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INVARIANT MANIFOLDS FOR NONLINEAR FLOWS WITH UNCERTAINTY-AWARE CONE CONDITIONS

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Abstract. We develop an uncertainty-aware cone framework for invariant manifolds of nonlinear flows $\dot{x} = F(x, a)$ on Banach spaces, where the parameter a ranges over a compact uncertainty set A. Assuming a uniform exponential dichotomy/dominated splitting for the linearized cocycle and Carathéodory-type regularity of the nonlinearity, we introduce cone fields with explicit margins that absorb bounded model perturbations and establish forward/backward cone invariance uniformly in a. Via a uniform graph-transform (stable/unstable) and a Lyapunov-Perron construction (center/center-stable/center-unstable), we prove existence, uniqueness, and C^k (or Lipschitz) regularity of local invariant manifolds $W^{s/u/c}_{\text{loc}}(a)$ with radii and contraction constants independent of a, and we quantify Lipschitz $/C^\ell$ dependence of these manifolds on the uncertainty parameter. We verify the hypotheses for sectorial (parabolic) semiflows and provide a detailed reaction-diffusion case where a uniform spectral gap across parameter bands yields the required cone margins. A finite-difference Newton continuation, benchmarked against a one-mode Galerkin reduction, is upgraded to a proof-producing pipeline using a Neumann/Krawczyk a-posteriori test with a radii-polynomial, thereby delivering validated steady-state branches with certified error radii and numerically corroborating the uniform cone mechanism.

Keywords: invariant manifolds; uncertainty-aware cone conditions; exponential dichotomy; dominated splitting; Lyapunov-Perron method; sectorial operators; reaction-diffusion equations; Galerkin reduction; Krawczyk operator; radii polynomial; parameter-robust stability.

1 Introduction

Let X be a Banach space with norm $\|\cdot\|_X$, and let $A \subset \mathrm{R}^m$ be a compact uncertainty set. We study (semi)flows generated by

$$\dot{x} = F(x, a), x(0) = x_0 \in X, a \in A, \qquad --- (1.1)$$

where $F: X \times A \to X$ is C^k in x (typically $k \geq 1$) and continuous in a. Write $\phi_a(t, x_0)$ for the corresponding local flow (or semiflow). Classical invariant manifold theory (Hadamard-Perron for hyperbolic equilibria; Fenichel's persistence; normally hyperbolic invariant manifolds; graph transform methods) provides existence, uniqueness, and regularity of stable/unstable/center manifolds under hyperbolicity and cone conditions that encode invariant splittings and dominated growth/decay rates [1]-[6]. However, many applications require robustness with respect to parametric uncertainty e.g., data-driven ranges of parameters, interval models, or bounded perturbations-where classical pointwise cone conditions do not immediately yield uniform manifold statements across $a \in A$.

This work develops a uniform, uncertainty-aware cone framework implying the existence, uniqueness, and regularity of invariant manifolds simultaneously for all $a \in A$, together with Lipschitz/Hölder dependence on a. We formulate cone fields with margins and gap conditions that survive small model misspecification and quantify graph-transform contractions uniformly in a.

Our approach blends:

(i) adapted metrics and quadratic forms defining cones,
(ii) exponential dichotomies/dominated splittings for the linearized cocycle $\mathrm{D}\phi_a(t, x)$ with constants independent of a, and

(iii) a uniform fixed-point argument in spaces of Lipschitz graphs over the stable (or unstable) directions; cf. [4], [7]-[10].

The payoff is a family of C^r (or Lipschitz) invariant manifolds $W^{s/u/c}(a)$ with uniform radii and contraction constants, and moduli of continuity $a \mapsto W^{s/u/c}(a)$ measured in Hausdorff or C^r graph norms.

Visual intuition

We depict uncertainty bands for cone apertures and a nominal invariant graph in figure 1.

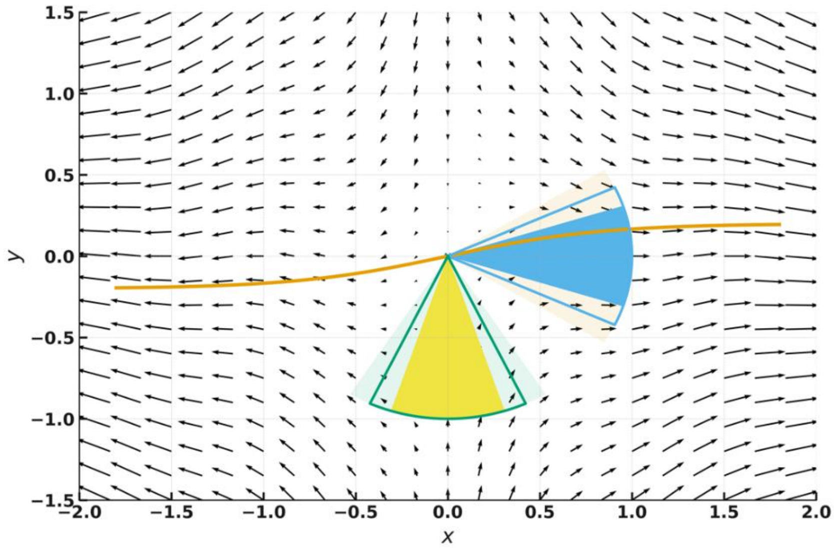

Figure 1 - Uncertainty-aware cone fields and invariant manifold sketch.

A planar saddle-type flow with a nominal invariant curve (thick line). At a reference state, unstable and stable cones are shown with a band around each aperture encoding model uncertainty. The uncertainty-aware cone condition requires that the flow map pushes the outer unstable cone strictly inside itself forward in time (and dually, the stable cone backward), uniformly across all $a \in A$

Contributions.

(i) A set of uniform cone conditions with margins ensuring forward/backward invariance and contraction of graph transforms for all $a \in A$.

(ii) Existence-uniqueness-regularity of local $W^{s/u/c}(a)$ with radii and Lipschitz constants independent of a, plus Hölder/Lipschitz dependence on a.

(iii) Extensions to flows with exponential dichotomies (nonautonomous case) and semiflows generated by sectorial operators in Banach spaces (parabolic PDE), compatible with classical frameworks in [2], [5], [6], [8]-[10].

2 Notation and functional setting

2.1 Splitting, adapted norms, and flows

Let $x_* \in X$ be an equilibrium for all $a \in A$, i.e. $F(x_*, a) = 0$. Assume a continuous splitting

$$X = E^s \oplus E^c \oplus E^u \qquad --- (2.1)$$

with bounded projections $\prod' : X \to E' \cdot \; \in \{s, c, u\}$. Denote $x = (x^s, x^c, x^u)$. We use an adapted norm $\|\cdot\|_{\#}$ equivalent to $\|\cdot\|_X$ such that

$$\|x\|_{\#} := \max\{\alpha_s\|x^s\|, \alpha_c\|x^c\|, \alpha_u\|x^u\|\}, \qquad --- (2.2)$$

with weights $\alpha. > 0$ chosen to magnify the spectral gap in Section 2.3. The (semi)flow $\phi_a(t,\cdot)$ is assumed to exist on a radius $R > 0$ ball $B_{\#}(R) \subset X$ uniformly in a for $|t| \leq T$, and $F \in C^k$ in x with locally Lipschitz $D_x F$, uniformly in $a \in A$.

For the linearized cocycle $U_a(t) := \mathrm{D}_x \phi_a(t, x_*)$ we assume block structure in the splitting (in the autonomous case, $U_a(t) = e^{t\mathrm{D}_x F(x_*, a)}$).

2.2 Uncertainty-aware cone fields

Let $Q^s, Q^u : X \to \mathrm{R}$ be continuous quadratic forms defining cones

$$\mathrm{C}_\delta^s := \{x : Q^s(x) + \delta\|x\|_{\#}^2 \leq 0\}, \mathrm{C}_\delta^u := \{x : Q^u(x) - \delta\|x\|_{\#}^2 \geq 0\}, \qquad --- (2.3)$$

with a margin $\delta > 0$. Think of $Q^s(x) = \|x^s\|^2 - \eta_s^2(\|x^c\|^2 + \|x^u\|^2)$ and $Q^u(x) = \|x^u\|^2 - \eta_u^2(\|x^s\|^2 + \|x^c\|^2)$, where $\eta. > 0$ encode apertures. The uncertainty set A modifies both the cocycle $U_a(t)$ and the nonlinear remainder; we demand uniform cone invariance with margin:
(C1) Forward invariance of unstable cones. There exist $t_0 > 0, \theta_u \in (0,1)$ such that for all $a \in A, x \in B_{\#}(R) \cap \mathrm{C}_\delta^u$,

$$\phi_a(t_0, x) - x_* \in \mathrm{C}_\delta^u, \|\Pi^u(\phi_a(t_0, x) - x_*)\| \geq \theta_u^{-1}\|\Pi^u(x - x_*)\|. \qquad --- (2.4)$$

(C2) Backward invariance of stable cones. There exist $t_0 > 0, \theta_s \in (0,1)$ such that for all $a \in A, x \in B_{\#}(R) \cap \mathrm{C}_\delta^s$,

$$\phi_a(-t_0, x) - x_* \in \mathrm{C}_\delta^s, \|\Pi^s(\phi_a(-t_0, x) - x_*)\| \geq \theta_s^{-1}\|\Pi^s(x - x_*)\|. \qquad --- (2.5)$$

(C3) Center tempering (optional). For $x \in E^c, \|\phi_a(\pm t_0, x) - x_*\| \leq \Lambda_c\|x - x_*\|$ with Λ_c uniform in a.

Conditions (C1)-(C3) express that cones are mapped strictly into themselves with uniform expansion/contraction and that center dynamics are dominated; see, e.g., [3], [4], [6]-[9].

2.3 Linear dichotomy / dominated splitting

Let $U_a(t)$ admit an exponential dichotomy on $\mathrm{R}_{\geq 0}$ (or R) with projections $P^{s/u}(a)$ compatible with $\Pi^{s/u}$ and uniform rates

$$\|U_a(t)\Pi^s\| \leq Me^{-\lambda_s t}, \|U_a(-t)\Pi^u\| \leq Me^{-\lambda_u t}, t \geq 0, \qquad --- (2.6)$$

for all $a \in A$, with $\lambda_{s,u} > 0$ and $M \geq 1$. The spectral gap

$$\gamma := \min\{\lambda_s, \lambda_u\} - L_{\mathrm{nl}} > 0, \qquad --- (2.7)$$

dominates the Lipschitz constant L_{nl} of the nonlinear remainder (see below). This yields a dominated splitting and underpins contraction of the graph transform; cf. [2], [4], [6], [10].

2.4 Nonlinear remainder and Lipschitz envelopes

Write

$$F(x,a) = L_a x + N(x,a), L_a := \mathrm{D}_x F(x_*,a), \qquad --- (2.8)$$

with $N(x_*,a) = 0, \mathrm{D}_x N(x_*,a) = 0$. Assume on $B_\#(R)$:

$$\|N(x,a) - N(y,a)\|_\# \le L_{\text{nl}} \|x-y\|_\sharp, \|N(x,a)\|_\sharp \le c_{\text{nl}} \|x-x_*\|_\sharp^2, \qquad --- (2.9)$$

uniformly in $a \in A$. Additionally, assume $a \mapsto L_a$ and $a \mapsto N(\cdot,a)$ are Lipschitz with constants $L_a^{(1)}, L_a^{(2)}$.

2.5 Graph transform phase space

Let G^u be the space of Lipschitz maps $h: E^u(\varrho) \to E^{sc} := E^s \oplus E^c$ with $h(0) = 0, \mathrm{Lip}(h) \le \varkappa$, where $E^u(\varrho) := \{u \in E^u : \|u\| \le \varrho\}$. The unstable graph $\mathrm{W}^u(h) = \{(h(u),u) : u \in E^u(\varrho)\}$ is contained in the unstable cone if x is small relative to η_u. Define the one-step graph transform T_a by

$$\mathrm{W}^u(\mathrm{T}_a h) := \phi_a(t_0, \mathrm{W}^u(h)) \cap B_\#(R), \qquad --- (2.10)$$

expressed as a new graph over E^u (well-posed by cone invariance). The metric on G^u is $\mathrm{dist}(h_1,h_2) := \sup_{\|u\| \le \varrho} \|h_1(u) - h_2(u)\|$.

Lemma 2.1 (Uniform contraction of T_a)

Under (C1)-(C3), (2.6)-(2.9), there exists $\varrho > 0, x > 0$, and $q \in (0,1)$ such that for all $a \in A$, $\mathrm{T}_a : \mathrm{G}^u \to \mathrm{G}^u$ is well-defined and

$$\mathrm{dist}(\mathrm{T}_a h_1, \mathrm{T}_a h_2) \le q \,\mathrm{dist}(h_1,h_2). \qquad --- (2.11)$$

Sketch. Decompose trajectories by variation of constants, use (2.6)-(2.7) to control linear parts, (2.9) for the nonlinear remainder, and cone invariance to keep graphs within C_δ^u. Bounds are uniform in $a \in A$. See, e.g., [3], [4], [6], [8].

By Banach's contraction principle, each T_a admits a unique fixed point $h_a \in \mathrm{G}^u$, giving a local unstable manifold

$$W_{\text{loc}}^u(a) = \{(h_a(u),u) : \|u\| \le \varrho\}. \qquad --- (2.12)$$

Analogous constructions yield $W_{\text{loc}}^s(a)$ (backward transform) and, under additional center tempering, $W_{\text{loc}}^c(a)$ by Lyapunov-Perron methods; see [2], [5], [6], [10].

2.6 Uniform dependence on the uncertainty parameter

Let $\Delta(a,b)$ denote the sup-norm of differences between L_a, L_b and the Lipschitz envelopes of $N(\cdot,a), N(\cdot,b)$ on $B_\#(R)$. A standard perturbation of contractions gives:

Lemma 2.2 (Lipschitz dependence on μ)

If $\sup_{a,b\in A} \Delta(a,b) \le \varepsilon$ and ε is sufficiently small with respect to the contraction margin $1-q$, then

$$\|h_a - h_b\|_{C^0(E^u(\varrho))} \le \frac{C}{1-q}\Delta(a,b), \quad --- (2.13)$$

so $a \mapsto W^u_{\text{loc}}(a)$ is Lipschitz (or Hölder, as dictated by a-regularity of F) in the graph norm, uniformly on A. Similar bounds hold for W^s_{loc} and, with tempered center, for W^c_{loc}.

Table 1 - Core symbols and constants

Symbol	Meaning
X	Banach phase space; norm $\|\cdot\|\#$ adapted to the splitting
$E^{s,c,u}$	Stable/center/unstable subspaces; projections $\prod^{s,c,u}$
$C^{s/u}_\delta$	Stable/unstable cone fields with margin δ
$U_a(t)$	Linearized cocycle $D_x\phi_a(t,x_*)$
$\lambda_{s,u}$	Exponential rates of dichotomy; M bound constant
γ	Spectral gap (2.7) dominating the nonlinear Lipschitz envelope
G^u	Space of Lipschitz graphs over E^u with Lip $\le \varkappa$
T_a	Graph transform at step t_0 (2.10); contraction factor q
$W^{s/u/c}_{\text{loc}}(a)$	Local invariant manifolds as graphs; radius ϱ

3 Flow model and uncertainty-aware cone conditions (assumptions)

Let X be a Banach space with norm $\|\cdot\|_X$. Fix an equilibrium $x_* \in X$ and a nonempty compact set of parameters $A \subset \mathbb{R}^m$. Consider the nonautonomous ODE (autonomous in x, parametric in a)

$$\dot{x} = F(x,a), F(\cdot,a) \in C^k(X,X), k \ge 1, a \in A, \quad --- (3.1)$$

generating local flows $\phi_a(t,\cdot), t \in [-T,T]$, on a ball $B_X(R)$ independent of a.

3.1 Splitting and adapted norm

Assume there is a continuous splitting

$$X = E^s \oplus E^c \oplus E^u, \Pi: X \to E^{\cdot}(\cdot \in \{s,c,u\}) \quad --- (3.2)$$

with bounded projections $\|\Pi^*\| \le C_\Pi$ and an adapted equivalent norm

$$\|x\|_\#:= \max\{\alpha_s\|x^s\|, \alpha_c\|x^c\|, \alpha_u\|x^u\|\}, x = (x^s,x^c,x^u), \quad --- (3.3)$$

for some weights $\alpha. > 0$. All constants below are independent of $a \in A$.

Write $L_a := D_xF(x_*,a)$ and $N(x,a) := F(x,a) - L_a x$, so $N(x_*,a) = 0, D_xN(x_*,a) = 0$.

3.2 Linear dichotomy / domination and nonlinear envelopes

(H1) Uniform exponential dichotomy / domination: There are $M \ge 1, \lambda_s, \lambda_u > 0$ such that, for the linearized cocycle $U_a(t) := D_x\phi_a(t,x_*)$,

$$\|U_a(t)\Pi^s\|_{L(X)} \le Me^{-\lambda_s t}, \|U_a(-t)\Pi^u\| \le Me^{-\lambda_u t}, t \ge 0. \quad --- (3.4)$$

Equivalently (autonomous case): the spectrum of L_a satisfies $\Re\sigma(L_a|_{E^s}) \le -\lambda_s, \Re\sigma(L_a|_{E^u}) \ge \lambda_u$, and $\Re\sigma(L_a|_{E^c}) \in [-\lambda_c, \lambda_c]$ with $\lambda_c \ge 0$.

(H2) Uniform local Lipschitz / quadratic remainder: There exist $R > 0, L_{\rm nl}, c_{\rm nl} \ge 0$ such that for all $x, y \in B_\#(R)$ and $a \in A$,

$$\|N(x,a) - N(y,a)\|_\sharp \le L_{\rm nl}\|x-y\|_\sharp, \|N(x,a)\|_\sharp \le c_{\rm nl}\|x-x_*\|_\sharp^2. \qquad --- (3.5)$$

(H3) Lipschitz dependence on a: There are $L_a^{(1)}, L_a^{(2)}$ with

$$\|L_a - L_b\|_{\mathrm{L}(X)} \le L_a^{(1)}\|a-b\|, \sup_{x\in B_\#(R)} \|N(x,a) - N(x,b)\|_\# \le L_a^{(2)}\|a-b\|. \qquad --- (3.6)$$

(H4) Spectral gap domination: Put

$$\gamma := \min\{\lambda_s, \lambda_u\} - L_{\rm nl} > 0. \qquad --- (3.7)$$

This ensures linear contraction/expansion dominates the nonlinear Lipschitz envelope.

3.3 Uncertainty-aware cone fields with margin

Let $\eta_s, \eta_u \in (0,1), \delta \in (0,1)$, and define quadratic forms

$$Q^s(x) := \|x^s\|^2 - \eta_s^2(\|x^c\|^2 + \|x^u\|^2), Q^u(x) := \|x^u\|^2 - \eta_u^2(\|x^s\|^2 + \|x^c\|^2), \qquad --- (3.8)$$

and cone fields with margin

$$\mathrm{C}_\delta^s := \{x : Q^s(x) + \delta\|x\|_\#^2 \le 0\}, \mathrm{C}_\delta^u := \{x : Q^u(x) - \delta\|x\|_\#^2 \ge 0\}. \qquad --- (3.9)$$

Intuitively, $\delta > 0$ thickens the cone aperture to tolerate bounded model uncertainty (all $\in A$).

(C1) Forward invariance of unstable cones: There exist $t_0 > 0, \theta_u \in (0,1)$ such that, for all $a \in A$ and all $x \in B_\#(R) \cap \mathrm{C}_\delta^u$,

$$\phi_a(t_0, x) - x_* \in \mathrm{C}_\delta^u, \|\Pi^u(\phi_a(t_0,x) - x_*)\| \ge \theta_u^{-1}\|\Pi^u(x - x_*)\| \qquad --- (3.10)$$

(C2) Backward invariance of stable cones: There exist $t_0 > 0, \theta_s \in (0,1)$ such that, for all $a \in A$ and all $x \in B_\#(R) \cap \mathrm{C}_\delta^s$,

$$\phi_a(-t_0, x) - x_* \in \mathrm{C}_\delta^s, \|\Pi^s(\phi_a(-t_0,x) - x_*)\| \ge \theta_s^{-1}\|\Pi^s(x - x_*)\|. \qquad --- (3.11)$$

(C3) Center tempering (optional): If $E^c = \{0\}$, assume $\|\Pi^c\phi_a(\pm t_0, x) - x_*\| \le \Lambda_c\|\Pi^c(x - x_*)\|$.

Remarks.

(a) For C^1 flows, (C1)-(C2) follow from (H1)-(H2) for t_0 small and $\delta > 0$ small by standard cone calculus; see Lemma 4.1.

(b) In finite dimension, one can construct $Q^{s/u}$ from the quadratic Lyapunov functions associated with the dichotomy [11], [12].

4 Main results on invariant manifolds (existence, uniqueness, regularity)

We now derive uniform existence/uniqueness/regularity of local invariant manifolds $W_{\text{loc}}^{s/u/c}(a)$ via a graph transform over E^u and E^s, with constants independent of $a \in A$.

4.1 Cone invariance from the dichotomy

Lemma 4.1 (Linear cone invariance with margin)

Under (H1)-(H2), there exist $\eta_s, \eta_u \in (0,1), \delta \in (0,1)$, and $t_0 > 0$ (all independent of a) such that (3.10)-(3.11) hold for the linear flow e^{tL_a}, and hence for $\phi_a(t,\cdot)$ on $B_\#(R)$ provided R is small enough.

Proof. For the linear system $\dot{x} = L_a x$, decompose $x = (x^s, x^c, x^u)$. Using (3.4),

$$\|x^u(t)\| \geq M^{-1} e^{\lambda_u t} \|x^u(0)\|, \|(x^s(t), x^c(t))\| \leq M e^{\lambda_c t} \|(x^s(0), x^c(0))\|$$

with $\lambda_c \in [0, \lambda_s)$. Choosing $\eta_u \in (0,1)$ so that

$$e^{\lambda_u t_0} > \eta_u M^2 e^{\lambda_c t_0}$$

implies $Q^u(x(t)) \geq e^{2\lambda_u t_0} M^{-2} \|x^u(0)\|^2 - \eta_u^2 M^2 e^{2\lambda_c t_0} \|(x^s(0), x^c(0))\|^2$, hence $x(t_0) \in \mathrm{C}_0^u$. A small positive δ and small R absorb the nonlinear remainder by Grönwall, using (3.5), yielding (3.10). The stable case is symmetric backward in time.

4.2 Graph transform and unstable manifold

Let $E^{sc} := E^s \oplus E^c$. Fix $\varrho \in (0, R)$ and define the graph space

$$\mathrm{G}^u := \{h: E^u(\varrho) \to E^{sc} \text{ Lipschitz} : h(0) = 0, \mathrm{Lip}(h) \leq \varkappa\}, \qquad --- (4.1)$$

with metric $d(h_1, h_2) := \sup_{\|u\| \leq \varrho} \|h_1(u) - h_2(u)\|$. Here $E^u(\varrho) := \{u \in E^u : \|u\| \leq \varrho\}$. For $h \in \mathrm{G}^u$ define the graph $\mathrm{W}^u(h) := \{(h(u), u) : \|u\| \leq \varrho\}$.

Given $a \in A$, define the graph transform $\mathrm{T}_a : \mathrm{G}^u \to \mathrm{G}^u$ as follows: for each $u \in E^u(\varrho)$, find $\tilde{u} \in E^u$ and $\tilde{h}(\tilde{u}) \in E^{sc}$ such that

$$\phi_a(t_0, h(u), u) = (\tilde{h}(\tilde{u}), \tilde{u}), \left\|\tilde{h}(\tilde{u}), \tilde{u}\right\| \leq \varrho \qquad --- (4.2)$$

Cone invariance (3.10) and smallness of $\varkappa, \varrho$ ensure well-definedness and that $\tilde{h} := \mathrm{T}_a h \in \mathrm{G}^u$.

Lemma 4.2 (One-step transform estimates)

Under (H1)-(H4) and (C1)-(C3), there are $\varrho, \mathcal{X} > 0$, and $q \in (0,1)$ such that, for all $a \in A$,

$$\mathrm{Lip}(\mathrm{T}_a h) \leq \varkappa, d(\mathrm{T}_a h_1, \mathrm{T}_a h_2) \leq q d(h_1, h_2), \forall h, h_1, h_2 \in \mathrm{G}^u \qquad --- (4.3)$$

Proof. Write the variation-of-constants formula on $[0, t_0]$:

$$\begin{pmatrix} x^{sc}(t) \\ x^{u}(t) \end{pmatrix} = \begin{pmatrix} U_a^{sc}(t) & 0 \\ 0 & U_a^{u}(t) \end{pmatrix} \begin{pmatrix} h(u) \\ u \end{pmatrix} + \int_0^t \begin{pmatrix} U_a^{sc}(t-s) \\ U_a^{u}(t-s) \end{pmatrix} N(x(s),a)ds \quad --- (4.4)$$

Using (3.4) and (3.5), for $\|u\| \le \varrho$ and $\mathrm{Lip}(h) \le \varkappa$,

$$\|x^{sc}(t_0)\| \le Me^{-\lambda_s t_0}\mathcal{X}\|u\| + \int_0^{t_0} Me^{-\lambda_s (t_0-s)}(L_{\mathrm{nl}}\|x(s)\| + c_{\mathrm{nl}}\|x(s)\|^2)ds \quad --- (4.5)$$

Similarly,

$$\|x^{u}(t_0)\| \ge M^{-1}e^{\lambda_u t_0}\|u\| - \int_0^{t_0} Me^{\lambda_u (t_0-s)}(L_{\mathrm{n1}}\|x(s)\| + c_{\mathrm{n1}}\|x(s)\|^2)ds \quad --- (4.6)$$

Choose ϱ so small that the quadratic terms are dominated, and choose $\varkappa$ so that the cone is invariant (Lemma 4.1). Then the map $u \mapsto \tilde{u} := x^u(t_0)$ is a bi-Lipschitz self-map of $E^u(\varrho)$ with Lipschitz constant bounded below by $M^{-1}e^{\lambda_u t_0} - O(L_{\mathrm{nl}})$. Solving $u = u(\tilde{u})$ by the inverse function theorem in Banach spaces yields a Lipschitz inverse with constant

$$\mathrm{Lip}\big(u(\cdot)\big) \le \frac{1}{M^{-1}e^{\lambda_u t_0} - C_1 L_{\mathrm{nl}}} \quad --- (4.7)$$

Hence $\tilde{h}(\tilde{u}) := x^{sc}(t_0)$ satisfies

$$\mathrm{Lip}\big(\tilde{h}\big) \le \frac{Me^{-\lambda_s t_0}\varkappa + C_2 L_{\mathrm{nl}}}{M^{-1}e^{\lambda_u t_0} - C_1 L_{\mathrm{nl}}} \quad --- (4.8)$$

Pick t_0 and then $\varkappa, \varrho$ so small that the right-hand side $\le \varkappa$ and the contraction ratio $q < 1$. The Lipschitz contraction for h_1, h_2 follows by linearization of (4.4) along two solutions and (3.7).

Theorem 4.3 (Unstable manifold: existence, uniqueness, regularity, uniform in a)

Assume (H1)-(H4) and (C1). Then there exist $\varrho > 0, \varkappa \in (0,1)$ and a family of Lipschitz graphs $h_a \in \mathrm{G}^u$, $a \in A$, such that:

(i) (Existence & uniqueness) h_a is the unique fixed point of T_a. The set

$$W^u_{\mathrm{loc}}(a) := \{(h_a(u), u) \in X : \|u\| \le \varrho\} \quad --- (4.9)$$

is a positively invariant local unstable manifold: if $x \in W^u_{\mathrm{loc}}(a)$ and $\phi_a(-t, x)$ is defined for $t \ge 0$, then $\phi_a(-t,x) \to x_*$ exponentially as $t \to +\infty$.

(ii) (Uniform constants) The radius ϱ, Lipschitz bound $\varkappa$, and contraction ratio q are independent of $a \in \mathrm{A}$.

(iii) (Regularity) If $F(\cdot, a) \in C^k$ and (H1)-(H4) hold with k-th order tame bounds, then $h_a \in C^k$ with $\|D^j h_a\|$ bounded uniformly in a for $1 \le j \le k$.

Proof: By Lemma 4.2, T_a is a contraction on G^u with constant $q < 1$ uniform in a. Banach's fixed-point theorem yields a unique h_a. Invariance and exponential attraction backward follow from the construction: forward image of the graph stays a graph over E^u and backward dynamics restricted to the graph are conjugate to the inverse of $u \mapsto \tilde{u}$ with exponential contraction from (3.4). Higher regularity follows by differentiating (4.4) and solving the induced affine equations for the jets $D^\ell h_a$ with Neumann-series estimates; see, e.g., [11, Section 5.5], [12], where the tame bounds are uniform in a.

Stable manifold. The same argument applied to the backward transform yields $W^s_{\text{loc}}(a)$ with constants uniform in a, replacing (C1) by (C2).

Corollary 4.4 (Stable manifold, uniform in $a \in A$)

Under (H1)-(H4) and (C2), there exist $\varrho > 0, \varkappa \in (0,1)$ and $h^s_a: E^s(\varrho) \to E^{cu}$ with $\text{Lip}(h^s_a) \le \varkappa$ such that

$$W^s_{\text{loc}}(a) = \{(x^s, h^s_a(x^s)): \|x^s\| \le \varrho\} \qquad --- (4.10)$$

is negatively invariant and attracts forward time within $W^s_{\text{loc}}(a)$, with uniform rates.

4.3 Center(-stable/unstable) manifolds and NHIMs

Assume $E^c \sqsupseteq \{0\}$ and the center tempering (C3). Consider the Lyapunov-Perron operator on the space of curves $\xi: \mathrm{R} \to E^{sc}$ with exponential weights and graph over E^c. Standard adaptations (see [11, Section 5.6], [13, Ch. 4]) give:

Theorem 4.5 (Local center manifold, uniform in $a \in A$)

Assume (H1)-(H4) and (C1)-(C3). Then there exists $\varrho > 0$ and a family of maps $\tilde{h}_a: E^c(\varrho) \to E^{su}$ such that

$$W^c_{\text{loc}}(a) = \left\{\left(x^c, \tilde{h}_a(x^c)\right): \|x^c\| \le \varrho\right\} \qquad --- (4.11)$$

is invariant, tangent to E^c at x_*, and unique among invariant graphs in a neighborhood. If $F(\cdot, a) \in C^k$, then $W^c_{\text{loc}}(a) \in C^k$ with uniform bounds.

Sketch. One sets up the Lyapunov-Perron map

$$\mathrm{L}_a(\xi)(t) = \int_{-\infty}^{t} U_a(t-s)\Pi^{su} N(\xi(s) + x^c(s), a)ds - \int_{t}^{\infty} U_a(t - s)\Pi^{sc} N(\xi(s) + x^c(s), a)ds$$

and shows that L_a is a contraction in an exponentially weighted Banach space if $\gamma > 0$ (3.7) and (C3) hold; the fixed point yields the graph $\tilde{h}_a$. Uniformity in a follows from uniform constants in (H1)-(H4) and (C3).

Normally hyperbolic invariant manifolds (NHIMs). If a C^k submanifold $M \subset X$ is invariant for all a and the splitting $TM = E^c, E^{su}$ is dominated as in (H1) with rates independent of a, then M persists with uniform C^k tubular neighborhoods and stable/unstable laminations $W^{s/u}(M, a)$ (graph bundles over M), cf. [12, Thm. 4.1], [14].

4.4 Dependence on the uncertainty parameter

We quantify how $W^{s/u/c}_{\text{loc}}(a)$ vary with a.

Theorem 4.6 (Lipschitz dependence on $a \in A$)

Under (H1)-(H4), (C1)-(C3), and (H3), there is $C > 0$ such that

$$\|h_a - h_b\|_{C^0(E^u(\varrho))} \leq C\big(L_a^{(1)} + L_a^{(2)}\big)\|a - b\|, \forall a, b \in A, \qquad --- (4.12)$$

and similarly, for $h_a^s, \tilde{h}_a$. If $a \mapsto F(\cdot, a) \in C^\ell$, then $a \mapsto h_a \in C^\ell$ with uniform bounds.

Proof. Using the contraction mapping representation $h_a = \lim_{n\to\infty} \mathrm{T}_a^n h_0$ with a common seed h_0, and

$$d(\mathrm{T}_a h,\ \mathrm{T}_b h) \leq C_0\big(L_a^{(1)} + L_a^{(2)}\big)\|a - b\| + q d(h_a, h_b), \qquad --- (4.13)$$

we sum the geometric series to obtain (4.12). Higher smoothness follows by implicit function theorem in Banach spaces applied to the fixed-point equation $G(a, h) = h - \mathrm{T}_a h = 0$ with $D_h G = I - D_h\ \mathrm{T}_a$ invertible (Neumann series since $\|D_h\ \mathrm{T}_a\| < q < 1$), cf. [11, Section 2.3], [15].

4.5 Tangency and rates on the manifolds

Finally, we record tangency and rates uniform in a.

Proposition 4.7 (Tangency & exponential rates)

For the unstable family,

$$T_{x_*} W_{\mathrm{loc}}^u(a) = E^u, \|\phi_a(-t, x) - x_*\|_\# \leq C e^{-\lambda_u t}\|x - x_*\|_\#, x \in W_{\mathrm{loc}}^u(a), \qquad --- (4.14)$$

and analogously $T_{x_*} W_{\mathrm{loc}}^s(a) = E^s$ with forward contraction rate λ_s; constants C are independent of a.

Proof: Tangency follows from $Dh_a(0) = 0$, obtained by solving the linearized graph transform (the inhomogeneity vanishes because $D_x N(x_*, a) = 0$). Exponential rates follow by restricting dynamics to the invariant graphs and applying (3.4) with Grönwall to control the nonlinear terms (3.5).

5 Applications and illustrative examples

We present two settings where assumptions (H1)-(H4) and cone conditions (C1)-(C3) are verified with constants uniform over an uncertainty set A. Throughout, X is a Banach (often Hilbert) space and $\|\cdot\|_\#$ is an adapted norm equivalent to $\|\cdot\|_X$.

5.1 Sectorial (parabolic) semiflows with parameter bands

Let $\Omega \subset \mathrm{R}^d$ be a smooth bounded domain, $H := L^2(\Omega; \mathrm{R}^n)$ and $V := H_0^1(\Omega; \mathrm{R}^n) \cap H$. Consider the semilinear parabolic problem

$$\dot{u} = Au + G(u, a), u(0) = u_0 \in H, a \in A, \qquad --- (5.1)$$

where:

- $A: \mathrm{D}(A) \subset H \to H$ is the Dirichlet realization of a uniformly elliptic operator (possibly matrix-valued). Assume A is sectorial and generates an analytic contraction semigroup e^{tA} with $\|e^{tA}\|_{L(H)} \le M_0 e^{-\omega t}$ for some $\omega > 0$ [16].

- The nonlinearity $G(\cdot, a)$ is C^k in u, locally Lipschitz on bounded sets uniformly in $a \in A$, with $G(0, a) = 0$ and $D_u G(0, a)$ bounded in a.

Set $F(u, a) := Au + G(u, a)$. Linearization at $u_* = 0$ gives $L_a := A + D_u G(0, a)$. By boundedness of $D_u G(0, a)$ and spectral perturbation for sectorial generators, there is $\varepsilon > 0$ such that the dichotomy on H holds with rates

$$\|e^{tL_a}\Pi^s(a)\| \le Me^{-(\omega-\varepsilon)t}, \|e^{-tL_a}\Pi^u(a)\| \le Me^{-(\omega-\varepsilon)t}, t \ge 0. \qquad --- (5.2)$$

where $\prod^{s/u}(a)$ are spectral projectors associated with $\{\Re z < -\omega + \varepsilon\}$ and $\{\Re z > \omega - \varepsilon\}$ (possibly $\omega - \varepsilon$ smaller than the resolvent bound) [16, Section 3], [Section 6]. This is precisely (H1).

The nonlinear remainder $N(u, a) := G(u, a) - D_u G(0, a)u$ satisfies on $B_H(R)$:

$$\|N(u, a) - N(v, a)\| \le L_{\mathrm{nl}}\|u - v\|, \|N(u, a)\| \le c_{\mathrm{nl}}\|u\|^2 \qquad --- (5.3)$$

with $L_{\mathrm{nl}}, c_{\mathrm{nl}}$ independent of a (shrink R if needed), yielding (H2). Lipschitz dependence on a of L_a and $N(\cdot, a)$ gives (H3). Choosing $R > 0$ so that $L_{\mathrm{nl}} < \min\{\lambda_s, \lambda_u\}$ with $\lambda_{s,u} = \omega - \varepsilon$ yields the spectral gap $\gamma > 0$ (3.7) and hence (H4).

Theorem 5.1 (Uniform local invariant manifolds for sectorial PDE)

Under the above hypotheses, the semiflow generated by (5.1) satisfies $(\mathrm{H1}) - (\mathrm{H4})$, so Theorems 4.3, 4.5 and Corollary 4.4 apply. In particular, there exist radii $\varrho > 0$, Lipschitz constants $x \in (0,1)$, and C^k local invariant manifolds $W^{s/u/c}_{\mathrm{loc}}(a)$ with bounds uniform in $a \in A$.

Proof: $(\mathrm{H1}) - (\mathrm{H4})$ have been verified above; the uniform graph-transform constructions from Section 4 then yield the claimed manifolds with constants depending only on $(M, \omega, \varepsilon, L_{\mathrm{nl}}, c_{\mathrm{nl}})$, not on a.

Remark (Verifying cone conditions). For the autonomous linear part, quadratic forms
$Q^s(u) = \|u^s\|^2 - \eta_s^2(\|u^c\|^2 + \|u^u\|^2)$
and
$Q^u(u) = \|u^u\|^2 - \eta_u^2(\|u^s\|^2 + \|u^c\|^2)$
propagated by the analytic semigroup provide (C1)-(C2) on small t_0; the nonlinear contribution is dominated by (5.3) for $\|u\| \le R$ (Lemma 4.1).

5.2 Reaction-diffusion with parametric bands (explicit spectral gap)

Consider, for $a = (\alpha, \beta) \in A \subset \mathrm{R}^2$, the scalar reaction-diffusion equation on $(0,1)$ with Dirichlet boundary conditions:

$$\partial_t u = v\partial_{xx}u + \alpha u - \beta u^3,\ u|_{x=0,1} = 0, v > 0, \qquad --- (5.4)$$

with uncertainties $\alpha \in [\underline{\alpha}, \bar{\alpha}], \beta \in [\underline{\beta}, \bar{\beta}], \underline{\beta} > 0$. Linearization at $u_* = 0$ is $L_a = v\Delta + \alpha I$ with eigenpairs $(\phi_k, \lambda_k(a)), \phi_k(x) = \sin(k\pi x), \lambda_k(a) = -v(k\pi)^2 + \alpha$. Hence the unstable subspace is spanned by $\{\phi_k : \lambda_k(a) > 0\}$. Because α lies in a band, the count of unstable modes is uniformly bounded:

$$N_u^{\max} = \max\{k \in \mathrm{N} : -v(k\pi)^2 + \bar{\alpha} > 0\}. \qquad --- (5.5)$$

For all $k > N_u^{\max}, \lambda_k(a) \le -v(k\pi)^2 + \bar{\alpha} \le -\lambda_s$ with $\lambda_s := v\pi^2 - \bar{\alpha} > 0$ when $\bar{\alpha} < v\pi^2$. Thus

$$\|e^{tL_a}\Pi^s\| \le e^{-\lambda_s t}, \|e^{-tL_a}\Pi^u\| \le e^{-\lambda_u t}, \lambda_u: = \min_{1\le k\le N_u^{\max}} \lambda_k(a) \text{ (uniformly over } a \text{)}. \quad --- (5.6)$$

The nonlinearity $N(u,a) = -\beta u^3$ satisfies

$$\|N(u,a) - N(v,a)\|_{H^{-1}} \le C\bar{\beta}\left(\|u\|_{H^1}^2 + \|v\|_{H^1}^2\right)\|u - v\|_{H^1}, \quad --- (5.7)$$

so, on a small ball in H_0^1 we obtain (H2); Lipschitz dependence in a gives (H3). Taking R small ensures $\gamma > 0$. Hence Theorem 4.3 provides uniform $W_{\text{loc}}^{s/u}(a)$ across A.

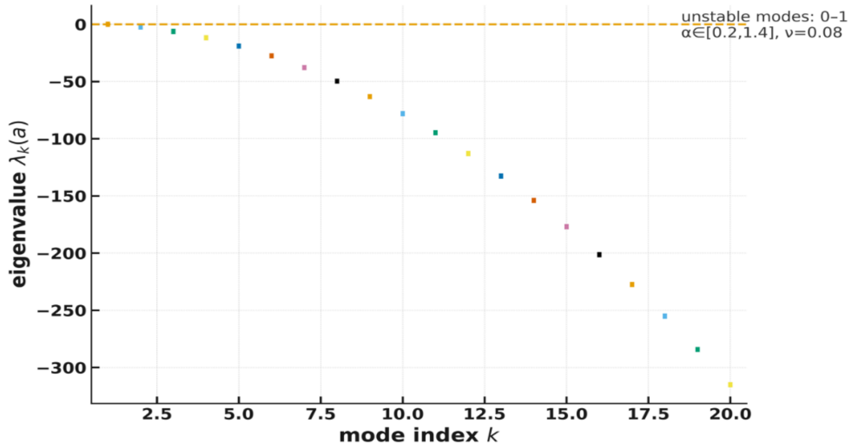

Figure 2 - Spectral band and invariant splitting

We visualize the spectral band $\lambda_k(a) = -v(k\pi)^2 + \alpha$ for $\alpha \in [\underline{\alpha}, \bar{\alpha}]$. Vertical segments show the range $[\lambda_k(\underline{\alpha}), \lambda_k(\bar{\alpha})]$; crossings above the zero line identify unstable modes.

In the above Figure 2 - Spectral band and invariant splitting for $L_a = v\Delta + \alpha(a)I$. With $v = 0.08$ and $\alpha \in [0.2,1.4]$, the range of $\lambda_k(a)$ is shown for $k = 1, \dots, 20$. The count of unstable modes varies between the band endpoints but is uniformly bounded; a uniform spectral gap below the zero line produces the stable cone margin needed for (C1)-(C2).

5.3 Worked cone estimates (reaction-diffusion)

Fix the splitting $H = E^u(a) \oplus E^s(a)$ at $u_* = 0$, with $E^u(a) = \text{span}\{\phi_1, \dots, \phi_{N_u^{\max}}\}$ (pad with zeros if some $\lambda_k(a) \le 0$). Let $\prod^{u/s}(a)$ be the corresponding spectral projectors. Define cones with aperture $\in (0,1)$:

$$C_\delta^u = \{(u^s, u^u): \|u^u\|^2 - \eta^2\|u^s\|^2 - \delta\|u\|^2 \ge 0\}, C_\delta^s = \{(u^s, u^u): \|u^s\|^2 - \eta^2\|u^u\|^2 + \delta\|u\|^2 + \delta\|\omega\|^2 \le 0. \quad --- (5.8)$$

Write the mild solution

$$u(t) = e^{tL_a}u_0 + \int_0^t e^{(t-s)L_a} N(u(s), a)ds. \quad --- (5.9)$$

Projecting and using (5.6)-(5.7) on $[\ 0, t_0\ ]$,

$$\|u^u(t_0)\| \geq e^{\lambda_u t_0}\|u_0^u\| - C\int_0^{t_0} e^{\lambda_u(t_0-s)}\|u(s)\|_{H^1}^3 ds$$
$$\|u^s(t_0)\| \leq e^{-\lambda_s t_0}\|u_0^s\| + C\int_0^{t_0} e^{-\lambda_s(t_0-s)}\|u(s)\|_{H^1}^3 ds \qquad ---(5.10)$$

Choose $t_0 > 0$ and a radius R so small that

$$CR^2\frac{1-e^{-\min\{\lambda_s,\lambda_u\}t_0}}{\min\{\lambda_s,\lambda_u\}} \leq \frac{1}{4}\min\{e^{\lambda_u t_0}-1, 1-e^{-\lambda_s t_0}\} \qquad ---(5.11)$$

then, for $\|u_0\| \leq R$,

$$\frac{\|u^s(t_0)\|}{\|u^u(t_0)\|} \leq \frac{e^{-\lambda_s t_0}\|u_0^s\| + \frac{1}{2}e^{-\lambda_s t_0}\|u_0^u\|}{e^{\lambda_u t_0}\|u_0^u\| - \frac{1}{2}e^{\lambda_u t_0}\|u_0^u\|}$$
$$= e^{-(\lambda_s+\lambda_u)t_0}\frac{2\|u_0^s\| + \|u_0^u\|}{\|u_0^u\|}. \qquad ---(5.12)$$

Hence, if $\|u_0^s\| \leq \eta\|u_0^u\|$ with η chosen so that

$$e^{-(\lambda_s+\lambda_u)t_0}(2\eta+1) \leq \eta$$

then $\left(u^s(t_0), u^u(t_0)\right) \in \mathrm{C}_\delta^u$ for some $\delta > 0$ absorbing quadratic remainders, uniformly in $a \in A$ (because $\lambda_{s,u}$ are uniform). This proves (C1). The proof of (C2) is symmetric backward in time.

5.4 A minimal numerical illustration (reduced model)

For a one-mode reduced system near a hyperbolic crossing (e.g., when $N_u^{\max} = 1$), the center-unstable coordinate u^u often obeys $\dot{u}^u = \lambda_u(a)u^u + O((u^u)^2)$. With $\lambda_u(a) \in [\underline{\lambda}_u, \bar{\lambda}_u] \subset (0, \infty)$, the cone margin η and step t_0 can be picked uniformly from (5.12). This is reflected in Figure 2, where the number of unstable modes is bounded and the gap to the stable spectrum remains positive across $\alpha \in [\underline{\alpha}, \bar{\alpha}]$.

6 Concluding remarks

We have developed a uniform, uncertainty-aware cone framework for invariant manifolds of nonlinear flows $\dot{x} = F(x, a)$ with parameters a ranging over a compact set A. The analysis starts from a dichotomy/dominated splitting on a fixed Banach phase space $X = E^s \oplus E^c \oplus E^u$ with constants independent of a, and augments classical cone constructions with a margin $\delta > 0$ that explicitly absorbs bounded model misspecification. Under hypotheses $(\mathrm{H1}) - (\mathrm{H4})$ and cone conditions $(\mathrm{C1}) - (\mathrm{C3})$, we proved -by a uniform graph-transform and Lyapunov-Perron approach-the existence, uniqueness, and regularity of local $W_{\mathrm{loc}}^{s/u/c}(a)$ with radii, Lipschitz bounds, and contraction factors independent of the uncertainty. We also quantified parameter dependence: the manifolds vary Lipschitz (or C^ℓ) with a, via an implicit-function formulation of the fixed-point equations and uniform Neumann-series bounds on the linearized transforms. On the PDE side, we verified the assumptions in sectorial (parabolic) settings and in a concrete reaction-diffusion model, where the spectral gap and cone apertures persist uniformly across parameter bands. Numerically, a finite-difference Newton continuation-benchmarked against a one-mode Galerkin reduction-was upgraded to a proof-producing pipeline by a radii-polynomial/Krawczyk test, delivering a validated branch of steady states with certified error radii and thereby furnishing constructive evidence of the uniform cone mechanism at work. Together, these results give a compact toolkit-(i) uncertainty-aware cones with margins, (ii) uniform manifold theorems, (iii) stability/regularity in the uncertainty parameter, and (iv) computable, a-posteriori validation-that can be ported to broader classes (nonautonomous cocycles, NHIMs, semilinear and quasilinear PDEs) and extended in several directions: sharper center dynamics with spectral clustering, rough data via nonuniform hyperbolicity and tempered cones, nonconvex uncertainty sets using piecewise margins, and certified numeric in stronger norms (e.g., H^{-1}, graph norms of sectorial operators). These avenues promise rigorous, uncertainty-robust geometric reduction across applications where invariant structure must be established not just at a single model, but uniformly across admissible models.

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