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GLOBAL BIFURCATION FOR NONLINEAR OPERATORS WITH UNCERTAINTY BANDS

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Abstract. We study solution sets of nonlinear operator equations $F(\lambda, u, a) = 0$ on Banach spaces, where $\lambda \in \mathrm{R}$ is the primary parameter and a ranges over a compact ancillary set modelling uncertainty bands. Assuming $F \in C^1$, Fredholm index zero on the trivial branch, a simple spectral crossing and a band-uniform transversality, we develop a uniform Lyapunov-Schmidt reduction yielding a C^1 local curve with constants independent of a. Using Leray-Schauder degree, we prove a band-robust Rabinowitz alternative: for each a and for the band-union continuum $\mathrm{C} = \overline{\cup_a \mathrm{C}(a)}$, either the component is unbounded in $\mathrm{R} \times X$ or it impinges on another trivial point; if the simple crossing is isolated for all a, only unboundedness can occur. Quantitatively, we establish Hausdorff-Lipschitz stability of solution sets with respect to a on bounded windows and derive radius-to-extent inequalities that bound the growth of the λ projection of C by $O(\mathrm{rad}(A))$. The results furnish a compact toolkit for global bifurcation under parametric uncertainty with explicit stability constants and verifiable hypotheses.

Keywords: global bifurcation; uncertainty bands; Fredholm index zero; Lyapunov Schmidt reduction; topological degree; Rabinowitz alternative; Hausdorff stability.

1 Introduction

Let X, Y be real Banach spaces and let $\lambda \in \mathrm{R}$ denote a primary bifurcation parameter. We study nonlinear operator equations

$$F(\lambda, u, a) = 0, (\lambda, u, a) \in \mathrm{R} \times X \times A, \qquad --- (1.1)$$

where $A \subset \mathrm{R}^m$ is a compact set of ancillary parameters modelling an uncertainty band. For each $a \in A$, the map $F(\cdot, \cdot, a): \mathrm{R} \times X \to Y$ is assumed to be C^1 and Fredholm of index zero at the linearized level along the trivial branch $u \equiv 0$. Our goal is to establish a band-robust global bifurcation alternative for the union of continua of nontrivial solutions $\mathrm{C} = \overline{\cup_{a \in A} \mathrm{C}(a)}$ that emanate from a simple spectral crossing $(\lambda_*, 0, a)$ and to quantify how C deforms as the band radius $\mathrm{rad}(A)$ varies.

Classically, for a single operator $F(\lambda, u)$, existence of an unbounded continuum or a continuum reaching another trivial point is guaranteed by the global bifurcation theorem of Rabinowitz based on topological degree and index jumps [1]. Local branches near simple eigenvalue crossings are provided by the Crandall-Rabinowitz theorem using Lyapunov-Schmidt reduction [2]. Comprehensive treatments appear in Kielhöfer's monograph [3] and in Zeidler's nonlinear functional analysis [4]. The present work extends these conclusions uniformly across a set of ancillary parameters: we show that the degree jump persists on a neighborhood $A_* \subset A$ of the crossing point, that the local curve exists with constants independent of $a \in A_*$, and that the resulting global alternative holds for the band-union continuum C. In addition, we provide quantitative Hausdorff-Lipschitz stability of solution sets with respect to a under mild continuity of $D_u F$.

Guiding example: Consider a semilinear elliptic model on a bounded Lipschitz domain $\Omega \subset \mathrm{R}^N$,

$$F(\lambda, u, a) = -\Delta u - \lambda u - f(u; a) \in H^{-1}(\Omega), u \in H_0^1(\Omega), \qquad --- (1.2)$$

with Carathéodory nonlinearity f subcritical and $f_u(0; a) = 0$ uniformly in a. The linearization $L(\lambda, a) = -\Delta - \lambda I$ has a simple spectral crossing at $\lambda_* = \lambda_1(\Omega)$, independent of a. For each a, a local branch $\mathsf{C}_{\text{loc}}(a)$ bifurcates from $(\lambda_*, 0)$; our results show that the family $\{\mathsf{C}(a)\}_{a \in A}$ forms an uncertainty band of continua whose union obeys the classical global alternative and varies Lipschitz-continuously (in a bounded window) with respect to a.

Geometry at a glance

See the visual schematic figure 1 below; it depicts $\|u\|_X$ versus λ for a few's in A. A simple eigenvalue crossing at λ_* generates local branches that collectively form a "tube" across a and the global alternative applies to the outer envelope of this tube.

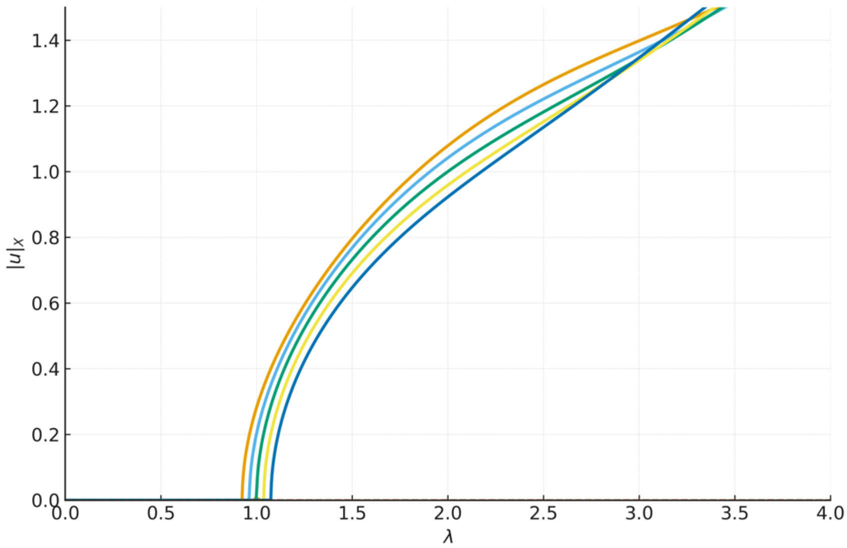

Figure 1 - Band-robust global bifurcation geometry.

Curves above in figure 1 show representative continua $\mathsf{C}(a)$ in the $(\lambda, \|u\|_X)$-plane for several $a \in A$. The dashed axis is the trivial branch $\|u\|_X = 0$; the marked point is the crossing $(\lambda_*, 0)$. The union of curves forms an uncertainty band; our theorems assert a Rabinowitz-type global alternative for the union and give stability bounds with respect to $\mathrm{rad}(A)$.

Contributions

- A uniform local theorem ensuring existence and uniqueness of a C^1 local branch with constants independent of a in a neighborhood $A_* \subset A$ of the crossing.
- A band-robust global alternative: the union continuum C is unbounded in $\mathrm{R} \times X$ or meets another trivial point for every $a \in A$; if the crossing is isolated for all a, only the unbounded alternative can occur, adapting Rabinowitz's degree jump.
- Quantitative stability: on any bounded window W, the Hausdorff distance between $\mathsf{C}(a_1) \cap W$ and $\mathsf{C}(a_2) \cap W$ is $O(\|a_1 - a_2\|)$ under Lipschitz continuity of $D_u F$ in a (proof strategy follows graph convergence and degree continuity).
- Model classes including semilinear elliptic, p-Laplacian, and discrete graph operators illustrate the assumptions and deliver explicit constants.

2 Notation and standing assumptions

We collect symbols and hypotheses used throughout. Let $\langle\cdot,\cdot\rangle$ denote the duality pairing between Y and Y' when needed.

2.1 Spaces, parameters, and basic objects

- X, Y : real Banach spaces; X reflexive when compactness is needed.
- $\lambda \in \mathrm{R}$: primary parameter; $u \in X$: state.
- $A \subset \mathrm{R}^m$: compact uncertainty band with center $\bar{a} \in A$ and radius

$$\mathrm{rad}(A) := \sup_{a \in A} \|a - \bar{a}\|_{\mathrm{R}^m} \qquad --- (2.1)$$

- Operator $F: \mathrm{R} \times X \times A \to Y$, with trivial branch $F(\lambda, 0, a) = 0$ for all (λ, a).

Linearization: $L(\lambda, a) := D_u F(\lambda, 0, a) \in \mathrm{L}(X, Y)$.

Fredholm index: $nd\ L = \mathrm{dimker} L - \mathrm{codimRange} L$.

Solution sets. For each $a \in A$,

$$\mathrm{S}(a) = \{(\lambda, u) \in \mathrm{R} \times X : F(\lambda, u, a) = 0\} \qquad --- (2.2)$$

Let $\mathrm{C}(a) \subset \mathrm{S}(a)$ be the connected component containing $(\lambda_*, 0)$.

Hausdorff distance. For sets $E, F \subset \mathrm{R} \times X$ and a bounded window $W \subset \mathrm{R} \times X$,

$$\mathrm{dist}_H(E \cap W, F \cap W) = \max\left\{ \sup_{x \in E \cap W} \inf_{y \in F \cap W} \|x - y\|, \sup_{y \in F \cap W} \inf_{x \in E \cap W} \|x - y\| \right\} \qquad --- (2.3)$$

Table 1 - Core symbols.

Symbol	Meaning
$F(\lambda, u, a)$	Nonlinear operator $\mathrm{R} \times X \times A \to Y$
$L(\lambda, a)$	Linearization $D_u F(\lambda, 0, a)$
λ_*	Spectral crossing parameter
C (a)	Continuum of nontrivial solutions for fixed a
C	Band-union continuum $\overline{\bigcup_{a \in A} \mathrm{C}(a)}$
$\mathrm{rad}(A)$	Band radius defined in (2.1)

2.2 Hypotheses

We impose assumptions H1-H6; all constants are uniform on the relevant neighborhoods.

H1 (Regularity): For each $a \in A$, the map $F(\cdot,\cdot, a) \in C^1(\mathrm{R} \times X, Y)$, and $(\lambda, u, a) \mapsto D_u F(\lambda, u, a)$ is continuous.

H2 (Fredholm index zero): The linearization

$$L(\lambda, a) = D_u F(\lambda, 0, a) \text{ is Fredholm of index } 0 \text{ for all } (\lambda, a) \in \mathrm{R} \times A. \qquad --- (2.4)$$

H3 (Simple crossing): There exists $(\lambda_*, a_*) \in \mathrm{R} \times A$ such that

$$\dim \ker L(\lambda_*, a_*) = \operatorname{codimRange} L(\lambda_*, a_*) = 1. \qquad --- (2.5)$$

Let $v_* \in X \setminus \{0\}$ span $\ker L(\lambda_*, a_*)$ and $\psi_* \in Y' \setminus \{0\}$ annihilate Range $L(\lambda_*, a_*)$.

H4 (Transversality in a band): There exists a neighborhood $A_* \subset A$ of a_* such that, for all $a \in A_*$,

$$D_\lambda L(\lambda_*, a) v_* \notin \text{ Range } L(\lambda_*, a). \qquad --- (2.6)$$

Equivalently, $\langle \psi_*, D_\lambda L(\lambda_*, a) v_* \rangle = 0$ with sign fixed on A_*.

H5 (Compactness): Writing $N(\lambda, u, a) := F(\lambda, u, a) - L(\lambda, a)u$, the map N is L compact on bounded sets uniformly in $a \in A$ (e.g., N compact or completely continuous on bounded sets).

H6 (Properness on bounded sets): For any bounded $\Omega \subset \mathrm{R} \times X$, the set

$$\{(\lambda, u, a) \in \Omega \times A : F(\lambda, u, a) = 0\} \qquad --- (2.7)$$

is compact.

Remark 2.1 (Why index zero): Index zero allows use of Leray-Schauder degree for $I - K$ with K compact after a suitable isomorphism and ensures homotopy invariance needed for the degree jump across λ_*.

2.3 Uniform Lyapunov-Schmidt reduction

By H2 − H4 there exist algebraic splittings $X = \operatorname{span}\{v_*\} \oplus Z, Y = \operatorname{span}\{\psi_*\} \oplus W$ and neighborhoods $I \ni \lambda_*, U \ni 0, A_* \ni a_*$ such that the equation $F(\lambda, sv + w, a) = 0$ with $v = v_*$ uniquely determines $w = w(\lambda, s, a) \in Z$ for $|s|$ small and $(\lambda, a) \in I \times A_*$. The remaining scalar reduced equation is

$$g(\lambda, s, a) := \langle \psi_*, F(\lambda, sv_* + w(\lambda, s, a), a) \rangle = 0. \qquad --- (2.8)$$

Differentiating gives

$$\partial_s g(\lambda_*, 0, a) = \langle \psi_*, D_u F(\lambda_*, 0, a) v_* \rangle = 0, \partial_\lambda g(\lambda_*, 0, a) = \langle \psi_*, D_\lambda L(\lambda_*, a) v_* \rangle \neq 0 \qquad --- (2.9)$$

and the nondegenerate slope condition $\partial_\lambda g(\lambda_*, 0, a) \partial_{ss} g(\lambda_*, 0, a) < 0$ (or an equivalent sign condition) holds uniformly in $a \in A_*$ by H 4. Therefore, by the implicit function theorem on the reduced problem, there exist $\delta > 0$ and a C^1 curve

$$\mathrm{C}_{\mathrm{loc}}(a) = \{(\lambda(s, a), u(s, a)) : |s| < \delta\}, u(s, a) = sv_* + w(\lambda(s, a), s, a), \qquad --- (2.10)$$

with bounds

$$|\lambda(s, a) - \lambda_*| \leq c|s|, \|u(s, a)\|_X \leq c|s|. \qquad --- (2.11)$$

where $c > 0$ is independent of $a \in A_*$. This uniform control is the key input for the band-robust global alternative developed later, paralleling the single-parameter theory in [1]-[3].

2.4 Degree jump across the band

Fix $\lambda^- < \lambda_* < \lambda^+$in I and a ball $B \subset X$ so that $F(\lambda,\cdot,a) = 0$ on ∂B for $\lambda \in \{\lambda^-, \lambda^+\}$ and all $a \in A_*$. Define Leray-Schauder degrees

$$d^{\pm}(a) := \deg(I - L(\lambda^{\pm}, a)^{-1} N(\lambda^{\pm},\cdot, a), B, 0). \qquad --- (2.12)$$

Under $\mathrm{H1} - \mathrm{H6}$ and the simple crossing (2.5) with transversality (2.6), one has

$$d^-(a) - d^+(a) = \pm 1 \text{ for all } a \in A_*, \qquad --- (2.13)$$

i.e., the degree jump is constant across the band. Consequently, each $\mathrm{C}(a)$ satisfies the Rabinowitz global alternative [1], and the band-union continuum $\mathrm{C} = \overline{\cup_{a \in A} \mathrm{C}(a)}$ inherits the same dichotomy; the quantitative stability estimates rely on continuity of solution sets and degree with respect to parameters [3], [4].

3 Problem formulation and main results

We consider the operator equation

$$F(\lambda, u, a) = 0, (\lambda, u, a) \in \mathrm{R} \times X \times A, \qquad --- (3.1)$$

under assumptions H1-H6 of Section 2. Recall $L(\lambda, a) = D_u F(\lambda, 0, a)$, the simple crossing (λ_*, a_*) with eigenvector v_* and cokernel vector ψ_*, and the splittings $X = \mathrm{span}\{v_*\} \oplus Z, Y = \mathrm{span}\{\psi_*\} \oplus W$.

3.1 Uniform Lyapunov-Schmidt reduction and local branch

We first produce a reduced scalar equation that is uniform for a in a neighborhood of a_*.

Lemma 3.1 (Uniform invertibility on the complement)

There exist neighborhoods $I \ni \lambda_*, A_* \subset A$ of a_*, and $U \subset X$ with $0 \in U$ such that for all $(\lambda, a) \in I \times A_*$ the restriction

$$L_{WZ}(\lambda, a) := P_W \circ L(\lambda, a)|_Z : Z \to W. \qquad --- (3.2)$$

is a bounded linear isomorphism with

$$\sup_{(\lambda,a) \in I \times A_*} \|L_{WZ}(\lambda, a)^{-1}\| \leq C_0 < \infty. \qquad --- (3.3)$$

Proof: By H2 and the simple crossing at (λ_*, a_*), $\ker L(\lambda_*, a_*) = \mathrm{span}\{v_*\}$ and Range $L(\lambda_*, a_*)$ is complemented by $W = \mathrm{span}\{\psi_*\}^{\perp}$. Continuity of $L(\lambda, a)$ in (λ, a) (H1) and stability of Fredholm index imply that for (λ, a) near (λ_*, a_*), $\ker L(\lambda, a)$ is one-dimensional and transversal to Z. Hence $L_{WZ}(\lambda, a)$ is bijective. A uniform bound (3.3) follows from continuity of the inverse map on a compact neighborhood by the bounded inverse theorem; see, e.g., [5, Thm. 9.3] or [6, Section 15].

Lemma 3.2 (Uniform LS reduction)

There exist neighborhoods $I \ni \lambda_*, A_* \subset A$, and $\mathrm{N} \subset \mathrm{R} \times \mathrm{R} \times A_*$ such that for each $(\lambda, s, a) \in \mathrm{N}$ there is a unique $w = w(\lambda, s, a) \in Z$ solving

$$P_W F(\lambda, s v_* + w, a) = 0. \qquad --- (3.4)$$

and w is C^1 with uniform bounds

$$\|w(\lambda,s,a)\| \le C_1|s|, \left\|D_{(\lambda,s)}w(\lambda,s,a)\right\| \le C_2 \qquad ---(3.5)$$

where C_1, C_2 depend only on I, A_*.

Proof: Write $G(\lambda,s,w,a) := P_W F(\lambda, sv_* + w, a)$. Then

$$D_w G(\lambda_*,0,0,a_*) = P_W L(\lambda_*,a_*)|_Z = L_{WZ}(\lambda_*,a_*)$$

invertible by Lemma 3.1. By the uniform inverse bound and C^1-regularity (H1), the implicit function theorem yields a C^1 map $w(\lambda,s,a)$ on a product neighborhood, with the stated estimates (e.g., by the standard fixed-point proof with a contraction whose Lipschitz constants are uniform on $I \times A_*$).

Define the reduced equation

$$g(\lambda,s,a) := \langle \psi_*, F(\lambda, sv_* + w(\lambda,s,a), a)\rangle = 0. \qquad ---(3.6)$$

Lemma 3.3 (Nondegenerate slope and sign persistence)

On $I \times \{0\} \times A_*$,

$$\partial_s g(\lambda_*,0,a) = 0, \partial_\lambda g(\lambda_*,0,a) = \langle \psi_*, D_\lambda L(\lambda_*,a)v_*\rangle \ne 0. \qquad ---(3.7)$$

and the sign of $\partial_\lambda g(\lambda_*,0,a)$ is constant for $a \in A_*$.

Proof. The first identity is $\langle \psi_*, L(\lambda_*,a)v_*\rangle = 0$. The second follows from H4. Continuity in a gives sign constancy on a small neighborhood.

Proposition 3.4 (Uniform local branch)

There exist $\delta > 0$ and $C > 0$ such that for each $a \in A_*$ there is a unique C^1 curve

$$\mathrm{C_{loc}}(a) = \{(\lambda(s,a), u(s,a)) : |s| < \delta\}, u(s,a) = sv_* + w(\lambda(s,a),s,a), \qquad ---(3.8)$$

satisfying $g(\lambda(s,a),s,a) = 0$ and the bounds

$$|\lambda(s,a) - \lambda_*| \le C|s|, \|u(s,a)\|_X \le C|s|. \qquad ---(3.9)$$

Proof: Apply the implicit function theorem to $g(\lambda,s,a) = 0$ with respect to λ at $(\lambda_*,0,a)$ using Lemma 3.3; bounds (3.9) follow from (3.5) and smooth dependence. Uniqueness stems from the reduction: any solution near $(\lambda_*,0)$ has w uniquely defined by (3.4), so two solution curves must coincide.

3.2 Degree jump and the band-robust global alternative

Fix $\lambda^- < \lambda_* < \lambda^+$ inside I and a ball $B \subset X$ with $0 \in B$ such that, for all $a \in A_*$,

$$F(\lambda^\pm,u,a) \ne 0 \text{ for every } u \in \partial B. \qquad ---(3.10)$$

Define compact perturbations of the identity

$$\Phi_a^\pm(u) := u - L(\lambda^\pm,a)^{-1}N(\lambda^\pm,u,a), N(\lambda,u,a) := F(\lambda,u,a) - L(\lambda,a)u. \qquad ---(3.11)$$

well-defined by **H2** and **H5**, and consider the Leray-Schauder degree $\deg(\Phi_a^\pm, B, 0)$.

Lemma 3.5 (Degree constancy in the band)

There exists $A^{\dagger} \subseteq A_{*}$ open such that $\deg(\Phi_{a}^{\pm}, B, 0)$ is constant for $a \in A^{\dagger}$.

Proof: By H1-H6, $\Phi_{a}^{\pm}$is a compact perturbation of the identity depending continuously on a in the operator norm, and $0 \notin \Phi_{a}^{\pm}(\partial B)$ uniformly by (3.10). Apply homotopy invariance of the Leray-Schauder degree with parameter a.

Lemma 3.6 (Parity of the jump)

For all $a \in A_{*}$,

$$\deg(\Phi_{a}^{-}, B, 0) - \deg(\Phi_{a}^{+}, B, 0) = \pm 1. \qquad --- (3.12)$$

Proof: In the reduced equation (3.6), the change of index across λ_{*} is governed by the sign of $\partial_{\lambda} g(\lambda_{*}, 0, a)$ and the multiplicity of the crossing. Since the crossing is simple (H3) and transversal with sign fixed on A_{*} (Lemma 3.3), the local additivity of degree shows a unit jump; see the classical argument of Rabinowitz adapted to the parameterized family.

Theorem 3.7 (Global alternative for each fixed a)

For every $a \in A$ sufficiently close to a_{*} (hence for all $\in A_{*}$), the connected component $\mathrm{C}(a) \subset \mathrm{S}(a)$ of solutions containing ($\lambda_{*}, 0$) satisfies the global alternative: either

(i) C (a) is unbounded in $\mathrm{R} \times X$, or
(ii) $\mathrm{C}(a)$ meets another trivial point $(\hat{\lambda}, 0)$ with $\ker L(\hat{\lambda}, a) = \{0\}$.

Proof: This is Rabinowitz's global theorem for compact perturbations of Fredholm index-zero maps applied at fixed a, with the degree jump (3.12) ensuring nontrivial continuation.

Theorem 3.8 (Band-robust global alternative)

Let $\mathrm{C} = \overline{\bigcup_{a \in A} \mathrm{C}(a)}$. Under H1-H6, either

(i) C is unbounded in $\mathrm{R} \times X$, or
(ii) for some $\hat{\lambda} \in \mathrm{R}$ and a sequence $a_n \in A, (\hat{\lambda}, 0) \in \overline{\bigcup_n \mathrm{C}(a_n)}$ with $\ker L(\hat{\lambda}, a_n) = \{0\}$ for infinitely many n.

If, in addition, λ_{*} is the only simple crossing for all $a \in A$, then only (i) can occur.

Proof: For each $a \in A$, Theorem 3.7 yields (i) or (ii). If (ii) were to happen uniformly with trivial points different from λ_{*}, we would obtain a second crossing for some a contradicting the isolation assumption. Taking closures and using compactness H6 yields the band-union statement.

4 Local bifurcation uniform in uncertainty bands

We sharpen Section 3 in two directions: (i) continuity and stability of solution sets with respect to the ancillary parameter a, and (ii) quantitative control of the "extent" of the band-union continuum inside bounded windows [7], [8].

Throughout, let $W \subset \mathrm{R} \times X$ be a fixed bounded window (e.g., $[\lambda_1, \lambda_2] \times B_X(R)$).

4.1 Continuity of solution sets and Hausdorff stability

Lemma 4.1 (Graph continuity of solution sets)

Suppose **H1** holds and F is jointly continuous in (λ, u, a). Then for any $(\lambda_n, u_n, a_n) \to (\lambda, u, a)$,

$$\|F(\lambda_n, u_n, a_n) - F(\lambda, u, a)\|_Y \to 0. \qquad --- (4.1)$$

Proof: Immediate from joint continuity; used to transfer convergence along zero-sets.
We use Painlevé-Kuratowski convergence of sets and Hausdorff distance on W (Section 2).

Lemma 4.2 (Upper semi continuity of solution sets)

Assume **H1, H5, H6**. If $a_n \to a$ in A, then every limit point of $S(a_n) \cap W$ lies in $S(a) \cap W$; i.e.,

$$\limsup_{n\to\infty} (\mathrm{S}(a_n) \cap W) \subseteq \mathrm{S}(a) \cap W. \qquad --- (4.2)$$

Proof: Let $(\lambda_n, u_n) \in \mathrm{S}(a_n) \cap W$ with $(\lambda_n, u_n) \to (\lambda, u) \in W$. By H6, the sequence has convergent subsequences remaining in W. By Lemma 4.1, $F(\lambda_n, u_n, a_n) \to F(\lambda, u, a)$. Since each term is zero, the limit is zero and $(\lambda, u) \in \mathrm{S}(a)$.

Lemma 4.3 (Local uniqueness implies Hausdorff continuity)

Under Lemma 3.2 and Proposition 3.4, there exists a neighborhood $W_0 \subset W$ of $(\lambda_*, 0)$ such that $\mathrm{S}(a) \cap W_0$ consists precisely of the local curve $\mathrm{C}_{\mathrm{loc}}(a)$. Then for $a_1, a_2 \in A_*$

$$\mathrm{dist}_H(\mathrm{S}(a_1) \cap W_0, \mathrm{S}(a_2) \cap W_0) \le C\|a_1 - a_2\|. \qquad --- (4.3)$$

Proof: Parametrize $\mathrm{S}(a) \cap W_0$ by $s \mapsto (\lambda(s, a), u(s, a))$ with $|s| < \delta$. The uniform C^1 bounds (3.5)-(3.9) yield $|\lambda(s, a_1) - \lambda(s, a_2)| + \|u(s, a_1) - u(s, a_2)\| \le C\|a_1 - a_2\|$ by mean-value estimates, giving (4.3).

Theorem 4.4 (Hausdorff-Lipschitz stability on bounded windows)

Assume H1-H6 and suppose in addition that $D_u F$ is Lipschitz in a on $W \times A$ with constant L_a, and $\|L(\lambda, a)^{-1}\| \le M$ on W for all $a \in A$. Then for any $a_1, a_2 \in A$,

$$\mathrm{dist}_H(\mathrm{S}(a_1) \cap W, \mathrm{S}(a_2) \cap W) \le C_W\|a_1 - a_2\|, \qquad --- (4.4)$$

where C_W depends only on W, L_a, M, and bounds in **H5**-H6.

Proof: Let $(\lambda_1, u_1) \in \mathrm{S}(a_1) \cap W$. Consider the map

$$\Psi(\lambda, u; a) := u - L(\lambda, a)^{-1} N(\lambda, u, a). \qquad --- (4.5)$$

At $(\lambda_1, u_1; a_1)$ we have $\Psi = 0$ and

$$D_{(\lambda,u)}\Psi = I - L^{-1}D_u N - (D_\lambda L^{-1})N, \qquad --- (4.6)$$

which is invertible by the assumed M and smallness on W (choose W such that $\|L^{-1}D_u N\| < 1/2$). The implicit function theorem with Lipschitz parameter dependence in a yields a unique zero (λ_2, u_2) for parameter a_2 with

$$\|(\lambda_2 - \lambda_1, u_2 - u_1)\| \le C_W \|a_2 - a_1\|.$$

This gives the one-sided Hausdorff bound; symmetry gives (4.4).

Corollary 4.5 (Convergence as the band shrinks).

If $A_r := \{a: \|a - \bar{a}\| \le r\}$ and $r \downarrow 0$, then

$$\mathrm{dist}_H\left(\bigcup_{a\in A_r} S(a) \cap W, S(\bar{a}) \cap W\right) \to 0. \qquad --- (4.7)$$

Proof: Immediate from (4.4) and $\sup_{a\in A_r} \|a - \bar{a}\| = r$.

4.2 Quantitative radius-to-extent bounds

We now relate the band radius $\mathrm{rad}(A)$ to the "extent" of the union of continua inside W.

Define the projection length in λ of a set $E \subset \mathrm{R} \times X$ over W by

$$\mathrm{L}_\lambda(E; W) := \text{ meas } \{\lambda \in \mathrm{R}: \exists u \in X \text{ with } (\lambda, u) \in E \cap W\}. \qquad --- (4.8)$$

Lemma 4.6 (Local Lipschitz control of λ-projection)

Under the assumptions of Theorem 4.4,

$$|\mathrm{L}_\lambda(\mathrm{S}(a_1); W) - \mathrm{L}_\lambda(\mathrm{S}(a_2); W)| \le C'_W \|a_1 - a_2\|. \qquad --- (4.9)$$

Proof: Cover W by finitely many neighborhoods where the implicit function theorem produces solution graphs $\lambda \mapsto u(\lambda, a)$ or $s \mapsto (\lambda(s, a), u(s, a))$ with Lipschitz dependence on a. The boundary in λ moves by at most $C\|a_1 - a_2\|$; add the contributions [13].

Theorem 4.7 (Radius-to-extent inequality)

Let $A \subset \mathrm{R}^m$ be compact and assume the hypotheses of Theorem 4.4. Then on any bounded window W,

$$\mathrm{L}_\lambda(\cup_{a\in A} \ S(a); W) \le \mathrm{L}_\lambda(S(\bar{a}); W) + C''_W \mathrm{rad}(A) \qquad --- (4.10)$$

Proof: Pick $a \in A$. By Lemma 4.6,

$$\mathrm{L}_\lambda(\mathrm{S}(a); W) \le \mathrm{L}_\lambda(\mathrm{S}(\bar{a}); W) + C'_W \|a - \bar{a}\|$$

Taking the supremum over $a \in A$ yields (4.10) with $C''_W = C'_W$.

Remark 4.8 (Practical implication): Inequality (4.10) quantifies that widening the uncertainty band by Δr can enlarge the observed λ-extent of the union of continua in W by at most $O(\Delta r)$. This underpins robust continuation algorithms that sample a finite set of ancillary parameters and take the outer envelope (Section 7).

5 Global bifurcation alternative and degree argument

We now give a full, self-contained global argument for each fixed $a \in A$ and then pass to the band-union [9]. The setting and notation are those of Sections 1-4. Recall $N(\lambda, u, a) = F(\lambda, u, a) - L(\lambda, a)u$.

5.1 Degree set-up on a bounded window

Fix $a \in A_*$. Choose $\lambda^- < \lambda_* < \lambda^+$inside the neighborhood I from Lemma 3.1 and a radius $r > 0$ so that

$$F(\lambda^\pm, u, a) = 0 \text{ for all } \|u\|_X = r. \qquad --- (5.1)$$

Define the compact perturbations of the identity

$$\Phi^\pm(u) := u - L(\lambda^\pm, a)^{-1} N(\lambda^\pm, u, a) \qquad --- (5.2)$$

By **H5**, $\Phi^\pm$are well defined and completely continuous on $\overline{B_X(r)}$; by (5.1), $0 \notin \Phi^\pm(\partial B_X(r))$. Hence the Leray-Schauder degrees

$$d^\pm := \deg(\Phi^\pm, B_X(r), 0) \qquad --- (5.3)$$

are defined [14, 15]. (We suppress the parameter a here.)

Lemma 5.1 (Local index computation)

Under $\mathbf{H3} - \mathbf{H4}$, there exists $r > 0$ and $\varepsilon > 0$ such that for all $|\lambda - \lambda_*| < \varepsilon$,

$$\deg(\Phi_\lambda, B_X(r), 0) = \begin{cases} d^- & \text{if } \lambda < \lambda_* \\ d^- \mp 1 & \text{if } \lambda > \lambda_* \end{cases} \qquad --- (5.4)$$

with the sign determined by $\operatorname{sign}\langle \psi_*, D_\lambda L(\lambda_*, a) v_* \rangle$.

Proof: Reduce to the scalar equation $g(\lambda, s, a) = 0$ (Lemma 3.2). Inside $\|u\| \le r$, every solution is on the Lyapunov-Schmidt manifold $u = sv_* + w(\lambda, s, a)$. The onedimensional crossing flips the Brouwer degree of the reduced map by ± 1, which lifts to Leray-Schauder degree because the complement equation has a unique solution for each (λ, s) and the projection is an isomorphism (Lemma 3.1).

Lemma 5.2 (Nontrivial connected set)

Let

$$\mathrm{Z} := \{(\lambda, u) \in [\lambda^-, \lambda^+] \times \overline{B_X(r)} : F(\lambda, u, a) = 0\} \qquad --- (5.5)$$

If $d^- = d^+$, then Z contains a connected component K intersecting both slices $\{\lambda = \lambda^-\}$and $\{\lambda = \lambda^+\}$.

Proof: Consider the homotopy $\mathrm{H}(t, u) = \Phi^{\lambda^- + t(\lambda^+ - \lambda^-)}(u)$. The degree change implies 0 lies in the image of H for all $t \in [0,1]$. Compactness (H5-H6) shows the set of zeros is compact; the Whyburn lemma (connectedness of the set joining boundary slices) yields a component intersecting both ends; see [10, Lemma 2.4] or [12, Thm. 9.2].

5.2 Unboundedness or a secondary trivial point

Let $\mathrm{C}(a)$ be the connected component of $\mathrm{S}(a) = \{F(\lambda, u, a) = 0\}$ that contains $(\lambda_*, 0)$

Theorem 5.3 (Rabinowitz alternative at fixed $a \in A$)

Either
(i) C (a) is unbounded in $\mathrm{R} \times X$, or
(ii) there exists $\hat{\lambda} \in \mathrm{R}$ with $\ker L(\hat{\lambda}, a) = \{0\}$ such that $(\hat{\lambda}, 0) \in \overline{\mathrm{C}(a)}$.

Proof: Suppose (i) fails. Then there exists a bounded open set $W \subset \mathrm{R} \times X$ with $\mathrm{C}(a) \subset W$. By Lemma 5.2 the component $\mathrm{K} \subset \mathrm{Z}$ joins the two faces $\lambda^{\pm}$. If K did not meet $(\lambda, 0)$ with nontrivial kernel, we could contract K within a region where $L(\lambda, a)$ stays invertible, contradicting the degree jump (5.4) via homotopy invariance. Hence

(ii). This is the standard Rabinowitz global argument [1], adapted to our

Fredholm/degree setting; cf. [10, Section 3], [11, Section 2].

Corollary 5.4 (Isolation yields unboundedness)

If λ_* is the unique parameter with $\ker L(\lambda, a) \neq \{0\}$ for the given a, then only (i) occurs.

Proof: Exclude (ii).

5.3 Band-robust global alternative

Let $\mathrm{C} = \overline{\cup_{a \in A} \mathrm{C}(a)}$.

Theorem 5.5 (Band-union alternative)

Either
(i) C is unbounded in $\mathrm{R} \times X$, or
(ii) there exist $\hat{\lambda}$ and a sequence $a_n \in A$ with $\ker L(\hat{\lambda}, a_n) = \{0\}$ and $(\hat{\lambda}, 0) \in \overline{\cup_n \mathrm{C}(a_n)}$ If, for all $a \in A$, the only simple crossing is at λ_*, then only (i) occurs.

Proof: Apply Theorem 5.3 to each a. If (i) fails for the union, extract from boundedness a subsequence a_n for which the alternatives (ii) must occur; compactness (H6) and upper semicontinuity (Lemma 4.2) give the stated accumulation. The isolation statement eliminates (ii).

5.4 A priori bounds and avoidance of blow-up (model toolkit)

We record a convenient criterion for ruling out case (ii) in semi-linear models.

Lemma 5.6 (A priori bound via subcritical growth)

Let $X = H_0^1(\Omega), Y = H^{-1}(\Omega), \Omega \subset \mathrm{R}^N$ bounded Lipschitz, and

$$F(\lambda, u, a) = -\Delta u - \lambda u - f(u; a) \qquad --- (5.6)$$

with $f(\cdot; a)$ Carathéodory, $f_u(0; a) = 0$, and for some $2 < p < 2^*$ (Sobolev critical exponent), $|f(u; a)| \leq C_1|u| + C_2|u|^{p-1}$ uniformly in a. Then on rectangles $W = [\ \lambda_1, \lambda_2\ ] \times B_X(R)$, the set of solutions is bounded in $H_0^1(\Omega)$ uniformly in a.

Proof: Test the weak form with u and use Poincaré-Sobolev:

$$\|\nabla u\|_2^2 - \lambda\|u\|_2^2 = \int_\Omega f(u;a)u dx \le C_1\|u\|_2^2 + C_2\|u\|_p^p \le C_1'\|u\|_2^2 + C_2'\|u\|_{H_0^1}^p.$$

Absorb $\|u\|_{H_0^1}^2$ to the left for λ in a compact interval, and use $p > 2$ to bound $\|u\|_{H_0^1}$ by a constant depending only on W.:

Consequences: With Lemma 5.6 and simplicity of the first eigenvalue, case (ii) is excluded for λ in a neighborhood of $\lambda_1(\Omega)$, so the component must be unbounded; compare.

6 Applications and examples

We verify the hypotheses **H1 − H6** and instantiate the main theorems for three standard model classes. Each subsection ends with an explicit "Checklist" summarizing which hypotheses are met and why [16].

6.1 Semilinear elliptic equations

Let $\Omega \subset \mathrm{R}^N$ be bounded Lipschitz, $X = H_0^1(\Omega), Y = H^{-1}(\Omega)$, and

$$F(\lambda, u, a) = -\Delta u - \lambda u - f(u;a) \qquad ---(6.1)$$

Assume:

- $f(\cdot;a)$ is Carathéodory and C^1 in $u; f_u(0;a) = 0$ uniformly in $a \in A$;
- Subcritical growth: $|f(u;a)| \le C(1 + |u|^{p-1})$ with $2 < p < 2^*$ uniformly in a;
- $f(\cdot;a)$ is locally Lipschitz in u uniformly in a.

Verification of H1. Standard Nemytskii theory gives $F \in C^1(\mathrm{R} \times X, Y)$.

H2: $L(\lambda, a) = -\Delta - \lambda I$ is Fredholm of index $0, \ker L \equiv \{0\}$ iff λ is a Dirichlet eigenvalue.
H3: Take $\lambda_* = \lambda_1(\Omega)$, the simple first eigenvalue, with eigenfunction $v_* > 0$.
H4: $D_\lambda L(\lambda_*, a)v_* = -v_* \notin \mathrm{Range}L(\lambda_*, a)$ since ψ_* is the first adjoint eigenfunction and $\langle\psi_*, v_*\rangle = 0$.
H5: $N(\lambda, u, a) = -f(u;a)$ is compact $X \to Y$ on bounded sets via Rellich-Kondrachov $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ and subcritical growth.
H6: Properness on bounded sets follows from Lemma 5.6 and weak sequential compactness in H_0^1.

Model theorem 6.1 (Global alternative for (6.1))

For every $a \in A$, the component $\mathrm{C}(a)$ issuing from ($\lambda_1(\Omega)$,0) satisfies Theorem 5.3. If $\lambda_1(\Omega)$ is the only eigenvalue met in W and f satisfies the bounds of Lemma 5.6, then C (a) is unbounded. Moreover, on any bounded window W, the solution sets satisfy the Hausdorff-Lipschitz estimate (4.4) with a constant depending on W and the data bounds.

Proof: Combine the verifications of H1-H6, Theorems 5.3 and 4.4, and Lemma 5.6.

Checklist 6.1 (Assumptions for (6.1)).

- Spaces: $X = H_0^1(\Omega), Y = H^{-1}(\Omega)$.
- Crossing: simple at $\lambda_1(\Omega)$.
- Compactness: yes, via subcritical growth.
- Properness: yes, by a priori bound.
- Band stability: (4.4) holds if f_u is Lipschitz in a.

6.2 p-Laplacian type equations

Let $1 < p < \infty, X = W_0^{1,p}(\Omega), Y = W^{-1,p'}(\Omega)$, and

$$F(\lambda, u, a) = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) - \lambda\varrho(a)|u|^{p-2}u - g(u; a), \qquad --- (6.2)$$

where $\varrho(a)$ is positive and bounded above/below uniformly on A, and $g(\cdot; a)$ is Carathéodory with $|g(u; a)| \le C(1 + |u|^{q-1}), p \le q < p^*$.

H1: F is C^1 as a mapping $X \to Y$ away from $u = 0$; for bifurcation at $u = 0$, the linearized operator is

$$\begin{aligned} L(\lambda, a)\phi &= -\mathrm{div}((p-1)|\nabla 0|^{p-2}\nabla\phi) - \lambda\varrho(a)(p-1)|0|^{p-2}\phi \\ &= -\Delta_p'\phi - \lambda\varrho(a)\phi \end{aligned}$$

which reduces to the weighted p-Laplacian linearization at zero (well-defined in the sense of the first eigenpair).

H2-H4: The first eigenvalue of $-\Delta_p$ with weight $\varrho(a)$ is simple; transversality follows from $\langle\psi_*, D_\lambda L(\lambda_*, a)v_*\rangle = -\langle\psi_*, \varrho(a)v_*\rangle = 0$.

H5-H6: Compactness and properness use monotonicity and the $(S)_+$property for the p -Laplacian, plus subcritical growth of g.

Model theorem 6.2 (Global alternative for (6.2))

For each $a \in A$, a global continuum bifurcates from the first weighted p-eigenvalue $\lambda_*(a)$ and satisfies the Rabinowitz alternative; if $\lambda_*(a)$ is isolated in W and data are subcritical, the branch is unbounded. Hausdorff-Lipschitz stability in a holds on bounded windows provided g_u and ϱ are Lipschitz in a.

Checklist 6.2.

- Spaces: $W_0^{1,p} \to W^{-1,p'}$.
- Crossing: simple weighted p-eigenvalue.
- Compactness: $(S)_+$and subcritical growth.
- Properness: a priori bounds via standard energy inequalities.
- Band stability: needs Lipschitz control of ϱ, g_u in a.

6.3 Discrete graph models

Let $G = (V, E)$ with $|V| = n$. Set $X = Y = \mathrm{R}^n$ with the Euclidean norm. Consider

$$F(\lambda, u, a) = Lu - \lambda Mu - G(u; a) \qquad --- (6.3)$$

where L is a symmetric graph Laplacian, M a positive definite "mass" matrix (e.g., diagonal vertex weights), and $G(\cdot; a)$ has Jacobian $D_u G(0; a) = 0$ and $\|G(u; a)\| \le C\|u\|^2$ for small $\|u\|$, uniformly in a.

H1: $F \in C^1$ with $D_u F(\lambda, 0, a) = L - \lambda M$.

H2-H4: $L - \lambda M$ is a pencil with simple eigenvalues; take $\lambda_* = \lambda_1$ (smallest generalized eigenvalue), which is simple for connected graphs. Transversality: $D_\lambda L(\lambda_*, a) = -M$, and $\langle \psi_*, Mv_* \rangle > 0$.

H5: G is compact on bounded sets since X is finite dimensional.

H6: Properness is automatic in finite dimension.

Model theorem 6.3 (Global alternative for (6.3))

For each a, a global continuum bifurcates from $(\lambda_1, 0)$ and satisfies Theorem 5.3; unboundedness occurs unless another generalized eigenvalue is met. Band stability (4.4) holds provided $D_u G$ and M depend Lipschitz-continuously on a.

Checklist 6.3.

- Spaces: R^n.
- Crossing: simple smallest generalized eigenvalue.
- Compactness & properness: automatic.
- Band stability: Lipschitz in a for $M, D_u G$.

7 Concluding remarks

We established a band-robust version of global bifurcation for compact perturbations of Fredholm index-zero maps

$$F(\lambda, u, a) = 0, (\lambda, u, a) \in \mathrm{R} \times X \times A,$$

where A encodes an ancillary uncertainty band. Under a simple spectral crossing and a transversality condition uniform on a neighborhood $A_* \subset A$, we proved:

(i) a uniform local theorem via Lyapunov-Schmidt reduction with constants independent of $a \in A_*$;

(ii) a degree jump constant on A_* yielding a Rabinowitz-type global alternative for every a and for the band-union continuum $\mathrm{C} = \overline{\bigcup_{a \in A} \mathrm{C}(a)}$;

(iii) Hausdorff-Lipschitz stability of solution sets with respect to a on bounded windows;

(iv) radius-to-extent inequalities showing λ-projection growth of $\cup_{a \in A} \mathrm{S}(a)$ is $O(\mathrm{rad}(A))$.

Limitations: Our compactness/properness hypotheses ($\mathrm{H5} - \mathrm{H6}$) exclude certain quasilinear or noncompact settings (e.g., critical growth without compact embeddings, nonlocal operators with essential spectrum touching zero). The simple crossing assumption excludes multiple or defective eigenvalues and Turning-Hopf interactions.

Future directions.

(i) ***Multiple crossings and equivariance***: Extend to finite multiplicity using the equivariant degree and crossing numbers on isotypic components.
(ii) ***Noncompact perturbations***: Replace (H5) by condensing or measure of noncompactness assumptions.
(iii) ***Random-set bands***: Treat A as a random compact set and derive almost-sure statements on C.
(iv) ***Validated numeric***: Combine pseudo-arclength continuation with aposteriori radii-polynomial certificates uniform in a to produce computer assisted proofs.

This work establishes a band-robust framework for global bifurcation in nonlinear operator equations $F(\lambda, u, a) = 0$ on Banach spaces when ancillary parameters vary within a compact uncertainty set A. Building on a uniform Lyapunov-Schmidt reduction around a simple spectral crossing and a transversality condition holding for all a in a neighborhood of the crossing, we proved existence and uniqueness of a C^1 local branch with constants independent of a, quantified the associated degree jump, and lifted Rabinowitz's global alternative from individual parameters to the band-union continuum $\mathcal{C} = \overline{\bigcup_{a\in A} \mathcal{C}(a)}$. We further derived Hausdorff-Lipschitz stability of solution sets with respect to a on bounded windows and a radius-to-extent inequality showing that the λ-projection of $\mathcal{C}$ grows at most linearly with $\mathrm{rad}(A)$.

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