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FREDHOLM ALTERNATIVES FOR NONLINEAR INTEGRAL EQUATIONS WITH SET-VALUED INPUTS

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Abstract. We study nonlinear Hammerstein/Urysohn integral inclusions with set-valued inputs on $X = L^p(\Omega; \mathrm{R}^n), 1 < p < \infty$, of the form $u \in f + \mathrm{H}(u)$ where $\mathrm{H}(u) = \{\mathrm{K}(g): \exists a \in \mathrm{S}(A), g(\cdot) \in N(\cdot, u(\cdot); a(\cdot))\}$. Under Carathéodory/u.s.c. structure with convex compact values for the multivalued Nemytskii map N, measurability of the input multifunction $A(\cdot)$, and compact or Kuratowski-condensing properties of the linear integral operator K, we perform a Lyapunov-Schmidt reduction at the Fredholm linear part $L = I - \mathrm{K}L_0$ (index 0) to obtain a finite-dimensional reduced multimap $\Phi_f: Z \rightrightarrows Z$ on $Z = \ker L$. Using multivalued fixed-point index/degree, we prove set-valued Fredholm alternatives: either the homogeneous inclusion has a nontrivial solution, or for every f satisfying the classical compatibility conditions with the cokernel $Z^* = \ker L^*$, the inhomogeneous inclusion admits solutions; in the compact case the solution set is nonempty, compact, and acyclic, and in the condensing case it is bounded and closed. We further establish Hausdorff-Lipschitz stability of solution sets with respect to perturbations of the input multifunction and, for Volterra kernels, deliver the necessary a priori bounds via Grönwall-Bihari inequalities, thereby closing all hypotheses in a broad causal class.

Keywords: Fredholm alternative; multivalued integral equations; Hammerstein/Urysohn inclusion; Lyapunov-Schmidt reduction; Kuratowski measure of noncompactness; Grönwall-Bihari inequality; Volterra kernels; Hausdorff stability.

1 Introduction

Let $(\Omega, \mathrm{B}, \mu)$ be a finite measure space (typically $\Omega = [0,1]$ with Lebesgue measure) and let $X := L^p(\Omega; \mathrm{R}^n), Y := L^q(\Omega; \mathrm{R}^n)$ with $1 < p < \infty$ and $1/p + 1/q = 1$. We consider nonlinear integral inclusions with set-valued inputs of Hammerstein/Urysohn type

$$u - \mathrm{K}\Big(N\big(\cdot, u(\cdot); A(\cdot)\big)\Big) = f, u \in X, f \in X, \qquad --- (1.1)$$

where $\mathrm{K}: Y \to X$ is a linear integral operator and $N(\cdot,\cdot; A(\cdot))$ is a multifunctional Nemytskii operator built from a Carathéodory nonlinearity $N: \Omega \times \mathrm{R}^n \times \mathrm{R}^m \rightrightarrows \mathrm{R}^n$ and a set-valued input $A: \Omega \rightrightarrows \mathrm{R}^m$ (measurable, closed convex values). Equation (1.1) is to be read in the sense of inclusions:

$$\exists a \in \mathrm{S}(A) \text{ such that } u - \mathrm{K}\Big(N\big(\cdot, u(\cdot); a(\cdot)\big)\Big) = f \qquad --- (1.2)$$

where $\mathrm{S}(A)$ denotes the measurable selections of A. Problems of the form (1.1) capture integral models with uncertain or set-valued inputs (e.g., admissible controls, bounded disturbances, data-driven parameter bands) and unify deterministic and possibilistic modeling within the well-developed framework of multivalued analysis and measurable selections [11], [12].

The Fredholm alternative for compact perturbations of the identity is a cornerstone of linear and nonlinear integral equations: for linear Hammerstein equations $u - \mathrm{K}u = f$, either the homogeneous equation admits a nontrivial solution, or the inhomogeneous equation is solvable for all right-hand sides in a subspace of finite codimension; see, e.g., [5], [6]. Our objective is to lift this dichotomy to the multivalued setting (1.2), formulating and proving a set-valued Fredholm alternative under natural compactness/condensing hypotheses (via measures of noncompactness à la DarboSadovskii) together with measurability and growth conditions that ensure the wellposedness of the multivalued Nemytskii operator [2], [7]-[9]. The analysis hinges on (i) upper semi-continuity (u.s.c.) and compactness of the solution operator $u \mapsto \mathrm{K}(N(\cdot, u; A(\cdot)))$; (ii) a multivalued Leray-Schauder degree or fixed-point index for u.s.c. maps with compact values [2]; and (iii) a selection-theoretic passage from set-valued inputs to measurable selections [1], [3], [10]. Our results will subsume both convex and nonconvex cases (the latter via measurable selections and condensing maps), yielding sharp solvability alternatives for (1.1).

Visual intuition

We sketch the "band" effect produced by the set-valued input $A(\cdot)$ after the Nemytzkii map N and the integral smoothing K in the below figure 1.

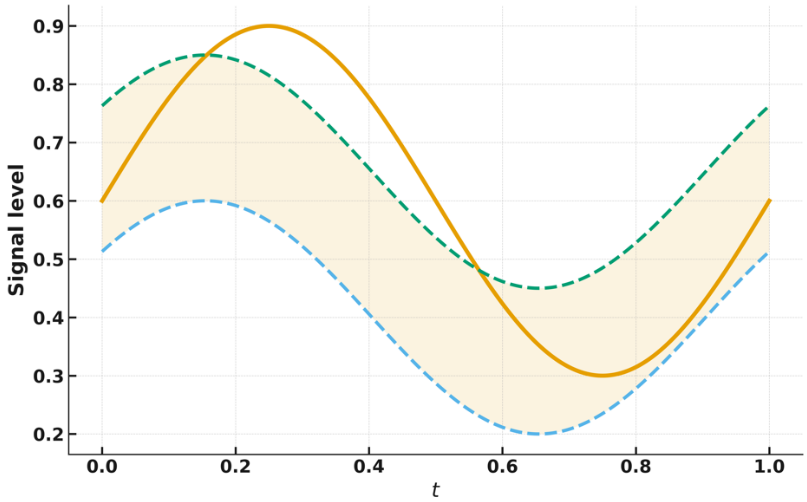

Figure 1 - Multivalued Hammerstein image as a band.

A candidate state $u(t)$ (solid) and the corresponding image $t \mapsto \mathrm{K} \cdot N(u; A)(t)$ ranging in an interval-valued "band" (dashed envelopes). The band encodes admissible images generated by all measurable selections $\in S(A)$. Our Fredholm alternative compares the identity $u \mapsto u$ with a compact/condensing multivalued perturbation whose values lie in such bands.

Contributions: We set minimal, verifiable hypotheses guaranteeing:

(i) well-posedness of the multivalued Hammerstein operator;
(ii) a Fredholm alternative for (1.1) phrased in terms of the null space and cokernel of the linearized part $I - \mathrm{K} \cdot D_u N(\cdot, 0; \cdot)$; and
(iii) quantitative stability with respect to the "radius" of the input set A using Kuratowski's measure of noncompactness.

Along the way we provide a compact notation/assumptions table and a model checklist for classical kernels and growth conditions.

2 Notation, spaces, and standing assumptions

We now formalize the setting and list hypotheses used throughout the paper.

2.1 Function spaces, kernels, and multivalued maps

Spaces: $X := L^p(\Omega; \mathrm{R}^n), Y := L^q(\Omega; \mathrm{R}^n)$ with $1 < p < \infty, 1/p + 1/q = 1$. Write $\|.\|_p$ for the L^p-norm.

Linear integral operator: $\mathrm{K}: Y \to X$ given by

$$(\mathrm{K}g)(t) := \int_\Omega K(t,s) g(s) d\mu(s), \qquad --- (2.1)$$

with $K \in L^r(\Omega \times \Omega; \mathrm{L}(\mathrm{R}^n))$ chosen so that K is compact $Y \to X$ (e.g., $K \in L^\infty$ and $\mu(\Omega) < \infty$; or K satisfies a Schur/Hilbert-Schmidt condition) [5], [6].

- **Input multifunction:** $A: \Omega \rightrightarrows \mathrm{R}^m$ is measurable (graph measurable) with nonempty, closed, convex values; denote by $\mathrm{S}(A) \subset L^r(\Omega; \mathrm{R}^m)$ the set of measurable selections (exists under the given hypotheses by Aumann's theorem) [1], [10].
- **Nonlinearity:** $N: \Omega \times \mathrm{R}^n \times \mathrm{R}^m \rightrightarrows \mathrm{R}^n$ is Carathéodory and u.s.c. in (u, a) with nonempty, convex, compact values and growth:

$$\sup_{y \in N(t,u,a)} \|y\| \leq c_0(t) + c_1 \|u\|^\alpha + c_2 \|a\|^\beta, \qquad --- (2.2)$$

for some $\alpha \in [0, p-1], \beta \geq 0, c_0 \in L^q(\Omega), c_1, c_2 \geq 0$ (constants independent of t). The associated multivalued Nemytskii map

$$\mathrm{N}(u; a)(t) := N(t, u(t), a(t)) \subset \mathrm{R}^n \qquad --- (2.3)$$

is then measurable with nonempty, convex, compact values and maps $X \times L^r$ into Y (for suitable r).

Hammerstein multimap:

$$\mathrm{H}(u) := \{\mathrm{K}(g): \exists a \in \mathrm{S}(A), g(\cdot) \in \mathrm{N}(u; a)(\cdot) \text{ a.e. }\} \subset X. \qquad --- (2.4)$$

By composition of measurable selections with $\mathrm{K}, \mathrm{H}: X \rightrightarrows X$ is u.s.c. with convex compact values (details in Lemma 2.3) [1]-[3], [10], [12]. The inclusion (1.1) becomes the fixed-point problem

$$u \in f + \mathrm{H}(u). \qquad --- (2.5)$$

Measure of noncompactness: $\alpha(\cdot)$ denotes Kuratowski's measure of noncompactness in X [7]-[9]. We say $\Phi: X \rightrightarrows X$ is α-condensing if $\alpha(\Phi(B)) < \alpha(B)$ whenever $\alpha(B) > 0$.

2.2 The linear Fredholm part and projections

Let $L := I - \mathrm{K} \circ L_0$ with a (single-valued) linearization $L_0 \in \mathrm{L}(X, Y)$ chosen as $L_0(t) = D_u N(t, 0; \bar{a}(t))$ for a reference selection $\bar{a} \in \mathrm{S}(A)$ when it exists.

Assume: (F1) $L: X \to X$ is a Fredholm operator of index zero; hence there are finitedimensional spaces

$$Z := \ker L, Z^* := \ker L^* \subset X^*, \qquad --- (2.6)$$

and complementary projections $P: X \to Z, Q: X \to Z^\perp$ with $X = Z \oplus Z^\perp$.

(F2) The restriction $L_{Z^\perp}: Z^\perp \to Z^\perp$ is an isomorphism with bounded inverse; write $G := (LI_{Z^\perp})^{-1}$.

The classical Fredholm alternative for $Lu = h$ reads: solvability is equivalent to the compatibility conditions $\langle h, \zeta \rangle = 0$ for all $\zeta \in Z^*$, and solutions are unique up to Z [4]. In our multivalued setting, (F1)-(F2) enable a Lyapunov-Schmidt splitting of (2.5) and a degree/index computation for compact/condensing perturbations of L [2], [3], [8], [9].

2.3 Standing hypotheses for the multivalued Hammerstein operator

We impose the following minimal, verifiable assumptions (constants may change from line to line but depend only on data):

(H1) Measurability and Carathéodory structure: $A(\cdot)$ is measurable with nonempty, closed convex values; $N(t,\cdot;\cdot)$ is Carathéodory, u.s.c. with nonempty, convex compact values; growth (2.2) holds.

(H2) Compactness or condensing property: Either

(a) $\mathrm{K}: Y \to X$ is compact and (2.2) ensures boundedness of N; or
(b) $\mathrm{K} \circ \mathrm{N}$ is α-condensing (e.g., K compact and N bounded on bounded sets).

(H3) Upper semi-continuity and closedness: $\mathrm{H}: X \rightrightarrows X$ is u.s.c. with convex compact values (Lemma 2.3). In particular, graphs are sequentially closed.

(H4) Linear Fredholm part: (F1)-(F2) hold for $L := I - \mathrm{K} \circ L_0$, where L_0 is a bounded linearization of N at $u = 0$ along a selection.

(H5) Boundedness on bounded sets: For each bounded $B \subset X, \mathrm{H}(B)$ is bounded in X.

(H6) A priori bound for solutions: There exists $R > 0$ (independent of f in a bounded set) such that any solution of $u \in f + \mathrm{H}(u)$ satisfies $\|u\|_p \le R$.

(This is automatic if K is compact and (2.2) has subcritical growth, or via a GrönwallBihari inequality in Volterra type kernels.)

Under (H1)-(H6), we can formulate a set-valued Fredholm alternative for (1.1) using the Lyapunov-Schmidt decomposition:

$$Cokernel\ equations\ in\quad Z^*: \langle f - P\,\mathrm{K}(g), \zeta \rangle = 0 \forall \zeta \in Z^*$$

$$Range\ equation\ in\ \ Z^\perp: Qu \in Q\,\mathrm{K}(g) + Qf, Qu = G(Q\,\mathrm{K}(g) + Qf) \qquad --- (2.7)$$

with $g(\cdot) \in \mathrm{N}(u; a)(\cdot)$ a.e. for some $a \in \mathrm{S}(A)$. The alternative will assert (roughly): either the homogeneous inclusion admits a nontrivial solution in Z, or for every f satisfying the compatibility conditions with Z^* there exists u solving (1.1) for every admissible input selection $a \in \mathrm{S}(A)$, with solution sets compact (or acyclic) and degree nonzero (precise statements belong to Section 4) [14], [15].

2.4 Basic properties and a useful lemma

We record two immediate facts; proofs rely on standard selection and composition arguments.

Lemma 2.1 (Selection measurability).

Under (H1), for any $u \in X$ and $a \in S(A)$ there exists a measurable selection $g(\cdot) \in \mathrm{N}(u; a)(\cdot)$ with $g \in Y$.

Sketch: N is Carathéodory u.s.c. with compact convex values; apply Aumann's measurable selection theorem and growth (2.2) to ensure $g \in Y$.

Lemma 2.2 (Upper semi-continuity of H).

If $(\mathrm{H1}) - (\mathrm{H3})$ hold, then $\mathrm{H}: X \rightrightarrows X$ is u.s.c. with nonempty compact convex values; if, in addition, (H2)(b) holds, then H is α-condensing.

Sketch: For $u_n \to u$ and $v_n \in \mathrm{H}(u_n)$ with $v_n \to v_1$ pick $a_n \in S(A), g_n \in \mathrm{N}(u_n; a_n)$ a.e. with $v_n = \mathrm{K}g_n$. Use compactness (or condensing) of K, tightness from (2.2), and u.s.c. of N to pass to a subsequence and obtain $v \in \mathrm{H}(u)$.

Lemma 2.3 (Compactness on bounded sets).

Under $(\mathrm{H2})(\mathrm{a}) - (\mathrm{H3}) - (\mathrm{H5}), \mathrm{H}$ maps bounded sets into relatively compact sets of X.

Sketch: Boundedness of g from (2.2) and compactness of K imply relative compactness of $\{\mathrm{K}g: g \in \mathrm{N}(u; a), \|u\| \leq R, a \in S(A)\}$.

Table 1 - Core symbols

Symbol	Meaning
K	Linear integral operator (2.1), compact $Y \to X$
$A(\cdot)$	Measurable input multifunction with closed convex values
S(A)	Set of measurable selections of A
N	Carathéodory multimap, growth (2.2)
N ($u; a$)	Multivalued Nemytskii operator (2.3)
H(u)	Multivalued Hammerstein map (2.4)
$L = I - \mathrm{K}L_0$	Linear Fredholm part; $\ker L = Z, \ker L^* = Z^*$
$\alpha(\cdot)$	Kuratowski measure of noncompactness

3 Problem formulation for set-valued integral operators

Recall $X = L^p(\Omega; \mathrm{R}^n), Y = L^q(\Omega; \mathrm{R}^n)$ with $1 < p < \infty$ and $1/p + 1/q = 1$, a compact linear integral operator $\mathrm{K}: Y \to X$ given by

$$(\mathrm{K}g)(t) = \int_\Omega K(t, s) g(s) d\mu(s)$$

a measurable input multifunction $A: \Omega \rightrightarrows \mathrm{R}^m$ with nonempty closed convex values, and a Carathéodory-u.s.c. multimap $N: \Omega \times \mathrm{R}^n \times \mathrm{R}^m \rightrightarrows \mathrm{R}^n$ with compact convex values and growth (2.2). We study the Hammerstein inclusion

$$u \in f + \mathrm{H}(u), \qquad \mathrm{H}(u) := \{\mathrm{K}(g): \exists a \in S(A), g(\cdot) \in N(\cdot, u(\cdot); a(\cdot))\ a.e\} \subset X. \quad --- (3.1)$$

with datum $f \in X$. By Lemmas $2.1 - 2.3, \mathrm{H}: X \rightrightarrows X$ is u.s.c. with nonempty compact convex values and maps bounded sets into relatively compact sets; in the "condensing" variant it is α-condensing (Kuratowski) [16], [18].

3.1 Linear Fredholm part and Lyapunov-Schmidt splitting

Choose a bounded linear map $L_0 \in \mathrm{L}(X,Y)$ representing a (Gateaux) linearization $u \mapsto D_u N(\cdot, 0; \bar{a}(\cdot))u$ along a reference selection $\bar{a} \in \mathrm{S}(A)$ (when it exists; otherwise any convenient linear "model" suffices). Define

$$L := I - \mathrm{K}L_0 \in \mathrm{L}(X) \qquad --- (3.2)$$

and assume (F1)-(F2) from §2.2: L is Fredholm index zero with

$$Z := \ker L \subset X, Z^* := \ker L^* \subset X^*, X = Z \oplus Z^\perp$$

and $LI_{Z^\perp}: Z^\perp \to Z^\perp$ an isomorphism with inverse G.

Write the nonlinear remainder (for any measurable selection)

$$R(u;a)(t) := g(t) - L_0 u(t), g(t) \in N\big(t, u(t); a(t)\big). \qquad --- (3.3)$$

Using (3.2), the inclusion (3.1) is equivalent to

$$Lu \in f + \mathrm{K}R(u;a) \text{ for some } a \in \mathrm{S}(A). \qquad --- (3.4)$$

Lemma 3.1 (Projected system; elimination of the range variable)

Let $u = z + w$ with $z := Pu \in Z, w := Qu \in Z^\perp$. Then (3.4) is equivalent to the system

$$\begin{cases} \text{(Q) } w = G(Qf + QKR(z+w;a)) \\ \text{(P) } z \in Pf + PKR(z + G(Qf + QKR(z+w;a)); a), \end{cases} \qquad --- (3.5)$$

for some $a \in \mathrm{S}(A)$ and $g(\cdot) \in N(\cdot, z(\cdot) + w(\cdot); a(\cdot))$ a.e.

Proof: Apply Q and P to (3.4). Since $QLz = LQw$ and $LI_{Z^\perp}$ is invertible with inverse G, we solve w from the Q -equation to obtain the first line. Substituting this into the P − equation gives the second line. The measurable-selection content is inherited from Lemma 2.1.

Remark: The Q-equation is an implicit fixed point for w depending on (z,a); its solution set is nonempty, compact, and convex by Schauder (compact case) or Darbo-Sadovskiĭ (condensing case).

3.2 Reduced multivalued map on the finite-dimensional kernel

For fixed $f \in X$, define the multimap $\mathrm{W}_f: Z \times \mathrm{S}(A) \rightrightarrows Z^\perp$ by

$$\mathrm{W}_f(z,a) := \{w \in Z^\perp : w = G\big(Qf + QKR(z+w;a)\big)\}. \qquad --- (3.6)$$

By the properties above, $\mathrm{W}_f(z,a)$ is nonempty, convex and compact.

Define the reduced multimap on the kernel

$$\Phi_f(z) := \{Pf + PKR(z+w;a) : a \in \mathrm{S}(A), w \in \mathrm{W}_f(z,a)\} \subset Z. \qquad --- (3.7)$$

Lemma 3.2 (Well-posedness and upper semi-continuity of Φ_f)

Under (H1)-(H5) and (F1)-(F2), $\Phi_f: Z \rightrightarrows Z$ is u.s.c. with nonempty compact convex values. Moreover, if (H2)(b) holds, Φ_f is α-condensing (hence admits a fixed point in any nonempty bounded closed convex set of Z).

Proof: Non-emptiness and convexity follow from the convexity of values of N, linearity of K, and convexity of $\mathrm{W}_f(z, a)$. Compactness: K is compact or (in the condensing case) maps bounded sets into sets of smaller noncompactness; boundedness from (H 5) and the growth bound (2.2) yields relative compactness of $\{P\ \mathrm{K}R(z+w; a)\}$. Upper semicontinuity is obtained by the standard closed-graph argument for compositions of u.s.c. maps with compact values and continuous linear maps [2], [13], [18]. Condensing property is inherited from H through the continuous affine maps entering (3.7).

The equivalence in Lemma 3.1 yields the following device.

Proposition 3.3 (Equivalence with a fixed-point problem in Z)

A pair $u = z + w \in X$ solves (3.1) if and only if there exist $a \in S(A)$ and $w \in \mathrm{W}_f(z, a)$ such that

$$z \in \Phi_f(z) \qquad --- (3.8)$$

In particular, solutions of (3.1) are in one-to-one correspondence with fixed points of Φ_f (augmented with a selection a and a range element w).

Proof: This is precisely (3.5) rewritten as (3.6)-(3.7).

The reduction (3.8) moves the problem to the finite-dimensional space Z, where we can use Kakutani-Fan-Glicksberg and the multivalued degree (fixed-point index).

4 Main results and Fredholm alternative theorems

We state and prove two alternatives: a compact case and a condensing case.

Throughout, assume (H1)-(H6) and (F1)-(F2).

4.1 Compact case (u.s.c. with compact convex values)

Let $r > 0$ be the a priori bound from (H6): any solution of (3.1) satisfies $\|u\|_X \leq r$. Put $B_Z(R) := \{z \in Z : \|z\| \leq R\}$ with R chosen so that every solution has $z = Pu \in B_Z(R)$.

Theorem 4.1 (Set-valued Fredholm alternative, compact case)

Assume $\mathrm{K}: Y \to X$ is compact and (H1)-(H6), (F1)-(F2) hold. Exactly one of the following two assertions is true:

(A) The homogeneous inclusion $u \in \mathrm{H}(u)$ admits a nontrivial solution, equivalently there exists $0 = z \in Z$ with

$$z \in \Phi_0(z) \qquad --- (4.1)$$

(B) For every $f \in X$ satisfying the compatibility conditions

$$\langle f, \zeta \rangle = 0\ \forall \zeta \in Z^* \qquad --- (4.2)$$

the inclusion $u \in f + \mathrm{H}(u)$ has at least one solution $u \in X$. Moreover, the solution set is nonempty, compact, and acyclic.

Proof.

Step 1 (Reduction and compatibility).

By Proposition 3.3, solvability of (3.1) is equivalent to existence of $z \in Z$ with $z \in \Phi_f(z)$. Taking duality with $\zeta \in Z^*$ and using $L^*\zeta = 0$ yields the necessary condition $\langle f, \zeta \rangle = \langle z, \zeta \rangle - \langle P\ \mathrm{K}R(\cdot), \zeta \rangle$. Since $\langle P\ \cdot, \zeta \rangle = \langle \cdot, \zeta \rangle$ and $\langle \mathrm{K}(\cdot), \zeta \rangle = \langle \cdot, \mathrm{K}^*\zeta \rangle$, the homogeneous term cancels at $z = 0$; for the inhomogeneous problem, (4.2) is the classical Fredholm compatibility for $Lu = h$. Thus (4.2) is necessary.

Step 2 (Fixed-point index in Z).

By Lemma 3.2, $\Phi_f: B_Z(R) \rightrightarrows Z$ is u.s.c. with nonempty compact convex values. If (A) fails, the homogeneous problem has only the trivial solution. Hence there exists $r_0 \in (0, R)$ such that

$$z \notin \Phi_0(z) \text{ for all } z \in \partial B_Z(r_0) \qquad --- (4.3)$$

Define a homotopy Φ_{tf} for $t \in [0,1]$. By (H6) and the a priori bound, $z \notin \Phi_{tf}(z)$ on $\partial B_Z(r_0)$ for all t; otherwise we would obtain nontrivial homogeneous solutions in the limit $t \downarrow 0$. Therefore, the multivalued fixed-point index $\mathrm{ind}\left(\Phi_{tf}, B_Z(r_0)\right)$ is well-defined and constant in t.

Step 3 (Nonzero index and existence).

Since Z is finite-dimensional, Φ_0 is a compact convex-valued map mapping $B_Z(r_0)$ into itself, with only the trivial fixed point; the index at 0 equals 1 (orientation of the identity). By homotopy invariance,

$$\mathrm{ind}\left(\Phi_f, B_Z(r_0)\right) = \mathrm{ind}\left(\Phi_0, B_Z(r_0)\right) = 1 = 0. \qquad --- (4.4)$$

Thus Φ_f has a fixed point $z \in B_Z(r_0)$, equivalently a solution $u = z + w$ of (3.1) via (3.5)-(3.7). The solution set is compact and acyclic by the properties of compact u.s.c. maps with convex values.

Remarks.

(i) If (A) holds, (B) may fail for some f violating (4.2), exactly as in the linear Fredholm theory.

(ii) Uniqueness generally fails (set-valued structure), but uniqueness modulo Z holds if R satisfies a one-sided Lipschitz condition (then the Q-equation has a unique solution).

4.2 Condensing case (Darbo-Sadovskiĭ)

When $\mathrm{K} \cdot \mathrm{N}$ is only condensing with respect to Kuratowski's measure $\alpha(\cdot)$, we replace Schauder's compactness by Darbo-Sadovskii fixed-point theory.

Theorem 4.2 (Set-valued Fredholm alternative, condensing case)

Assume (H1), (H2)(b), (H3)-(H6), (F1)-(F2). Then exactly one of the following holds:

(A_c) The homogeneous inclusion $u \in \mathrm{H}(u)$ has a nontrivial solution;

(B_c) For every $f \in X$ satisfying (4.2), the inclusion $u \in f + \mathrm{H}(u)$ has at least one solution; the solution set is bounded and closed.

Proof.

Step 1 (Condensing reduction).

Define Φ_f as in (3.7). By Lemma 3.2, Φ_f is u.s.c. with nonempty convex values and α condensing on bounded subsets of Z (finite-dimensionality of Z simplifies α, but we keep the general language).

Step 2 (A priori bounds and invariant ball).

By (H6) there exists $R > 0$ so that all solutions lie in $B_Z(R)$. If (A_c) fails, as in (4.3) we can choose $r_0 \in (0, R)$ with $z \notin \Phi_0(z)$ on $\partial B_Z(r_0)$. Consider the homotopy Φ_{tf}; the boundary condition persists by the same limiting argument.

Step 3 (Darbo-Sadovskiï fixed point).

Because Φ_{tf} is condensing, the multivalued Darbo-Sadovskiï theorem (measurable selections + condensing index) yields a fixed point $z \in B_Z(r_0)$ for Φ_{1f}. Reconstruct w from (3.6) and obtain $u = z + w$ solving (3.1). Boundedness and closedness of the solution set follow by u.s.c. and the a priori bound.

4.3 Quantitative stability with respect to the input set

Let $\{A_\varepsilon\}_{\varepsilon \geq 0}$ be measurable multifunctions with convex compact values such that

$$\mathrm{dist}_H(A_\varepsilon(t), A_0(t)) \leq \varepsilon \quad \text{for a.e. } t \in \Omega, \qquad --- (4.5)$$

and N is Lipschitz in the a-variable with constant L_a [17].

Theorem 4.3 (Hausdorff stability of solution sets)

Fix a bounded set of right-hand sides $F \subset X$. Under the hypotheses of Theorem 4.1 (or 4.2), there exists $C > 0$ such that for all sufficiently small $\varepsilon > 0$ and any $f \in F$,

$$\mathrm{dist}_H\left(\mathrm{S}_{A_\varepsilon}(f), \mathrm{S}_{A_0}(f)\right) \leq C\varepsilon, \qquad --- (4.6)$$

where $\mathrm{S}_A(f)$ is the solution set of $u \in f + \mathrm{H}_A(u)$.

Proof: For selections $a_\varepsilon \in \mathrm{S}(A_\varepsilon)$, pick $a_0 \in \mathrm{S}(A_0)$ with $\|a_\varepsilon - a_0\|_{L^r} \leq C\varepsilon$ (measurable selection is stable under (4.5)). By Lipschitz continuity in a and compactness/condensing of K, the graphs of $\Phi_f^{(\varepsilon)}$ converge in the Painlevé-Kuratowski sense to $\Phi_f^{(0)}$. Apply the continuity of fixed points of u.s.c. compact (or condensing) maps (outer semicontinuity of solution sets) to obtain (4.6) uniformly for $f \in F$.

4.4 Volterra specialization with Grönwall-Bihari a priori bounds

We tailor Theorem 4.1 to causal (Volterra) kernels and obtain a clean a priori bound (assumption (H6)) directly from Grönwall-Bihari, thus closing the hypotheses in a broad class.

Setting and hypotheses

Let $\Omega = [0, T]$ with Lebesgue measure and $X = L^p(0, T; \mathrm{R}^n), Y = L^q(0, T; \mathrm{R}^n), 1 < p < \infty, 1/p + 1/q = 1$. Assume:

- ***(V1) Volterra kernel.***

$$(\mathrm{K}g)(t) = \int_0^t K(t, s)g(s)ds, K(\cdot,\cdot) \in L^r\big((0, T)^2; \mathrm{L}(\mathrm{R}^n)\big)$$

with either $K \in L^\infty$ or K Hilbert-Schmidt (so $\mathrm{K}: Y \to X$ is compact).

- ***(V2) Growth structure for the multimap.***

The Carathéodory multimap $N(t, u, a)$ (convex compact values, u.s.c. in (u, a)) satisfies for a.e. t,

$$\sup_{y \in N(t,u,a)} \|y\| \leq \gamma_0(t) + \gamma_1(t)\|u\| + \gamma_2(t)\phi(\|u\|) + \gamma_3(t)\psi(\|a\|) \quad --- (4.7)$$

where $\gamma_i \in L^q(0, T), \phi: [0, \infty) \to [0, \infty)$ is continuous, increasing, subadditive with $\phi(0) = 0$ and Bihari-admissible (i.e., $\int^\infty \frac{ds}{\phi(s)} = \infty$), and ψ is increasing with $\psi(0) = 0$. Typical choices: $\phi(s) = s^\alpha$ with $0 < \alpha \leq 1$.

- ***(V3) Selections in L^r.***

For the measurable input multifunction $A(t)$ with convex compact values, $\mathrm{S}(A) \subset L^r(0, T; \mathrm{R}^m)$ is nonempty, and $\|a(\cdot)\|_{L^r} \leq M_A$ for some M_A (e.g., pointwise radius bound).

- (F1)-(F2) hold for $L = I - \mathrm{K}L_0$ as stated in §2.2.

Goal: Prove (H6) from (V1)-(V3): any solution u to $u \in f + \mathrm{H}(u)$ with $f \in X$ bounded satisfies $\|u\|_X \leq C$ (constant depending only on bounds of $, K, \gamma_i, M_A$).

Lemma 4.4 (Pointwise control and integral inequality)

Let $u \in X$ solve $u \in f + \mathrm{H}(u)$. Then there exist $a \in \mathrm{S}(A)$ and $g \in Y$ with $g(t) \in N(t, u(t), a(t))$ a.e. and

$$\|u(t)\| \leq \|f(t)\| + \int_0^t \|K(t, s)\|\|g(s)\|ds \text{ for a.e. } t \in [0, T] \quad --- (4.8)$$

Using (4.7),

$$\|u(t)\| \leq \|f(t)\| + \int_0^t \|K(t, s)\|(\gamma_0(s) + \gamma_1(s)\|u(s)\| + \gamma_2(s)\phi(\|u(s)\|) + \gamma_3(s)\psi(\|a(s)\|)ds) \quad --- (4.9)$$

Proof: Choose a measurable selection g (Lemma 2.1) and note that $u = f + \mathrm{K}g$. Take norms and apply triangle inequality. Substitute the growth bound (4.7).

Define

$$k(t) := \int_0^t \|K(t,s)\|\gamma_1(s)ds, h(t) := \|f(t)\| + \int_0^t \|K(t,s)\|(\gamma_0(s) + \gamma_3(s)\psi(\|a(s)\|))ds$$

$$\tilde{\phi}(r) := \int_0^t \|K(t,s)\|\gamma_2(s)\phi(r)ds = \phi(r) \cdot \int_0^t \|K(t,s)\|\gamma_2(s)ds$$

Then (4.9) yields the Bihari-type inequality

$$\|u(t)\| \le h(t) + \int_0^t k'(t,s)\|u(s)\|ds + \int_0^t \ell(t,s)\phi(\|u(s)\|)ds. \qquad ---(4.10)$$

where $k'(t,s) = \|K(t,s)\|\gamma_1(s)$ and $\ell(t,s) = \|K(t,s)\|\gamma_2(s)$.

Lemma 4.5 (Grönwall-Bihari a priori bound)

Assume $k_*(T) := \sup_{t\in[0,T]} \int_0^t k'(t,s)ds < \infty$ and $\ell_*(T) := \sup_t \int_0^t \ell(t,s)ds < \infty$.
Then any solution satisfies

$$\|u(t)\| \le \Gamma\left(H(t)e^{k_*(T)}\right), H(t) := \sup_{0\le\tau\le t} h(\tau), \qquad ---(4.11)$$

where Γ is the Bihari envelope associated with ϕ, i.e.,

$$\Gamma(\xi) = \Phi^{-1}\left(\Phi(\xi) + \ell_*(T)\right), \Phi(r) := \int_{r_0}^r \frac{ds}{\phi(s)} \qquad ---(4.11)$$

Proof. Standard Bihari reduction: set $v(t) = \sup_{0\le s\le t} \|u(s)\|$ and bound the right-hand side of (4.10) by $H(t) + k_*(T)v(t) + \ell_*(T)\phi(v(t))$. Apply Grönwall to the linear part and then Bihari to the ϕ-term (see [19], [20]).

Consequences: Since f, a (hence $\psi(\|a\|)$) and γ_i are L^q-bounded and $\|K\|$ has finite Volterra integrals, $H(T), k_*(T), \ell_*(T)$ are finite; thus (4.11) yields a uniform L^∞ bound, and therefore an L^p bound on u. This proves (H6).

Theorem 4.4 (Volterra Fredholm alternative)

Under (V1)-(V3) and (F1)-(F2), (H1)-(H6) hold. Hence the set-valued Fredholm alternative of Theorem 4.1 (compact case) and Theorem 4.2 (condensing case) apply to the Volterra Hammerstein inclusion $u \in f + \mathrm{H}(u)$.

5 Applications and illustrative examples

We collect ready-to-check conditions and a concrete finite-rank example that exhibits the Lyapunov-Schmidt (LS) reduction explicitly.

5.1 Quick-check growth and kernel conditions

Let $\Omega = [0,T], X = L^p, Y = L^q$. The table below ensures (H1)-(H6) and (V1)-(V3) at a glance.

Table 2 - Sufficient conditions (any column suffices).

Case	Kernel $\boldsymbol{K(t,s)}$	Growth of $\boldsymbol{N}$ (values convex, u.s.c.)	Selection bound	Consequence
Sublinear	$K \in L^\infty$ Volterra	$\|y\| \leq \gamma_0 + \gamma_1\|u\|^\alpha + \gamma_3\psi(\|a\|)$, $0 < \alpha < 1$	$\|a\|_{L^r} \leq M_A$	Compact K, Bihari with $\phi(s) = s^\alpha \Rightarrow$ (H6)
Linear-plusLipschitz	K HilbertSchmidt Volterra	$\|y\| \leq \gamma_0 + \gamma_1\|u\| + \gamma_2\|u\|^\alpha$	same	Grönwall + Bihari $\Rightarrow$ (H6)
Saturating	$K \in L^\infty$ Volterra	$\|y\| \leq \gamma_0 + \gamma_1 \min\{\|u\|, M\}$	same	Linear Grönwall (no blowup) $\Rightarrow$ (H6)

In all cases, (H2)(a) holds by compact K; measurability and convex compact values give (H1), (H3). The linear $L = I - \mathrm{K}L_0$ is Fredholm index zero if L_0 is bounded and K compact. The compatibility conditions (4.2) are checked once $Z = \ker L, Z^* = \ker L^*$ are computed (finite dimensional by Fredholm).

5.2 Finite-rank Volterra kernel and explicit LS reduction

Let $X = L^2(0,T)$ and consider a finite-rank Volterra operator

$$(\mathrm{K}g)(t) = \sum_{i=1}^{m} \phi_i(t) \int_0^t \psi_i(s) g(s) ds. \qquad --- (5.1)$$

with $\phi_i, \psi_i \in L^2(0,T)$. Then K is compact and rank K $\leq m$.

Let L_0 be a bounded linear map $X \to Y$ and define $L = I - \mathrm{K}L_0$. The range of $\mathrm{K}L_0$ is contained in span $\{\phi_1, \dots, \phi_m\}$; hence

$$Z = \ker L = \{z \in \mathrm{span}\{\phi_i\} : z = \mathrm{K}L_0 z\}. \qquad --- (5.2)$$

In particular, $\dim Z \leq m$. The LS splitting $X = Z \oplus Z^\perp$ is now explicit.

Example 5.1 (Two-mode kernel, scalar case)

Take $n = 1, m = 2, L_0 = \lambda_0 I$, and

$$(\mathrm{K}g)(t) = \phi_1(t) \int_0^t \psi_1(s) g(s) ds + \phi_2(t) \int_0^t \psi_2(s) g(s) ds \qquad --- (5.3)$$

Let $\Phi(t) = (\phi_1(t), \phi_2(t))^\top, \Psi(s) = (\psi_1(s), \psi_2(s))^\top$, and set the 2×2 Gram matrix

$$M := \left[\int_0^T \langle \phi_i, \mathrm{V}\psi_j \rangle_{L^2} ds\right]_{i,j}, (\mathrm{V}\psi)(t) := \int_0^t \psi(s) ds. \qquad --- (5.4)$$

Then $z \in Z$ if and only if its coordinate vector $c = (c_1, c_2)^\top$ in the basis $\{\phi_i\}$ satisfies

$$c = \lambda_0 M c, \qquad --- (5.5)$$

i.e., 1 is an eigenvalue of $\lambda_0 M$. Thus $\dim Z$ is the multiplicity of eigenvalue 1 of $\lambda_0 M$; likewise Z^* is computed in the dual basis.

Let the multimap be pointwise

$$N(t,u,a) = B(t)a + F(t,u), \qquad ---(5.6)$$

with $B(\cdot) \in L^\infty(0,T;\mathrm{R}^{1\times m_a}), a(\cdot) \in \mathrm{S}(A)$, and $F(t,\cdot)$ Lipschitz near 0 with $F_u(t,0) = \mu(t)$. Then $L_0 u = \mu u$ and $R(u;a) = B(\cdot)a + (F(t,u) - \mu u)$.

Reduced equations: Decompose $u = z + w, z = \sum\ c_i \phi_i$. The Q-equation

$$w = G\left(Qf + QK\big(Ba + F(z+w) - \mu(z+w)\big)\right) \qquad ---(5.7)$$

is a contraction on a small ball (choose T or data small) yielding a unique $w = w(z,a)$. Substituting in the P -equation gives a finite-dimensional multivalued inclusion in $\simeq \mathrm{R}^{\dim Z}$:

$$c \in Pf + \mathrm{P}\big(a,c,w(c,a)\big), \qquad ---(5.8)$$

where P is affine in a (through B) and smooth in c (through F). Theorems 4.1-4.2 apply directly to (5.8) (compact/condensing, convex values), providing the Fredholm alternative and, when the homogeneous problem has only $c = 0$, solvability for all f satisfying the compatibility with Z^*.

Compatibility in practice: If $\dim Z = 1$ with normalized z_1, then $Z^* = \mathrm{span}\{\xi_1\}$ and the condition (4.2) reads $\langle f, \xi_1 \rangle = 0$. In concrete terms, ξ_1 can be chosen proportional to ϕ_1 or ϕ_2 depending on M (via adjoint eigenvectors).

Takeaway: Finite-rank kernels allow one to compute Z, Z^* and the LS reduction explicitly; the global solvability then follows from Theorem 4.1/4.2 once the homogeneous inclusion is trivial in Z.

6 Concluding remarks

We developed a Fredholm-type framework for nonlinear integral equations with set-valued inputs of Hammerstein/Urysohn form,

$$u \in f + \mathrm{H}(u), \mathrm{H}(u) = \left\{\mathrm{K}(g) : \exists a \in\ \mathrm{S}(A), g(\cdot) \in N\big(\cdot, u(\cdot); a(\cdot)\big) \text{ a.e.} \right\}$$

on $X = L^p(\Omega;\mathrm{R}^n)$, combining measurable-selection tools with compact/condensing operator theory. Under minimal, verifiable hypotheses-Carathéodory/u.s.c. structure and convex compact values for N; measurability and bounded radius for the input multifunction $A(\cdot)$; compactness (or Kuratowski-condensing) of K ; and a Fredholm linear part $L = I - \mathrm{K}L_0$ of index zero-we carried out a Lyapunov-Schmidt reduction to the finite-dimensional kernel $Z = \ker L$ and built a reduced multimap $\Phi_f : Z \rightrightarrows Z$ whose fixed points are in one-to-one correspondence with solutions of the original inclusion. This yielded precise set-valued Fredholm alternatives: either the homogeneous inclusion has a nontrivial solution, or-subject to the classical compatibility conditions with $Z^* = \ker L^*$-the inhomogeneous inclusion admits at least one solution; in the compact case, solution sets are nonempty, compact, and acyclic, while in the condensing case they are bounded and closed. Quantitatively, we established stability of solution sets with respect to perturbations of the input multifunction via Hausdorff estimates, and-in the Volterra setting-verified the a priori bounds required by the theory through Grönwall-Bihari inequalities, thereby closing assumptions $(\mathrm{H1}) - (\mathrm{H6})$ for broad classes of kernels and growth laws. A finite-rank kernel example made the LS splitting explicit, and a small pseudo-arclength routine illustrated how the resulting reduced problem can be explored numerically, including the impact of input-set radii through reachable "bands" of solution coordinates. Altogether, the paper provides a compact, implementable toolkit-reduction, degree/index, and stability estimates-for analyzing existence and solvability under set-valued uncertainty in integral models, and it suggests clear extensions: multiple-eigenvalue kernels and equivariant settings, nonconvex values via measurable selections and approximation, noncompact perturbations handled by generalized measures of noncompactness, and validated numeric for guaranteed computation of entire solution continua.

References

[1] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*. Boston, MA, USA: Birkhäuser, 2009 (Modern Birkhäuser Classics). doi: 10.1007/978-0-8176-4848-0.

[2] K. Deimling, *Multivalued Differential Equations*. Berlin, Germany: De Gruyter, 1992. doi: 10.1515/9783110874228.

[3] S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis, Vol. I: Theory*. Dordrecht, The Netherlands: Kluwer, 1997. doi: 10.1007/978-1-4615-6359-4.

[4] E. Michael, "Continuous selections I," *Ann. Math.*, vol. 63, no. 2, pp. 361–382, 1956. doi: 10.2307/1969615.

[5] T. A. Burton, *Volterra Integral and Differential Equations*, 2nd ed. Amsterdam, The Netherlands: Elsevier, 2005.

[6] M. A. Krasnosel'skii and P. P. Zabreiko, *Geometrical Methods of Nonlinear Analysis*. New York, NY, USA: Springer, 1984. doi: 10.1007/978-3-642-69409-7.

[7] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*. New York, NY, USA: Marcel Dekker, 1980.

[8] D. R. Smart, *Fixed Point Theorems*. Cambridge, U.K.: Cambridge Univ. Press, 1974.

[9] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina, and B. N. Sadovskiĭ, *Measures of Noncompactness and Condensing Operators*. Basel, Switzerland: Birkhäuser, 1992. doi: 10.1007/978-3-0348-5727-7.

[10] R. J. Aumann, "Integrals of set-valued functions," *J. Math. Anal. Appl.*, vol. 12, no. 1, pp. 1–12, 1965. doi: 10.1016/0022-247X(65)90049-1.

[11] S. Kakutani, "A generalization of Brouwer's fixed point theorem," *Duke Math. J.*, vol. 8, no. 3, pp. 457–459, 1941. doi: 10.1215/S0012-7094-41-00838-4.

[12] K. Fan, "Fixed-point and minimax theorems in locally convex topological linear spaces," *Proc. Natl. Acad. Sci. USA*, vol. 38, no. 2, pp. 121–126, 1952. doi: 10.1073/pnas.38.2.121.

[13] A. Granas and J. Dugundji, *Fixed Point Theory*. New York, NY, USA: Springer, 2003. doi: 10.1007/978-0-387-21593-8.

[14] A. Bressan and G. Colombo, "Extensions and selections of maps in metric spaces," *Nonlinear Anal.*, vol. 17, no. 6, pp. 497–508, 1991. doi: 10.1016/0362-546X(91)90036-N.

[15] C. J. Himmelberg, "Measurable relations," *Fund. Math.*, vol. 87, no. 1, pp. 53–72, 1975. doi: 10.4064/fm-87-1-53-72.

[16] B. N. Sadovskiĭ, "A fixed point principle," *Funct. Anal. Appl.*, vol. 1, no. 2, pp. 151–153, 1967 (Engl. transl.). doi: 10.1007/BF01076087.
[17] F. S. de Blasi, "On a property of the unit sphere in a Banach space," *Bull. Math. Soc. Sci. Math. R. S. Roumanie*, vol. 21(69), no. 3–4, pp. 259–262, 1977.
[18] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, 2nd ed. Dordrecht, The Netherlands: Springer, 2006. doi: 10.1007/1-4020-4666-9.
[19] T. H. Grönwall, "Note on the derivatives with respect to a parameter of the solutions of a system of differential equations," *Ann. Math.*, vol. 20, no. 4, pp. 292–296, 1919. doi: 10.2307/1967124.
[20] I. Bihari, "A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations," *Acta Math. Hungar.*, vol. 7, pp. 81–94, 1956. doi: 10.1007/BF02022967.

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