots u})$,
then we have $\widetilde{X}_n^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_n^{uuu\ldots u}) \succcurlyeq \widetilde{P}_n^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_n^{uuu\ldots u})$ and
$\widetilde{X}_n^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_n^{uuu\ldots u}) \succcurlyeq \frac{\widetilde{1}}{\widetilde{1}+\widetilde{R}}\widetilde{E}_n^Q(\widetilde{X}_{n+1}^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_{n+1}^{uuu\ldots u}))$, for $n = 0, 1, \ldots, N$.
We prove equation (4.5) by reverse induction on $n$.
When $n = N$: Suppose
$\widetilde{X}_N^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_N^{uuu\ldots u}) \succcurlyeq \widetilde{P}_N^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_N^{uuu\ldots u})$ and
$\widetilde{V}_N^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_N^{uuu\ldots u}) \succcurlyeq \widetilde{P}_N^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_N^{uuu\ldots u})$, then we have $\widetilde{X}_N^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_N^{uuu\ldots u}) \succcurlyeq \widetilde{V}_N^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_N^{uuu\ldots u})$.
Assume equation (4.5) is true for $n+1$.
i.e., we have $\widetilde{X}_{n+1}^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_{n+1}^{uuu\ldots u}) \preceq \widetilde{V}_{n+1}^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_{n+1}^{uuu\ldots u})$.
We now show equation (4.5) is true for $n$.
Since $\widetilde{X}_n^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_n^{uuu\ldots u}) \succcurlyeq$
$\frac{\widetilde{1}}{\widetilde{1}+\widetilde{R}}(\widetilde{E}_n^Q(\widetilde{X}_{n+1}^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_{n+1}^{uuu\ldots u})))$, we have
$\widetilde{X}_n^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_n^{uuu\ldots u}) \succcurlyeq \frac{\widetilde{1}}{\widetilde{1}+\widetilde{R}}(\widetilde{E}_n^Q(\widetilde{V}_{n+1}^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_{n+1}^{uuu\ldots u})))$.
If $\widetilde{X}_n^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_n^{uuu\ldots u}) \succcurlyeq \widetilde{P}_n^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_n^{uuu\ldots u})$ and
$\widetilde{X}_n^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_n^{uuu\ldots u}) \succcurlyeq \frac{\widetilde{1}}{\widetilde{1}+\widetilde{R}}\left(\widetilde{E}_n^Q(\widetilde{V}_{n+1}^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_{n+1}^{uuu\ldots u}))\right)$, then we have
$\widetilde{X}_n^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_n^{uuu\ldots u}) \succcurlyeq \max\left(\widetilde{P}_n^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_n^{uuu\ldots u}),\ \frac{\widetilde{1}}{\widetilde{1}+\widetilde{R}}\widetilde{E}_n^Q(\widetilde{V}_{n+1}^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_{n+1}^{uuu\ldots u}))\right)$.
$\implies \widetilde{X}_n^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_n^{uuu\ldots u}) \succcurlyeq \widetilde{V}_n^{\mathcal{A}P/\mathcal{A}C}(\widetilde{F}_n^{uuu\ldots u})$, for $n = 0, 1, \ldots, N$
and it is true for other states. In a similar fashion, we can prove equation (4.5) is true for $n = 1$ and $n = 0$. □

**Definition 4.10.** The fuzzy intrinsic values of European fuzzy put option model price on fuzzy stocks is defined by $\widetilde{V}_N^{\mathcal{EP}}(\widetilde{S}_{N,i}) \approx \max\left\{\widetilde{K} - \widetilde{S}_{N,i}, \widetilde{0}\right\}$, for $i = 0, 1, \ldots, N$ and $\widetilde{V}_n^{\mathcal{E}P}(\widetilde{S}_{n,i}) \approx \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})}\left(\widetilde{E}_n^Q(\widetilde{V}_{n+1}^{\mathcal{E}P}(\widetilde{S}_{n+1,i}))\right)$, where the expected fuzzy intrinsic values of European fuzzy put option price on fuzzy stocks at time $n+1$ is defined as
$\widetilde{E}_n^Q(\widetilde{V}_{n+1}^{\mathcal{E}P}(\widetilde{S}_{n+1,i})) \approx (\widetilde{P}_u\widetilde{V}_{n+1}^{\mathcal{E}P}(\widetilde{U}\widetilde{S}_{n,i}) + \widetilde{P}_d\widetilde{V}_{n+1}^{\mathcal{E}P}(\widetilde{D}\widetilde{S}_{n,i}))$, for $n = 0, 1, 2, \ldots, N-1$ and $i = 0, 1, \ldots, n$.

**Definition 4.11.** The fuzzy intrinsic values of European fuzzy call option model price on fuzzy stocks is defined by $\widetilde{V}_N^{\mathcal{EC}}(\widetilde{S}_{N,i}) \approx \max\left\{\widetilde{S}_{N,i} - \widetilde{K}, \widetilde{0}\right\}$, for $i = 0, 1, \ldots, N$ and $\widetilde{V}_n^{\mathcal{E}C}(\widetilde{S}_{n,i}) \approx \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})}\left(\widetilde{E}_n^Q(\widetilde{V}_{n+1}^{\mathcal{E}C}(\widetilde{S}_{n+1,i}))\right)$, where the expected fuzzy intrinsic values of European fuzzy call option price on fuzzy stocks at time $n+1$ is defined as
$\widetilde{E}_n^Q(\widetilde{V}_{n+1}^{\mathcal{E}C}(\widetilde{S}_{n+1,i})) \approx (\widetilde{P}_u\widetilde{V}_{n+1}^{\mathcal{E}C}(\widetilde{U}\widetilde{S}_{n,i}) + \widetilde{P}_d\widetilde{V}_{n+1}^{\mathcal{E}C}(\widetilde{D}\widetilde{S}_{n,i}))$, for $n = 0, 1, 2, \ldots, N-1$ and $i = 0, 1, \ldots, n$.

**Proposition 4.12.** *The discounted fuzzy intrinsic process* $\left\{\dfrac{\widetilde{V}_n^{\mathcal{E}P}}{(\widetilde{1}+\widetilde{R})^n}\right\}_{n=0}^{N}$ *for both buyers and sellers of European fuzzy put/call options models on fuzzy future contract is a* $Q-$ *fuzzy martingale.*

*Proof.* For proving the same, we have to show that

$$\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{n+1}}\widetilde{E}_n^Q(\widetilde{V}_{n+1}^{\mathcal{E}P/\mathcal{E}C}(\widetilde{F}_{n+1}^{uuu\ldots u}))\approx\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{n}}\widetilde{V}_{n}^{\mathcal{E}P/\mathcal{E}C}(\widetilde{F}_{n}^{uuu\ldots u}) \tag{4.6}$$

for $n=0,1,\ldots,N$ and it can be true for other states by reverse induction on $n$.
Assume (4.6) does not hold for $n=p+1$ and it is false for other states when $\widetilde{U}\widetilde{P}_u+\widetilde{D}\widetilde{P}_d\approx\widetilde{1}+\widetilde{R}$. Then we have the following two assumptions:
Assumption(i):

$$\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{p+2}}\widetilde{E}_{p+1}^Q(\widetilde{V}_{p+2}^{\mathcal{E}P/\mathcal{E}C}(\widetilde{F}_{p+2}^{uuu\ldots u}))\succeq\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{p+1}}\widetilde{V}_{p+1}^{\mathcal{E}P/\mathcal{E}C}(\widetilde{F}_{p+1}^{uuu\ldots u}) \tag{4.7}$$

Assumption (ii):

$$\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{p+2}}\widetilde{E}_{p+1}^Q(\widetilde{V}_{p+2}^{\mathcal{E}P/\mathcal{E}C}(\widetilde{F}_{p+2}^{uuu\ldots u}))\preceq\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{p+1}}\widetilde{V}_{p+1}^{\mathcal{E}P/\mathcal{E}C}(\widetilde{F}_{p+1}^{uuu\ldots u}) \tag{4.8}$$

Now to prove equation (4.6) is not true for $n=p$.
$\frac{1}{(\widetilde{1}+\widetilde{R})^{p+1}}\widetilde{E}_p^Q(\widetilde{V}_{p+1}^{\mathcal{E}P/\mathcal{E}C}(\widetilde{F}_{p+1}^{uuu\ldots u}))\approx$
$\frac{1}{(\widetilde{1}+\widetilde{R})^{p+1}}\widetilde{E}_p^Q(\widetilde{V}_{p+1}^{\mathcal{E}P/\mathcal{E}C}((\widetilde{1}+\widetilde{R})^{N-(p+1)}\widetilde{S}_{p+1}^{uuu\ldots u}))$
$\Longrightarrow\ \frac{1}{(\widetilde{1}+\widetilde{R})^{p+1}}\widetilde{E}_p^Q(\widetilde{V}_{p+1}^{\mathcal{E}P/\mathcal{E}C}(\widetilde{F}_{p+1}^{uuu\ldots u}))\approx\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{p+1}}\ (\widetilde{P}_u\widetilde{V}_{p+1}^{\mathcal{E}P/\mathcal{E}C}((\widetilde{1}+\widetilde{R})^{N-(p+1)}\widetilde{U}\widetilde{S}_{p}^{uuu\ldots u})+\widetilde{P}_d\widetilde{V}_{p+1}^{\mathcal{E}P/\mathcal{E}C}((\widetilde{1}+\widetilde{R})^{N-(p+1)}\widetilde{D}\widetilde{S}_{p}^{uuu\ldots u}))$
$\Longrightarrow\ \frac{1}{(\widetilde{1}+\widetilde{R})^{p+1}}\widetilde{E}_p^Q(\widetilde{V}_{p+1}^{\mathcal{E}P/\mathcal{E}C}(\widetilde{F}_{p+1}^{uuu\ldots u}))\approx\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{p+1}}\ (\widetilde{P}_u\widetilde{V}_{p+1}^{\mathcal{E}P/\mathcal{E}C}((\widetilde{1}+\widetilde{R})^{N-(p+1)}\widetilde{S}_{p+1}^{uuu\ldots u})+\widetilde{P}_d\widetilde{V}_{p+1}^{\mathcal{E}P/\mathcal{E}C}((\widetilde{1}+\widetilde{R})^{N-(p+1)}\widetilde{S}_{p+1}^{uuu\ldots d}))$
$\Longrightarrow\ \frac{1}{(\widetilde{1}+\widetilde{R})^{p+1}}\widetilde{E}_p^Q(\widetilde{V}_{p+1}^{\mathcal{E}P/\mathcal{E}C}(\widetilde{F}_{p+1}^{uuu\ldots u}))\succeq\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{p}}\widetilde{V}_{p}^{\mathcal{E}P/\mathcal{E}C}((\widetilde{1}+\widetilde{R})^{N-p}\widetilde{S}_{p}^{uuu\ldots u})$
Since
$\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{p+2}}(\widetilde{P}_u\widetilde{V}_{p+2}^{\mathcal{E}P/\mathcal{E}C}((\widetilde{1}+\widetilde{R})^{N-(p+2)}\widetilde{S}_{p+2}^{uuu\ldots u})+\widetilde{P}_d\widetilde{V}_{p+2}^{\mathcal{E}P/\mathcal{E}C}((\widetilde{1}+\widetilde{R})^{N-(p+2)}\widetilde{S}_{p+2}^{uuu\ldots d}))\succeq\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{p+1}}\widetilde{V}_{p+1}^{\mathcal{E}P/\mathcal{E}C}((\widetilde{1}+\widetilde{R})^{N-(p+1)}\widetilde{S}_{p+1}^{uuu\ldots u})$,
by equation (4.7)
$\Longrightarrow\ \frac{1}{(\widetilde{1}+\widetilde{R})^{p+1}}\widetilde{E}_p^Q(\widetilde{V}_{p+1}^{\mathcal{E}P/\mathcal{E}C}(\widetilde{F}_{p+1}^{uuu\ldots u}))\succeq$
$\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{p}}\widetilde{V}_{p}^{\mathcal{E}P/\mathcal{E}C}(\widetilde{F}_{p}^{uuu\ldots u})$.
We can also prove equation (4.6) is false for other states.
In a similar way, by using equation (4.8), we can show
$\frac{1}{(\widetilde{1}+\widetilde{R})^{p+1}}\widetilde{E}_p^Q(\widetilde{V}_{p+1}^{\mathcal{E}P/\mathcal{E}C}(\widetilde{F}_{p+1}^{uuu\ldots u}))\preceq$
$\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{p}}\widetilde{V}_{p}^{\mathcal{E}P/\mathcal{E}C}(\widetilde{F}_{p}^{uuu\ldots u})$.
Likewise, it can be proved that it is false for other states.
Thus, we get a contradiction and hence equation (4.6) is true for $n=p$ which implies equations (4.7) and (4.8) are false for $n=p+1$.
Hence, the same holds for $n=0$: Similarly, we can show (4.6) is true when $\widetilde{U}\widetilde{P}_u+\widetilde{D}\widetilde{P}_d\succcurlyeq\widetilde{1}+\widetilde{R}$ and $\widetilde{U}\widetilde{P}_u+\widetilde{D}\widetilde{P}_d\preccurlyeq\widetilde{1}+\widetilde{R}$, using reverse induction on $n$. □

**Definition 4.13.** Let $\{\widetilde{X}_n\}_{n=0}^N$ be a discrete time fuzzy stochastic process with respect to the filtration $\{\mathcal{M}_n\}_{n=0}^N$ and let $\tau$ be a stopping time. Then the stopped fuzzy stochastic process $\{\widetilde{X}\}^\tau$ is defined by $\widetilde{X}_n^\tau(w)\approx\widetilde{X}_{\tau(w)\wedge n}(w)$ for any positive integer $n$ and any $w\in\Omega$, where $\tau(w)\wedge n=\min\{\tau(w),\ n\}$.

**Theorem 4.14.** *The stopped discounted fuzzy intrinsic values of American fuzzy put/call options models on fuzzy future contracts process for buyers and sellers* $\left\{ \frac{\widetilde{V}^{\mathcal{AP}/\mathcal{AC}}_{n\wedge\tau}}{(\widetilde{1}+\widetilde{R})^{n\wedge\tau}} \right\}_{n=0}^{N}$ *are* $Q-$ *fuzzy supermartingales with respect to* $Q$*.*

*Proof.* If $\tau \geq n+1$, then

$$\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{n+1}} \widetilde{E}^Q_n(\widetilde{V}^{\mathcal{AP}/\mathcal{AC}}_{(n+1)\wedge\tau}(\widetilde{F}^{uuu\ldots u}_{n+1})) \approx$$
$$\frac{1}{(\widetilde{1}+\widetilde{R})^{n+1}} \widetilde{E}^Q_n(\widetilde{V}^{\mathcal{AP}/\mathcal{AC}}_{(n+1)\wedge\tau}((\widetilde{1}+\widetilde{R})^{N-(n+1)} \widetilde{S}^{uuu\ldots u}_{n+1}))$$
$$\implies \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{n+1}} \widetilde{E}^Q_n(\widetilde{V}^{\mathcal{AP}/\mathcal{AC}}_{(n+1)\wedge\tau}(\widetilde{F}^{uuu\ldots u}_{n+1})) \approx \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{n+1}}$$
$$\left(\widetilde{P}_u \widetilde{V}^{\mathcal{AP}/\mathcal{AC}}_{(n+1)\wedge\tau}((\widetilde{1}+\widetilde{R})^{N-(n+1)} \widetilde{U} \widetilde{S}^{uuu\ldots u}_n) + \widetilde{P}_d \widetilde{V}^{\mathcal{AP}/\mathcal{AC}}_{(n+1)\wedge\tau}((\widetilde{1}+\widetilde{R})^{N-(n+1)} \widetilde{D} \widetilde{S}^{uuu\ldots u}_n)\right)$$
$$\implies \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{n+1}} \widetilde{E}^Q_n(\widetilde{V}^{\mathcal{AP}/\mathcal{AC}}_{(n+1)\wedge\tau}(\widetilde{F}^{uuu\ldots u}_{n+1})) \approx \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{n+1}}$$
$$\left(\widetilde{P}_u \widetilde{V}^{\mathcal{AP}/\mathcal{AC}}_{(n+1)\wedge\tau}((\widetilde{1}+\widetilde{R})^{N-(n+1)} \widetilde{S}^{uuu\ldots u}_{n+1}) + \widetilde{P}_d \widetilde{V}^{\mathcal{AP}/\mathcal{AC}}_{(n+1)\wedge\tau}((\widetilde{1}+\widetilde{R})^{N-(n+1)} \widetilde{S}^{uuu\ldots d}_{n+1})\right)$$
$$\implies \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{n+1}} \widetilde{E}^Q_n(\widetilde{V}^{\mathcal{AP}/\mathcal{AC}}_{(n+1)\wedge\tau}(\widetilde{F}^{uuu\ldots u}_{n+1})) \approx \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{n+1}}$$
$$\left(\widetilde{P}_u \widetilde{V}^{\mathcal{AP}/\mathcal{AC}}_{n+1}((\widetilde{1}+\widetilde{R})^{N-(n+1)} \widetilde{S}^{uuu\ldots u}_{n+1}) + \widetilde{P}_d \widetilde{V}^{\mathcal{AP}/\mathcal{AC}}_{n+1}((\widetilde{1}+\widetilde{R})^{N-(n+1)} \widetilde{S}^{uuu\ldots d}_{n+1})\right)$$
$$\implies \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{n+1}} \widetilde{E}^Q_n(\widetilde{V}^{\mathcal{AP}/\mathcal{AC}}_{(n+1)\wedge\tau}(\widetilde{F}^{uuu\ldots u}_{n+1})) \succeq$$
$$\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{n}} \widetilde{V}^{\mathcal{AP}}_n((\widetilde{1}+\widetilde{R})^{(N-n)} \widetilde{S}^{uuu\ldots u}_n)$$
$$\implies \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{n+1}} \widetilde{E}^Q_n(\widetilde{V}^{\mathcal{AP}/\mathcal{AC}}_{(n+1)\wedge\tau}(\widetilde{F}^{uuu\ldots u}_{n+1})) \succeq \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{n}} \widetilde{V}^{\mathcal{AP}}_n(\widetilde{F}^{uuu\ldots u}_n)$$
$$\implies \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{n+1}} \widetilde{E}^Q_n(\widetilde{V}^{\mathcal{AP}/\mathcal{AC}}_{(n+1)\wedge\tau}(\widetilde{F}^{uuu\ldots u}_{n+1})) \succeq \frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{n}} \widetilde{V}^{\mathcal{AP}/AC}_{n\wedge\tau}(\widetilde{F}^{uuu\ldots u}_n)$$

Similarly, we can prove it for other states.
We can also show equation (4.5) is true for $n = 1$ and $n = 0$.
In a similar manner, we can prove that

$$\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{n+1}} \widetilde{E}^Q_n(\widetilde{V}^{\mathcal{AP}/\mathcal{AC}}_{(n+1)\wedge\tau}(\widetilde{F}^{uuu\ldots u}_{n+1})) \approx$$
$$\frac{\widetilde{1}}{(\widetilde{1}+\widetilde{R})^{n}} \widetilde{V}^{\mathcal{AP}/\mathcal{AC}}_{n\wedge\tau}(\widetilde{F}^{uuu\ldots u}_n), \text{ if } \tau \leq n.$$ □

*Remark* 4.15. An analogue of Theorem 4.14 can be proved showing that the stopped discounted fuzzy intrinsic values of European fuzzy put and call options models on fuzzy future contracts processes are $Q-$ fuzzy martingale using Proposition 4.12.

## 5. Fuzzy put/call options pricing models on fuzzy future contracts using continuous time fuzzy stochastic process

This section is dealt with the fuzzy put/ call options models on fuzzy future contracts for both buyers and sellers using continuous time fuzzy stochastic process. The fuzzy stock price and fuzzy future price processes are modeled using continuously compounded fuzzy risk-free interest rate to model continuous time fuzzy stochastic process which is approximated by a discrete time fuzzy process with sufficiently large finite $n$ (see Figure 1).

**Definition 5.1.** The fuzzy future price at time $n$ is denoted $\widetilde{F}_{n,i}$ and defined by $\widetilde{F}_{n,i} \approx e^{\widetilde{R}\triangle t}\widetilde{S}_{n,i}$, for $n = 0, 1, \ldots, N$ and $i = 0, 1, \ldots, N$, where $\Delta T = \frac{T}{n}$. Hence, the fuzzy future price grows with continuously compounded fuzzy risk-free interest rate.

**Example of modeling the fuzzy stock price and fuzzy future price process using continuous time fuzzy stochastic process:**
Here, we divide the number of time steps into three in the fuzzy binomial tree model. Using the three-period fuzzy binomial tree model, we estimate the fuzzy stock prices and fuzzy future prices of INTC shares involving GLOFN for the data considered in

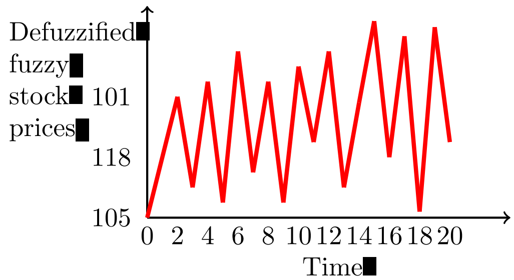


FIGURE 1. Evolution of Defuzzified fuzzy stock prices during $n = 0$ to $n = 20$

Table (1) . The same is computed using the codes of fuzzy stock and fuzzy future prices given in MATLAB program . Defuzzify the same, by choosing $k = 0.3$. The values are represented in the form of a three-period binomial tree (see Figures 2 and 3).

**Bearish Scenario 2020:**

| **Symbol** | $S_0$ | $K$ | **Premium** | **Expiry** | **Style** | $R$ |
|---|---|---|---|---|---|---|
| MSFT | $49.57 | $54.50 | $1.25 | 1/10/2020 | P / C | 0.11 |

TABLE 1. Quotes of American type INTC (2020) shares

We infer that the defuzzified fuzzy stock and fuzzy future prices of INTC shares evolves with

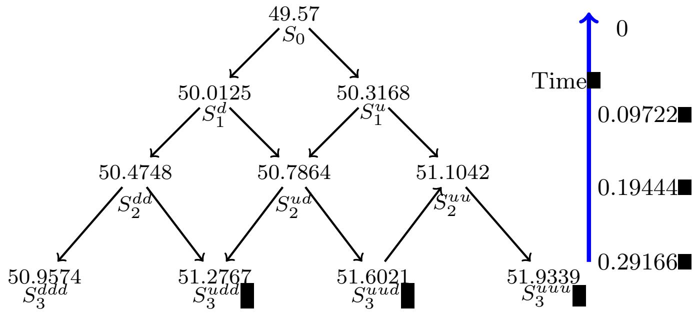


FIGURE 2. Defuzzified fuzzy stock prices: three - period binomial tree

respect to the continuous time.

**Fuzzy risk-neutral probability measures involving continuously compounded fuzzy risk-free interest rate:**

Following the treatment of fuzzy risk-neutral probability measures involving GLOFN discussed, the following possible no arbitrage conditions satisfied by the two up and down fuzzy jump factors and the continuously compounded fuzzy risk-free interest rate of the fuzzy stock are defined as follows:

i.e., $D_i < e^{R_k \triangle t} < U_j$ for $i = 1, 2, 3, 4$, $k = 1, 2, 3, 4, 5, 6, 7, 8$ and $j = 5, 6, 7, 8$.

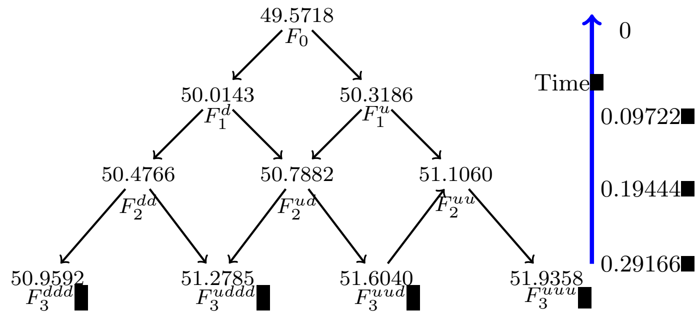


FIGURE 3. Defuzzified fuzzy future prices: three - period binomial tree

**Definition 5.2.** The up and down fuzzy risk-neutral probability measures $Q^*(\widetilde{P}_u^*, \widetilde{P}_d^*)$ for buyers are defined by

$$\widetilde{P}_u^* \approx (\frac{e^{\widetilde{r}_1\triangle t}-d_4}{u_5-d_4}, \frac{e^{\widetilde{r}_2\triangle t}-d_4}{u_5-d_4}, \frac{e^{\widetilde{r}_3\triangle t}-d_4}{u_5-d_4}, \frac{e^{\widetilde{r}_4\triangle t}-d_4}{u_5-d_4}, \frac{e^{\widetilde{r}_5\triangle t}-d_4}{u_5-d_4}, \frac{e^{\widetilde{r}_6\triangle t}-d_4}{u_5-d_4}, \frac{e^{\widetilde{r}_7\triangle t}-d_4}{u_5-d_4}, \frac{e^{\widetilde{r}_8\triangle t}-d_4}{u_5-d_4}; k)$$

$$\widetilde{P}_d^* \approx (\frac{u_5-e^{\widetilde{r}_8\triangle t}}{u_5-d_4}, \frac{u_5-e^{\widetilde{r}_7\triangle t}}{u_5-d_4}, \frac{u_5-e^{\widetilde{r}_6\triangle t}}{u_5-d_4}, \frac{u_5-e^{\widetilde{r}_5\triangle t}}{u_5-d_4}, \frac{u_5-e^{\widetilde{r}_4\triangle t}}{u_5-d_4}, \frac{u_5-e^{\widetilde{r}_3\triangle t}}{u_5-d_4}, \frac{u_5-e^{\widetilde{r}_2\triangle t}}{u_5-d_4}, \frac{u_5-e^{\widetilde{r}_1\triangle t}}{u_5-d_4}; k)$$

**Definition 5.3.** The up and down fuzzy risk-neutral probability measures $Q^*(\widetilde{P}_u^*, \widetilde{P}_d^*)$ for sellers are defined by

$$\widetilde{P}_u^* \approx (\frac{e^{\widetilde{r}_1\triangle t}-d_1}{u_8-d_1}, \frac{e^{\widetilde{r}_2\triangle t}-d_1}{u_8-d_1}, \frac{e^{\widetilde{r}_3\triangle t}-d_1}{u_8-d_1}, \frac{e^{\widetilde{r}_4\triangle t}-d_1}{u_8-d_1}, \frac{e^{\widetilde{r}_5\triangle t}-d_1}{u_8-d_1}, \frac{e^{\widetilde{r}_6\triangle t}-d_1}{u_8-d_1}, \frac{e^{\widetilde{r}_7\triangle t}-d_1}{u_8-d_1}, \frac{e^{\widetilde{r}_8\triangle t}-d_1}{u_8-d_1}; k)$$

$$\widetilde{P}_d^* \approx (\frac{u_5-e^{\widetilde{r}_8\triangle t}}{u_8-d_1}, \frac{u_8-e^{\widetilde{r}_7\triangle t}}{u_8-d_1}, \frac{u_8-e^{\widetilde{r}_6\triangle t}}{u_8-d_1}, \frac{u_8-e^{\widetilde{r}_5\triangle t}}{u_8-d_1}, \frac{u_8-e^{\widetilde{r}_4\triangle t}}{u_8-d_1}, \frac{u_8-e^{\widetilde{r}_3\triangle t}}{u_8-d_1}, \frac{u_8-e^{\widetilde{r}_2\triangle t}}{u_8-d_1}, \frac{u_8-e^{\widetilde{r}_1\triangle t}}{u_8-d_1}; k).$$

*Remark* 5.4. Analogues of Definitions 5.2 and 5.3, give rise to fuzzy risk-neutral probability measures involving general trapezoidal and triangular fuzzy numbers for both buyers and sellers under the respective no-arbitrage assumptions.

**Definition 5.5.** The fuzzy intrinsic values of American fuzzy put option model on fuzzy future contract is defined by b $\widetilde{V}_N^{\mathcal{AP}}(\widetilde{F}_{N,i}) \approx \max\left\{\widetilde{K} - \widetilde{F}_{N,i}, \widetilde{0}\right\}$, for $i = 0, 1, \ldots, N$ and
$\widetilde{V}_{n,i}^{\mathcal{A}P}(\widetilde{F}_{n,i}) \approx \max\left\{\widetilde{K} - \widetilde{F}_{n,i}, \frac{\widetilde{1}}{e^{\widetilde{R}\triangle t}} \widetilde{E}_n^{Q^*}(\widetilde{V}_{n+1}^{\mathcal{A}P}(\widetilde{F}_{n+1,i}))\right\}$,
for $i = 0, 1, \ldots, n$ and
$n = N-1, N-2, \ldots, 0$ where, $\widetilde{E}_n^{Q^*}(\widetilde{V}_{n+1}^{\mathcal{A}P}(\widetilde{F}_{n+1,i})) \approx \widetilde{P}_u^* \widetilde{V}_{n+1}^{\mathcal{A}P}(\widetilde{F}_{n+1,i}) + \widetilde{P}_d^* \widetilde{V}_{n+1}^{\mathcal{A}P}(\widetilde{F}_{n+1,i})$ denotes the expectation with respect to the fuzzy risk-neutral probability measure $Q^*$.

**Definition 5.6.** The fuzzy intrinsic values of European fuzzy put option model on fuzzy future contract are defined as $\widetilde{V}_N^{\mathcal{EP}}(\widetilde{F}_{N,i}) \approx \max\left\{\widetilde{K} - \widetilde{F}_{N,i}, \widetilde{0}\right\}$, for $i = 0, 1, \ldots, N$ and
$\widetilde{V}_n^{\mathcal{E}P}(\widetilde{F}_{n+1,i}) \approx \left\{\frac{\widetilde{1}}{e^{\widetilde{R}\triangle t}} \widetilde{E}_n^{Q^*}(\widetilde{V}_{n+1}^{\mathcal{E}P}(\widetilde{F}_{n+1,i}))\right\}$,
for $i = 0, 1, \ldots, n$ and $n = N-1, N-2, \ldots, 0$ where, $\widetilde{E}_n^{Q^*}(\widetilde{V}_{n+1}^{\mathcal{E}P}(\widetilde{F}_{n,i})) \approx \widetilde{P}_u^* \widetilde{V}_{n+1}^{\mathcal{E}P}(\widetilde{F}_{n+1,i}) + \widetilde{P}_d^* \widetilde{V}_{n+1}^{\mathcal{E}P}(\widetilde{F}_{n+1,i})$ denotes the expectation with respect to the fuzzy risk-neutral probability measure $Q^*$.

**Definition 5.7.** The fuzzy intrinsic values of American fuzzy call option model on fuzzy future contract are defined by $\widetilde{V}_N^{\mathcal{AC}}(\widetilde{F}_{N,i}) \approx \max\left\{\widetilde{F}_{N,i} - \widetilde{K}, \widetilde{0}\right\}$, for $i = 0, 1, \ldots, N$ and
$\widetilde{V}_n^{\mathcal{AC}}(\widetilde{F}_{n,i}) \approx \max\left\{\widetilde{F}_{n,i} - \widetilde{K}, \frac{\widetilde{1}}{e^{\widetilde{R}\triangle t}} \widetilde{E}_n^{Q^*}(\widetilde{V}_{n+1}^{\mathcal{AC}}(\widetilde{F}_{n+1,i}))\right\}$,
for $i = 0, 1, \ldots, n$ and
$n = N-1, N-2, \ldots, 0$ where, $\widetilde{E}_n^{Q^*}(\widetilde{V}_{n+1}^{\mathcal{AC}}(\widetilde{F}_{n+1,i})) \approx \widetilde{P}_u^* \widetilde{V}_{n+1}^{\mathcal{AC}}(\widetilde{F}_{n+1,i}) + \widetilde{P}_d^* \widetilde{V}_{n+1}^{\mathcal{AC}}(\widetilde{F}_{n+1,i})$ denotes the expectation with respect to the fuzzy risk-neutral probability measure $Q^*$.

**Definition 5.8.** The fuzzy intrinsic values of European fuzzy call option model on fuzzy future contract are defined as $\widetilde{V}_N^{\mathcal{EC}}(\widetilde{F}_{N,i}) \approx \max\left\{\widetilde{F}_{N,i} - \widetilde{K}, \widetilde{0}\right\}$, for $i = 0, 1, \ldots, N$ and
$\widetilde{V}_n^{\mathcal{E}C}(\widetilde{F}_{n,i}) \approx \left\{\frac{\widetilde{1}}{e^{\widetilde{R}\triangle t}} \widetilde{E}_n^{Q^*}(\widetilde{V}_{n+1}^{\mathcal{E}C}(\widetilde{F}_{n+1,i}))\right\}$,
for $i = 0, 1, \ldots, n$ and $n = N-1, N-2, \ldots, 0$ where, $\widetilde{E}_n^{Q^*}(\widetilde{V}_{n+1}^{\mathcal{E}C}(\widetilde{F}_{n+1,i})) \approx \widetilde{P}_u^* \widetilde{V}_{n+1}^{\mathcal{E}C}(\widetilde{F}_{n+1,i}) + \widetilde{P}_d^* \widetilde{V}_{n+1}^{\mathcal{E}P}(\widetilde{F}_{n+1,i})$ denotes the expectation with respect to the fuzzy risk-neutral probability measure $Q^*$.

*Remark* 5.9. We call $\frac{\widetilde{1}}{e^{\widetilde{R}\triangle t}} \widetilde{E}_n^{Q^*}(\widetilde{V}_{n+1}(\widetilde{F}_{n,i}))$ as the expected discounted fuzzy intrinsic values under $Q^*$ and given by $\frac{\widetilde{1}}{e^{\widetilde{R}\triangle t}} \widetilde{E}_n^{Q^*}(\widetilde{V}_{n+1}(\widetilde{F}_{n,i})) \approx \frac{\widetilde{1}}{e^{\widetilde{R}\triangle t}} \left(\widetilde{P}_u^* \widetilde{V}_{n+1}^{\mathcal{E}P}(\widetilde{F}_{n+1,i}) + \widetilde{P}_d^* \widetilde{V}_{n+1}^{\mathcal{E}P}(\widetilde{F}_{n+1,i})\right)$,
for $i = 0, 1, \ldots, n$ and $n = N-1, N-2, \ldots, 0$.

**Definition 5.10.** A continuous time fuzzy stochastic price process $\widetilde{\mathcal{X}^*} \approx \left\{\widetilde{X}_n, \mathcal{M}_n\right\}_{n=0}^{\infty}$ is a $Q^*$ fuzzy martingale under $Q^*$ with respect to a filtration $\{\mathcal{M}_n\}_{n=0}^{\infty}$ if
(i) $\widetilde{E}_n^{Q^*}(\widetilde{X}_{n+1}) \approx \widetilde{X}_n$, where $\widetilde{E}_n^{Q^*}$ is the expectation with respect to the fuzzy risk-neutral probability measure $Q^*$.
(ii) Further if (i) is replaced by $\widetilde{E}_n^{Q^*}(\widetilde{X}_{n+1}) \preccurlyeq \widetilde{X}_n$, then $\widetilde{\mathcal{X}^*} \approx \left\{\widetilde{X}_n, \mathcal{M}_n\right\}_{n=0}^{\infty}$ is a $Q^*-$ fuzzy supermartingale with respect to $\{\mathcal{M}_n\}_{n=0}^{\infty}$
(iii) and if (i) is replaced by $\widetilde{E}_n^{Q^*}(\widetilde{X}_{n+1}) \succcurlyeq \widetilde{X}_n$, then $\widetilde{\mathcal{X}^*} \approx \left\{\widetilde{X}_n, \mathcal{M}_n\right\}_{n=0}^{\infty}$ is a $Q^*-$ fuzzy submartingale related to a filtration $\{\mathcal{M}_n\}_{n=0}^{\infty}$ .

*Remark* 5.11. In order to prove Proposition 5.14 and Remarks 5.13, 5.15, 5.16, 5.17, 5.18, using continuous time fuzzy stochastic process, we choose $n$ as a finite large number in the fuzzy binomial tree model.

**Definition 5.12.** The expected fuzzy future price at time $n$ in continuous time is defined as $\widetilde{E}_n^Q(\widetilde{F}_{n+1}^{uuu\ldots u}) \approx \widetilde{E}_n^Q(e^{\widetilde{R}\triangle t}\widetilde{S}_{n+1}^{uuu\ldots u})$.
*i.e.*, $\widetilde{E}_n^Q(\widetilde{F}_{n+1}^{uuu\ldots u}) \approx e^{\widetilde{R}\triangle t}\widetilde{P}_u \widetilde{S}_n^{uuu\ldots u}\widetilde{U} + e^{\widetilde{R}\triangle t}\widetilde{P}_d \widetilde{S}_n^{uuu\ldots u}\widetilde{D}$. Similarly, it can be defined for other states.

*Remark* 5.13. The discounted fuzzy stock price process turns out to be a $Q^*-$ fuzzy martingale, $Q^*-$ fuzzy supermartingale and $Q^*-$ fuzzy submartingale using continuous time fuzzy stochastic process.

We prove Proposition 5.14 using Definitions 5.12 and Remark 5.11.

**Proposition 5.14.** *The discounted fuzzy future price process is a $Q^*-$ fuzzy martingale, $Q^*-$ fuzzy supermartingale and $Q^*-$ fuzzy submartingale with respect to the fuzzy risk-neutral probability measure $Q^*$ whenever*
*(i)* $\widetilde{U}\widetilde{P}_u + \widetilde{D}\widetilde{P}_d \approx e^{\widetilde{R}\triangle t}$;
*(ii)* $\widetilde{U}\widetilde{P}_u + \widetilde{D}\widetilde{P}_d \succcurlyeq e^{\widetilde{R}\triangle t}$;
*(iii)* $\widetilde{U}\widetilde{P}_u + \widetilde{D}\widetilde{P}_d \preccurlyeq e^{\widetilde{R}\triangle t}$.

*Proof.* To show that the discounted fuzzy future price process is a $Q^*-$ fuzzy submartingale with respect to $Q^*$ under condition (ii), we have to show that

$$\frac{\widetilde{1}}{(e^{\widetilde{R}\triangle t})^{n+1}}\widetilde{E}_n^Q(\widetilde{F}_{n+1}^{uuu\ldots u}) \succcurlyeq \frac{\widetilde{1}}{(e^{\widetilde{R}\triangle t})^{n}}\widetilde{F}_n^{uuu\ldots u},\ for\ n = 0,1,\ldots,N \tag{5.1}$$

and it holds for other states. We prove equation (5.1) by reverse induction on $n$.
Assume equation (5.1) fails to hold for $n = p+1$ and it does not hold for other states, then we have

$$\frac{\widetilde{1}}{(e^{\widetilde{R}\triangle t})^{p+2}}\widetilde{E}_{p+1}^Q(\widetilde{F}_{p+2}^{uuu\ldots u}) \preceq \frac{\widetilde{1}}{(e^{\widetilde{R}\triangle t})^{p+1}}\widetilde{F}_{p+1}^{uuu\ldots u} \tag{5.2}$$

and it does not hold for other states.
Now to prove equation (5.1) is false for $n = p$.
$\widetilde{E}_p^Q(\widetilde{F}_{p+1}^{uuu\ldots u}) \approx \widetilde{E}_p^Q(e^{\widetilde{R}\triangle t}\widetilde{S}_{p+1}^{uuu\ldots u})$
$\implies \widetilde{E}_p^Q(\widetilde{F}_{p+1}^{uuu\ldots u}) \approx e^{\widetilde{R}\triangle t}\widetilde{E}_p^Q(\widetilde{S}_{p+1}^{uuu\ldots u}) \implies \widetilde{E}_p^Q(\widetilde{F}_{p+1}^{uuu\ldots u}) \preceq (e^{\widetilde{R}\triangle t})^2\widetilde{E}_p^Q(\widetilde{S}_{p+1}^{uuu\ldots u})$
$\implies \frac{\widetilde{1}}{e^{\widetilde{R}\triangle t}}\widetilde{E}_p^Q(\widetilde{F}_{p+1}^{uuu\ldots u}) \preceq e^{\widetilde{R}\triangle t}\widetilde{S}_p^{uuu\ldots u} \implies \frac{\widetilde{1}}{e^{\widetilde{R}\triangle t}}\widetilde{E}_p^Q(\widetilde{F}_{p+1}^{uuu\ldots u}) \preceq \widetilde{F}_p^{uuu\ldots u}$
$\implies \frac{\widetilde{1}}{(e^{\widetilde{R}\triangle t})^{p+1}}\widetilde{E}_p^Q(\widetilde{F}_{p+1}^{uuu\ldots u}) \preceq \frac{\widetilde{1}}{(e^{\widetilde{R}\triangle t})^{p}}\widetilde{F}_p^{uuu\ldots u}$.
Similarly, we can prove equation (5.1) does not hold for other states, which is a contradiction.
Since equation (5.1) is true for $n = p$ will imply equation 5.2 is false for $n = p+1$.
When $n = 1$: if $\widetilde{U}\widetilde{P}_u + \widetilde{D}\widetilde{P}_d \approx e^{\widetilde{R}\triangle t}$, then we have $\widetilde{E}_1^Q(\widetilde{F}_2^u) \approx \widetilde{E}_1^Q(e^{\widetilde{R}\triangle t}\widetilde{S}_2^u)$.
$\implies \widetilde{E}_1^Q(\widetilde{F}_2^u) \approx e^{\widetilde{R}\triangle t}\widetilde{E}_1^Q(\widetilde{S}_2^u) \implies \widetilde{E}_1^Q(\widetilde{F}_2^u) \preceq (e^{\widetilde{R}\triangle t})^2\widetilde{S}_1^u$
$\implies \frac{\widetilde{1}}{e^{\widetilde{R}\triangle t}}\widetilde{E}_1^Q(\widetilde{F}_2^u) \succcurlyeq e^{\widetilde{R}\triangle t}\widetilde{S}_1^u \implies \frac{\widetilde{1}}{e^{\widetilde{R}\triangle t}}\widetilde{E}_1^Q(\widetilde{F}_2^u) \succcurlyeq \widetilde{F}_1^u$
$\implies \frac{\widetilde{1}}{(e^{\widetilde{R}\triangle t})^2}\widetilde{E}_1^Q(\widetilde{F}_2^u) \succcurlyeq \frac{\widetilde{1}}{(e^{\widetilde{R}\triangle t})^1}\widetilde{F}_1^u$.
In like manner, we can prove it for down state.
i.e., $\frac{\widetilde{1}}{(e^{\widetilde{R}\triangle t})^2}\widetilde{E}_1^Q(\widetilde{F}_2^d) \succcurlyeq \frac{\widetilde{1}}{(e^{\widetilde{R}\triangle t})^1}\widetilde{F}_1^d$.
When $n = 0$: if $\widetilde{U}\widetilde{P}_u + \widetilde{D}\widetilde{P}_d \approx e^{\widetilde{R}\triangle t}$, then we have $\widetilde{E}_0^Q(\widetilde{F}_1) \approx \widetilde{E}_0^Q(e^{\widetilde{R}\triangle t}\widetilde{S}_1)$.
$\implies \widetilde{E}_0^Q(\widetilde{F}_1) \approx e^{\widetilde{R}\triangle t}\widetilde{E}_0^Q(\widetilde{S}_1) \implies \widetilde{E}_0^Q(\widetilde{F}_1) \preceq (e^{\widetilde{R}\triangle t})^2\widetilde{S}_0$
$\implies \frac{\widetilde{1}}{e^{\widetilde{R}\triangle t}}\widetilde{E}_0^Q(\widetilde{F}_1) \succcurlyeq e^{\widetilde{R}\triangle t}\widetilde{S}_0 \implies \frac{\widetilde{1}}{(e^{\widetilde{R}\triangle t})^1}\widetilde{E}_0^Q(\widetilde{F}_1) \succcurlyeq \frac{\widetilde{1}}{(e^{\widetilde{R}\triangle t})^0}\widetilde{F}_0$. Similarly, we can prove equation (5.1) is true when (i) and (iii) holds by reverse induction on $n$. □

*Remark* 5.15. The discounted fuzzy intrinsic values of buyers in American fuzzy put and call option models on fuzzy future contracts processes respectively are $Q^*-$ fuzzy supermartingale and $Q^*-$ fuzzy martingale. Also, the same processes respectively are $Q^*-$ fuzzy martingale and $Q^*-$ fuzzy supermartingale in sellers model.

*Remark* 5.16. Analogues to Proposition 4.9, we can prove that $\left\{\frac{\widetilde{V}_n^{\mathcal{A}P/AC}}{\frac{\widetilde{1}}{e^{\widetilde{R}\triangle t}}}\right\}_{n=0}^{N}$ is the smallest $Q^*-$ fuzzy supermartingale dominating $\left\{\frac{\widetilde{X}_n^{\mathcal{A}P/AC}}{\frac{\widetilde{1}}{e^{\widetilde{R}\triangle t}}}\right\}_{n=0}^{N}$ with respect to the fuzy risk-neutral probability measures $Q^*$.

*Remark* 5.17. Similar to Proposition 4.12, we can prove that the discounted fuzzy intrinsic values of European fuzzy put option and fuzzy call option models on fuzzy future contracts processes are $Q^*-$ fuzzy martingale for both buyers and sellers.

*Remark* 5.18. Parallel to Theorem 4.14, the stopped discounted fuzzy intrinsic values of American fuzzy put and call option models on fuzzy future contracts processes for buyers and sellers $\left\{\frac{\widetilde{V}_{n\wedge\tau}^{\mathcal{A}P/AC}}{e^{(\widetilde{R}\triangle t)^{n\wedge\tau}}}\right\}_{n=0}^{N}$ are $Q^*-$ fuzzy supermartingale. Similarly using Remark4.15, the stopped discounted fuzzy intrinsic values of European fuzzy put and call option models on fuzzy future contracts processes for buyers and sellers $\left\{\frac{\widetilde{V}_{n\wedge\tau}^{\mathcal{E}P/EC}}{e^{(\widetilde{R}\triangle t)^{n\wedge\tau}}}\right\}_{n=0}^{N}$ are $Q^*-$ fuzzy martingale.

## 6. Conclusions and Future Work

This study examined certain characteristics of fuzzy options derivatives, which enable traders to adjust their strategies based on prospective profits, thereby reducing risk.

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Department of Mathematics, School of Engineering, Presidency University, Bengaluru, Karnataka, India.

*Email address*: meenakshivinodkumar@gmail.com

Department of Mathematics, School of Engineering, Presidency University, Bengaluru, Karnataka, India.

*Email address*: sathish.s @presidencyuniversity.in

Department of Computing Technologies, School of Computing, SRM Institute of Science and Technology, Kattankulathur, Chengalpattu, Tamil Nadu, India, 603203

*Email address*: p.rajasekaran537@gmail.com

Department of ECE, Vel Tech Rangarajan Dr.Sagunthala R & D Institute of Science and Technology, Chennai, India, 600062

*Email address*: satheshraaj@gmail.com

School of Computer Science and Engineering, Vellore Institute of Technology, Vellore, 632014, Tamilnadu, India

*Email address*: dhoni.praba@gmail.com *