Global and Stochastic Analysis
Vol. 12 No. 6 (November, 2025)

Received: 20th July 2025 *Revised: 30th September 2025* *Accepted: 04th November 2025*

A STUDY OF FIXED POINTS FOR ALMOST $\mathcal{B}$-TYPE CONTRACTION

[1]SANEHLATA, [2]POOJA VATS, AND [3]VIZENDER SINGH

Abstract. The notion of the $\mathcal{B}$-contraction was introduced in recent years by Bijender *et al*. The contraction of this type is a legitimate generalization of the conventional contraction in literature. The present work established the new concept of almost $\mathcal{B}$-contraction including generalized $\mathcal{B}$-contraction. Further, these contractions are utilized to prove some fixed point theorems. Some illustrative examples are provided to support these results. Finally, an application is presented to solve the Volterra-type integral equations.

1. Introduction and Preliminaries

The Banach contraction principle gave rise to the fruitful theory of invariant point, which has been extensively researched. The utility of the concept relied on the existence of solutions to mathematical issues derived from economics and engineering. After the existence of the solutions has been proven, the numerical methodology for obtaining the approximated solution will be defined. The considered spaces, which are specified using intuitive axioms, greatly influence the fixed points of the function. Various metric spaces, such as G-metric spaces, partial metric spaces, b-metric spaces, fuzzy metric spaces, probabilistic metric spaces, etc., are proposed. Different kinds of fixed-point theorems originate from different spaces. In other words, the literature has many distinct sorts of fixed-point theorems.

An unambiguous methodology of the fixed-point hypothesis is included in the doctoral thesis of Banach. Due to the simplicity and effectiveness of BCP, it has been widely used in approaching of tackling current challenges in various areas of mathematical sciences. Over the decades, the Banach contraction concept has been developed in various approaches. In several generalizations, the map's contractive character is weakened, see, for example, [2]-[9], and sometimes the topology is weakened in several extensions, see [13]-[26] and others. Nadler [25] advanced the study of BCP from single-valued to multi-valued type contractive mappings, for more results see, [28]-[33] and references cited therein. Recently an interesting generalization of BCP was given by Bijender et al., [10]. First, we will review the idea of $\mathcal{B}$-contraction as established by Bijender et al., [10], and summarize his conclusions.

2000 *Mathematics Subject Classification.* Primary 54H25; Secondary 47H10.

Key words and phrases. $\mathcal{B}$-contraction, almost $\mathcal{B}$-contraction, generalized $\mathcal{B}$-contraction, fixed point, complete metric space, integral equation.

Definition 1.1. [10] Let $\mathcal{B}$ be the collection of mappings $\phi : \mathbb{R}^+ \to \mathbb{R}$ holds the following axioms:

(M-1) for all $a, b \in \mathbb{R}^+$ such that $a < b, \phi(a) < \phi(b)$, i.e., strictly increasing,
(M-2) $\lim_{n\to\infty} \phi(a_n) = 0 \Leftrightarrow \lim_{n\to\infty} a_n = 0$, where $\{a_n\}_{n\in\mathbb{N}}$ is sequence with $a_n > 0$,
(M-3) Also, the function ϕ has continuity on $(0, \infty)$.

Definition 1.2. [10] A self-mapping T on a metric space (E, d) is said to be a $\mathcal{B}$-contraction if there exists $\alpha \in (0, 1)$ and $\phi \in \mathcal{B}$ such that

$$\forall \varsigma, \vartheta \in E,\ d(T\varsigma, T\vartheta) > 0 \Rightarrow \phi(d(T\varsigma, T\vartheta)) \leq \alpha\phi(d(\varsigma, \vartheta)). \tag{1.1}$$

Further, Bijender et al. [10] observed the various types of mappings ϕ in 1.1 and obtained a broad array of contractions, some of which are distinguished in the existing studies, as seen in the illustrations below:

Example 1.3. [10] Consider $\phi(\varsigma) = \sqrt{\varsigma}, \varsigma > 0$ then ϕ satisfies (M-1)-(M-3) and the contractive condition 1.1 for the ϕ-contraction T, is reduced to

$$d(T\varsigma, T\vartheta) \leq \alpha^2 d(\varsigma, \vartheta) \text{ for all } \varsigma, \vartheta \in E, T\varsigma \neq T\vartheta.$$

Example 1.4. [10] Let $\phi(\varsigma) = \varsigma^n, \varsigma > 0$, where ϕ satisfies (M-1)-(M-3) and the contractive condition 1.1 for the ϕ-contraction T, is reduced to

$$d(T\varsigma, T\vartheta) \leq \alpha^{\frac{1}{n}} d(\varsigma, \vartheta) \text{ for all } \varsigma, \vartheta \in E, T\varsigma \neq T\vartheta.$$

Example 1.5. [10] Let $\phi(\varsigma) = \dfrac{\varsigma}{\varsigma + 1}, \varsigma > 0$, then ϕ satisfies (M-1)-(M-3) and the contractive condition 1.1 for the ϕ - contraction T, is reduced to

$$d(T\varsigma, T\vartheta) \leq \alpha \frac{(d(T\varsigma, T\vartheta) + 1)}{(d(\varsigma, \vartheta) + 1)} d(\varsigma, \vartheta), \text{ for all } \varsigma, \vartheta \in E, T\varsigma \neq T\vartheta.$$

For several other instances of the function $\phi \in \mathcal{B}$ one may refer to [10]. Furthermore, Bijender et al. showed that each $\mathcal{B}$-contraction T implies conventional contraction, i.e.,

$$d(T\varsigma, T\vartheta) \leq \lambda d(\varsigma, \vartheta) \text{ for all } \varsigma, \vartheta \in E, T\varsigma \neq T\vartheta, \lambda \in (0, 1),$$

this implies that $\mathcal{B}$-contraction T is continuous. On similar lines authors in [10] observed that if $\phi_1, \phi_2 \in \mathcal{B}$ with $\phi_1(\varsigma) \leq \phi_2(\varsigma)$ for all $\varsigma > 0$ and $H = \phi_2 - \phi_1$ is non-decreasing, then every ϕ_1-contraction T is an ϕ_2-contraction.

Theorem 1.6. [10] *Let E be a non-empty set and mapping $d : E \times E \to \mathbb{R}^+$ such that pair (E, d) is complete metric space. If self map T on E is $\mathcal{B}$-contraction, then there exists unique $\varsigma \in E$ such that $T\varsigma = \varsigma$.*

The main objective of the present work is to introduce the new concept of almost $\mathcal{B}$-contraction, generalized $\mathcal{B}$-contraction and to utilize them to prove fixed point theorems in the setting of complete metric structure.

2. Main Results

To begin, consider the concept of almost contraction in metric space (for more details, see [11, 12])

Definition 2.1. Assume that E be a non-empty set and (E, d) be a complete metric space. A self-mapping $T : E \to E$ is said to be an almost contraction if there exists $0 < \delta < 1$ and $L \geq 0$ with

$$d(T\varsigma, T\vartheta) \leq \delta d(\varsigma, \vartheta) + Ld(\vartheta, T\varsigma) \text{ for all } \varsigma, \vartheta \in E. \tag{2.1}$$

Remark 2.2. Because of symmetric relation of distance, the almost contraction condition admits the undermentioned property

$$d(T\varsigma, T\vartheta) \leq \delta d(\varsigma, \vartheta) + Ld(\varsigma, T\vartheta) \text{ for all } \varsigma, \vartheta \in E. \tag{2.2}$$

So, it is necessary to check both 2.1 and 2.2 for a mapping T to be almost contraction.

In [11, 12], Berinde demonstrated that contractions such as Banach, Zamfirescu, and Chatterjea are almost contractions. Berinde [11] showed that an almost contraction mapping on a complete metric space structure (E, d) ensures unique fixed point.

Now, we introduced the following definition.

Definition 2.3. Let $T : E \to E$ be a self mapping defined on metric space (E, d). Then T is almost $\mathcal{B}$-contraction there exists $\alpha \in (0, 1)$ and $L \geq 0$ satisfying

$$\forall\ \varsigma, \vartheta \in E\ [d(T\varsigma, T\vartheta) > 0 \Rightarrow \phi(d(T\varsigma, T\vartheta)) \leq \alpha\phi(d(\varsigma, \vartheta) + Ld(\varsigma, T\vartheta))] \tag{2.3}$$

also, it is necessary to check for almost contraction

$$\forall\ \varsigma, \vartheta \in E\ [d(T\varsigma, T\vartheta) > 0 \Rightarrow \phi(d(T\varsigma, T\vartheta)) \leq \alpha\phi(d(\varsigma, \vartheta) + Ld(\vartheta, T\varsigma))]. \tag{2.4}$$

Remark 2.4. By assuming $\phi \in \mathcal{B}$ is an identity mapping, one can find that each almost contraction implies almost $\mathcal{B}$-contraction but not conversely.

Consider (E, d) be a complete metric space, where $E = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ with usual metric d. If map $T : E \to E$ such that

$$T\varsigma = \begin{cases} \frac{1}{n+1} & if\ \varsigma = \frac{1}{n}, \\ 0 & if\ \varsigma = 0, \end{cases}$$

then for $\varsigma = \dfrac{1}{n},\ \vartheta = \dfrac{1}{n+1}$, we have $d(\vartheta, T\varsigma) = 0$ and

$$\sup_{\varsigma,\vartheta \in E, \varsigma \neq \vartheta} \frac{d(T(\frac{1}{n}), T(\dfrac{1}{n+1}))}{d(\dfrac{1}{n}, \dfrac{1}{n+1})} = 1,$$

which is contradiction, hence there is no $0 < \delta < 1$and $L \geq 0$ satisfying condition 2.1 but T is almost $\mathcal{B}$-contraction.

Theorem 2.5. *Consider $E \neq \phi$ be a set and mapping $d : E \times E \to \mathbb{R}^+$ such that pair (E, d) is complete metric space. If self map T on E is an almost $\mathcal{B}$-contraction, then $\exists\ p \in E$ such that $Tp = p$.*

Proof. Let us assume $\varsigma_0 \in E$ be arbitrary and a sequence $\{\varsigma_n\}_{n=1}^{\infty}$ that for all $n \in \mathbb{N} \cup \{0\}$,

$$\varsigma_n = T\varsigma_{n-1}, \tag{2.5}$$

if for some $n_{k_0} \in \mathbb{N} \cup \{0\}$ such that $\varsigma_{n_k+1} = \varsigma_{n_k}$, then result is obvious. So, assume that $\varsigma_{n+1} \neq \varsigma_n$ for every $n \in \mathbb{N} \cup \{0\}$. Take $\Upsilon_n = d(\varsigma_{n+1}, x_n), n \in \mathbb{N} \cup \{0\}$. Then $\Upsilon_n > 0 \ \forall \ n \in \mathbb{N} \cup \{0\}$. From 2.5, we have $\phi(\Upsilon_n) = \phi(d(\varsigma_{n+1}, \varsigma_n)) = \phi(d(T\varsigma_n, T\varsigma_{n-1})) \leq \alpha\phi(d(\Upsilon_{n-1}))$. Similarly repeating this process, we get $\phi(\Upsilon_n) \leq \alpha^n \phi(\Upsilon_0)$, as n approaches to ∞, $\phi(\Upsilon_n) \to 0$, using (M-2), gives

$$\lim_{n\to\infty} \Upsilon_n = 0 \Rightarrow \lim_{n\to\infty} d(x_n, T\varsigma_n) = 0. \tag{2.6}$$

Now, to show that sequence $\{\varsigma_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, let us assume $\exists \ \epsilon > 0$ and $\{r_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ two sequences, with r_n, s_n both in $\mathbb{N}$ such that

$$r_n > s_n > n, \ d(\varsigma_{r_n}, \varsigma_{s_n}) \geq \epsilon, \ d(\varsigma_{r_{n-1}}, \varsigma_{s_n}) < \epsilon, \ \forall n \in \mathbb{N}. \tag{2.7}$$

Thus, we obtain

$$\begin{aligned} \epsilon \leq d(\varsigma_{r_n}, \varsigma_{s_n}) &\leq d(\varsigma_{r_n}, \varsigma_{r_{n-1}}) + d(\varsigma_{r_{n-1}}, \varsigma_{s_n}) \\ &\leq d(\varsigma_{r_n}, \varsigma_{r_{n-1}}) + \epsilon \\ &= d(\varsigma_{r_{n-1}}, T\varsigma_{r_{n-1}}) + \epsilon. \end{aligned} \tag{2.8}$$

Considering 2.6 and the preceding inequality, we obtain

$$\lim_{n\to\infty} d(\varsigma_{r_n}, \varsigma_{s_n}) = \epsilon. \tag{2.9}$$

Therefore, from 2.6, there exists $n \in \mathbb{N}$, satisfying

$$d(\varsigma_{r_m}, T\varsigma_{r_m}) < \frac{\epsilon}{3} \text{ and } d(\varsigma_{s_m}, T\varsigma_{s_m}) < \frac{\epsilon}{3}, \ \forall \ m \geq n, \tag{2.10}$$

further, we shall prove that

$$d(T\varsigma_{r_m}, T\varsigma_{s_m}) = d(\varsigma_{r_{m+1}}, \varsigma_{s_{m+1}}) > 0, \ \forall \ m \geq n. \tag{2.11}$$

To serve this purpose, let $\exists \ p \geq n$ so that

$$d(\varsigma_{r_{p+1}}, \varsigma_{s_{p+1}}) = 0, \tag{2.12}$$

using 2.7, 2.10 and 2.12, we have

$$\begin{aligned} \epsilon &\leq d(\varsigma_{r_p}, \varsigma_{s_p}) \leq d(\varsigma_{r_p}, \varsigma_{r_{p+1}}) + d(\varsigma_{r_{p+1}}, \varsigma_{s_p}) \\ &\leq d(\varsigma_{r_p}, \varsigma_{r_{p+1}}) + d(\varsigma_{r_{p+1}}, \varsigma_{s_{p+1}}) + d(\varsigma_{s_{p+1}}, \varsigma_{s_p}) \\ &= d(\varsigma_{r_p}, T\varsigma_{r_p}) + d(\varsigma_{r_{p+1}}, \varsigma_{s_{p+1}}) + d(\varsigma_{s_p}, T\varsigma_{s_p}) \\ &< \frac{\epsilon}{3} + 0 + \frac{\epsilon}{3} = \frac{2\epsilon}{3}, \end{aligned}$$

this leads to contradiction, hence 2.11 is true. Thus

$$\phi(d(T\varsigma_{r_m}, T\varsigma_{s_m})) \leq \alpha\phi(d(\varsigma_{r_m}, \varsigma_{s_m})). \tag{2.13}$$

By (M-3), 2.10 and 2.13, we get

$$\phi(\epsilon) \leq \alpha\phi(\epsilon).$$

It is a contraction, so our assumption is wrong. This shows that the sequence $\{\varsigma_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Furthermore, using completeness of (E, d) implies that the sequence $\{\varsigma_n\} \to p \in E$ i.e., $\lim_{n\to\infty} \varsigma_n = p$. From (M-1) and 2.1, it is easy to say that $d(T\varsigma, T\vartheta) < d(\varsigma, \vartheta) + Ld(\vartheta, T\varsigma)$, for all $\varsigma, \vartheta \in E$ with $T\varsigma \neq T\vartheta$. Therefore, $\forall \ \varsigma, \vartheta \in E$,

$$d(T\varsigma, T\vartheta) \leq d(\varsigma, \vartheta) + Ld(\vartheta, T\varsigma). \tag{2.14}$$

Thus, $d(Tp, \varsigma_{n+1}) = d(Tp, T\varsigma_n) \leq d(\varsigma_n, p) + Ld(p, T\varsigma_n) = d(\varsigma_n, p) + Ld(p, \varsigma_{n+1})$, as $n \to \infty$, $d(p, Tp) = 0$ and so $Tp = p$. □

As shown with example in remark 2.4, almost $\mathcal{B}$-contraction does not implies almost contraction. Also, mapping T is almost $\mathcal{B}$-contraction but not $\mathcal{B}$-contraction. This ensures that, the above Theorem 2.5 as extension over the Theorem 1.6.

Example 2.6. Assume $E = [0,1] \cup \{2,3\}$ with usual metric d. Let mapping $T : E \to E$ be a self-mapping defined as $T\varsigma = \begin{cases} \frac{1-\varsigma}{2} & \text{if } \varsigma \in [0,1], \\ \varsigma & \text{if } \varsigma \in \{2,3\}. \end{cases}$
Since $d(T2, T3) = 1 = d(2,3)$, then for $\alpha \in (0,1)$, we have $\phi(d(T2, T3)) > \alpha\phi(d(2,3))$. Which shows that, the self-mapping T is not $\mathcal{B}$-contraction. Now consider $\phi(\varsigma) = \varsigma$, then T is almost $\mathcal{B}$-contraction with $\alpha = [\frac{1}{2}, 1)$ and $L = 1$.

In theorem 2.5, we proved that almost $\mathcal{B}$-contraction, possesses a fixed point. Means that Theorem 2.5 does not guarantee the uniqueness of fixed point for the self-mapping T. For the uniqueness of fixed point of T, we have to consider some extra properties, which is mentioned in the next result.

Theorem 2.7. *Let the self-mapping $T : E \to E$ be an almost $\mathcal{B}$-contraction defined on complete metric space (E, d) and $G^* : \mathbb{R}^+ \to \mathbb{R}$ be such that it satisfies the conditions (M-1) to (M-3), with $L_1 \geq 0$ and $\alpha_1 \in (0,1)$. Suppose that T also satisfies*

$$\forall \varsigma, \vartheta \in E, [d(T\varsigma, T\vartheta) > 0 \Rightarrow G^*(d(T\varsigma, T\vartheta)) \leq \alpha_1 G^*(d(\varsigma, \vartheta) + L_1 d(\varsigma, T\varsigma))], \quad (2.15)$$

then T possesses a unique fixed point.

Proof. Assume ς_1 and ς_2 be two fixed points of self-mapping T. Suppose that $d(\varsigma_1, \varsigma_2) > 0$. From 2.15, we have

$$\begin{aligned} G^*(d(\varsigma_1, \varsigma_2)) &= G^*(d(T\varsigma_1, T\varsigma_2)) \\ &\leq \alpha_1[G^*(d(\varsigma_1, \varsigma_2) + Ld(\varsigma_1, T\varsigma_1))] \\ &= \alpha_1 G^*(d(\varsigma_1, \varsigma_2)), \end{aligned}$$

which contradicts our assumption. Thus, the fixed point is unique. □

Definition 2.8. Let $T : E \to E$ be a self-mapping defined on metric space (E, d). Then the self-mapping T is said to be generalized $\mathcal{B}$-contraction if for $\phi \in \mathcal{B}$ with $\alpha \in (0,1)$ satisfies

$$\forall_{\varsigma,\vartheta} \in E, \ [d(T\varsigma, T\vartheta) > 0 \Rightarrow \phi(d(T\varsigma, T\vartheta)) \leq \alpha\phi(\mathcal{M}(\varsigma, \vartheta))], \quad (2.16)$$

where, $\mathcal{M}(\varsigma, \vartheta) = \max\{d(\varsigma, \vartheta), d(\varsigma, T\varsigma), d(\vartheta, T\vartheta)\}$.

Theorem 2.9. *Consider $E \neq \Phi$ be a set and mapping $d : E \times E \to \mathbb{R}^+$ such that pair (E, d) is complete metric space. If self-mapping T on E is generalized $\mathcal{B}$-contraction, then there exists a unique $\varsigma \in E$ such that $T\varsigma = \varsigma$.*

Proof. Assume ς_0 be arbitrary and fixed in E. Let a sequence $\{\varsigma_n\}_{n=1}^{\infty}$ be such that $\forall$ n in $\mathbb{N} \cup \{0\}$,

$$\varsigma_n = T\varsigma_{n-1}, n \in \mathbb{N} \quad (2.17)$$

If for some $n_{k_0} \in \mathbb{N} \cup \{0\}$ such that $\varsigma_{n_k+1} = \varsigma_{n_k}$, then result is obvious. So, assume that $\varsigma_{n+1} \neq \varsigma_n$ for every $n \in \mathbb{N} \cup \{0\}$. Take $\Upsilon_n = d(\varsigma_{n+1}, x_n), n \in \mathbb{N} \cup \{0\}$. Then $\Upsilon_n > 0 \; \forall \; n \in \mathbb{N} \cup \{0\}$. From 2.17, we obtained $\phi(\Upsilon_n) = \phi(d(\varsigma_{n+1}, \varsigma_n)) = \phi d(T\varsigma_n, T\varsigma_{n-1})) \leq \alpha\phi(\mathcal{M}(\varsigma_n, \varsigma_{n-1})) = \alpha\phi(\max\{\Upsilon_n, \Upsilon_{n-1}\})$ for some $n \in \mathbb{N}$. If $\Upsilon_n \geq \Upsilon_{n-1}$, then $\phi(d_n) \leq \alpha\phi(d_n)$, this leads to contradiction, so this implies that $\Upsilon_n < \Upsilon_{n-1}, \forall n \in \mathbb{N}$. Hence, we have $\phi(\Upsilon_n) \leq \alpha\phi(\Upsilon_{n-1})$. Similarly, repeating this process, we get $\phi(\Upsilon_n) \leq \alpha^n\phi(\Upsilon_0)$, as n approaches to ∞, $\phi(\Upsilon_n) \to 0$, along with use of (M-2), provides

$$\lim_{n\to\infty} \Upsilon_n = 0 \Rightarrow \lim_{n\to\infty} d(\varsigma_n, T\varsigma_n) = 0. \tag{2.18}$$

Further, to show that the sequence $\{\varsigma_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, let us assume that $\exists \; \epsilon > 0$ and the sequences $\{r_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$, where r_n, s_n in $\mathbb{N}$ such that

$$r_n > s_n > n, \; d(\varsigma_{r_n}, \varsigma_{s_n}) \geq \epsilon, \; d(\varsigma_{r_{n-1}}, \varsigma_{s_n}) < \epsilon, \; \forall n \in \mathbb{N}. \tag{2.19}$$

Thus, we get

$$\begin{aligned} \epsilon \leq d(\varsigma_{r_n}, \varsigma_{s_n}) &\leq d(\varsigma_{r_n}, \varsigma_{r_{n-1}}) + d(\varsigma_{r_{n-1}}, \varsigma_{s_n}) \\ &\leq d(\varsigma_{r_n}, \varsigma_{r_{n-1}}) + \epsilon \\ &= d(\varsigma_{r_{n-1}}, T\varsigma_{r_{n-1}}) + \epsilon. \end{aligned} \tag{2.20}$$

Considering 2.18 and the preceding inequality, we obtain

$$\lim_{n\to\infty} d(\varsigma_{r_n}, \varsigma_{s_n}) = \epsilon. \tag{2.21}$$

Therefore, from 2.18, there exists $n \in \mathbb{N}$ satisfying

$$d(\varsigma_{r_m}, T\varsigma_{r_m}) < \frac{\epsilon}{3} \text{ and } d(\varsigma_{s_m}, T\varsigma_{s_m}) < \frac{\epsilon}{3}, \; \forall \; m \geq n. \tag{2.22}$$

Further, we shall prove that

$$d(T\varsigma_{r_m}, T\varsigma_{s_m}) = d(\varsigma_{r_{m+1}}, \varsigma_{s_{m+1}}) > 0, \; \forall \; m \geq n. \tag{2.23}$$

For this assume that, $\exists \; p \geq n$ such that

$$d(\varsigma_{r_{p+1}}, \varsigma_{s_{p+1}}) = 0, \tag{2.24}$$

using 2.19, 2.22 and 2.24, we have

$$\begin{aligned} \epsilon &\leq d(\varsigma_{r_p}, \varsigma_{s_p}) \leq d(\varsigma_{r_p}, \varsigma_{r_{p+1}}) + d(\varsigma_{r_{p+1}}, \varsigma_{s_p}) \\ &\leq d(\varsigma_{r_p}, \varsigma_{r_{p+1}}) + d(\varsigma_{r_{p+1}}, \varsigma_{s_{p+1}}) + d(\varsigma_{s_{p+1}}, \varsigma_{s_p}) \\ &= d(\varsigma_{r_p}, T\varsigma_{r_p}) + d(\varsigma_{r_{p+1}}, \varsigma_{s_{p+1}}) + d(\varsigma_{s_p}, T\varsigma_{s_p}) \\ &< \frac{\epsilon}{3} + 0 + \frac{\epsilon}{3} = \frac{2\epsilon}{3}. \end{aligned} \tag{2.25}$$

this leads to contradiction, hence 2.23 is true. Thus

$$\phi(d(T\varsigma_{r_m}, T\varsigma_{s_m})) \leq \alpha\phi(\mathcal{M}(\varsigma_{r_m}, \varsigma_{s_m})) \tag{2.26}$$

where

$$\mathcal{M}(\varsigma_{r_m}, \varsigma_{s_m}) = \max\{d(\varsigma_{r_m}, \varsigma_{s_m}), d(\varsigma_{r_m}, T\varsigma_{r_m}), d(\varsigma_{s_m}, T\varsigma_{s_m}).$$

By (M-3), 2.21 and 2.26, we get

(i) If $\mathcal{M}(\varsigma_{r_m}, \varsigma_{s_m}) = d(\varsigma_{r_m}, \varsigma_{s_m}) \Rightarrow \phi(\epsilon) \leq \alpha\phi(\epsilon)$, contradiction exists,

(ii) If $\mathcal{M}(\varsigma_{s_m}, \varsigma_{s_m}) = d(\varsigma_{r_m}, T\varsigma_{r_m}) \Rightarrow \phi(\epsilon) \leq \alpha\phi(0)$, again a contradiction exists,

(iii) If $\mathcal{M}(\varsigma_{s_m}, \varsigma_{s_m}) = d(\varsigma_{s_m}, T\varsigma_{s_m}) \Rightarrow \phi(\epsilon) \leq \alpha\phi(0)$, again a contradiction exists.

Hence our assumption is wrong, proving that the sequence $\{\varsigma_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence and utilizing completeness of (E, d), sequence $\{\varsigma_n\}_{n\in\mathbb{N}}$ is convergent to $\varsigma \in E$. If $T\varsigma \neq \varsigma$, then one can find a particular $n_1 \in \mathbb{N}$ and subsequence $\{\varsigma_{n_k}\}$ of $\{\varsigma_n\}$ such that $d(T\varsigma_{n_k}, T\varsigma_n) > 0$, $\forall\ n_k \geq n_1$. Therefore, we have

$$\begin{aligned}\phi(d(\varsigma_{n_{k+1}}, T\varsigma)) = \phi(d(T\varsigma_k, T\varsigma)) &\leq \phi(M(\varsigma_k, \varsigma))\\ &\leq \phi(\max\{d(\varsigma_{n_k}, \varsigma), d(\varsigma_{n_k}, \varsigma_{n_{k+1}}), d(\varsigma, T\varsigma)\}),\end{aligned}$$

Using the continuity of ϕ and employing limit as k approaches to ∞, one get

$$\phi(d(\varsigma, T\varsigma)) \leq \alpha\phi(d(\varsigma, T\varsigma)),$$

this leads to a contradiction, thus ς is a fixed point of T i.e., $T\varsigma = \varsigma$. Let $\vartheta(\neq \varsigma) \in E$ be another fixed point of E, i.e., $T\vartheta = \vartheta$. Therefore,

$$\begin{aligned}\phi(d(\varsigma, \vartheta)) = \phi(d(T\varsigma, T\vartheta)) &\leq \alpha\phi(M(\varsigma, \vartheta))\\ &= \alpha\phi(\max\{d(\varsigma, \vartheta), d(\varsigma, T\varsigma), d(\vartheta, T\vartheta)\})\\ &= \alpha\phi(d(\varsigma, \vartheta)),\end{aligned}$$

this leads to contradiction of our assumption. It ensures the uniqueness of fixed point of the self-mapping T. □

3. Application

Here, this section discusses further use of fixed point methods in solving the undermentioned Volterra-type integral problem.

$$\varsigma(t) = \int_0^t \mathcal{K}(t, s, \varsigma(s))ds + g(t),\ t \in [a, b], a > 0, b > 0. \tag{3.1}$$

Let $E = C[a, b]$ be the set of all real valued, continuous functions defined on $[a, b]$ and for arbitrary $\varsigma \in E$ define $||\varsigma||_\infty = \sup_{t\in[a,b]}\{|\varsigma(t)|\}$. and the metric $d : E \times E \to \mathbb{R}^+ \cup \{0\}$ is given by

$$d(\varsigma, \vartheta) = ||\varsigma - \vartheta||_\infty = \sup_{t\in[a,b]} |\varsigma(t) - \vartheta(t)|. \tag{3.2}$$

Also the metric space (E, d) is complete and the operator $T : E \to E$ is defined by

$$(T\varsigma)t = \int_0^t \mathcal{K}(t, s, \varsigma(s))ds + g(t), t \in [a, b], a > 0, b > 0.$$

then integral equation 3.1 has a solution iff operator T has a fixed point.

Theorem 3.1. *Consider an operator $T : C([a, b], \mathbb{R}) \to C([a, b], \mathbb{R})$ such that the following requirements are fulfilled:*

(i) $\mathcal{K} : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ *and* $g : [a, b] \to \mathbb{R}$,

(ii) $\int_0^t \mathcal{K}(t, s, .) : \mathbb{R} \to \mathbb{R}$ *increases monotonically,* $\forall\ t, s \in [a, b]$,

(iii) *there exists* $\alpha \in (0,1)$ *such that*

$$|\mathcal{K}(t,s,\varsigma) - \mathcal{K}(t,s,\vartheta)| \leq \frac{\sqrt{\alpha}}{t}|\mathcal{M}(\varsigma,\vartheta)|$$

where $\mathcal{M}(\varsigma,\vartheta) = \max\{d(\varsigma,\vartheta), d(\varsigma,T\varsigma), d(\vartheta,T\vartheta)\}$, *for all* $t,s \in [a,b]$ *and* $\varsigma, v \in (C[a,b],\mathbb{R})$.

Then, the integral equation 3.1 possesses a unique solution.

Proof. Let

$$T(\varsigma)(t) = \int_0^t \mathcal{K}(t,s,\varsigma(s))ds + g(s), t \in [a,b].$$

Now, using (iii), $\forall\ \varsigma,\vartheta \in (C[a,b],\mathbb{R})$, one get

$$\begin{aligned} |T(\varsigma)(t) - T(\vartheta)(t)| &\leq \int_0^t |\mathcal{K}(t,s,\varsigma(s)) - \mathcal{K}(t,s,\vartheta(s))|ds \\ &\leq \int_0^t \frac{\sqrt{\alpha}}{t}|\mathcal{M}(\varsigma,\vartheta)|ds \\ &\leq \frac{\sqrt{\alpha}}{t}|\mathcal{M}(\varsigma,\vartheta)|\int_0^t ds, \end{aligned}$$

therefore, we get

$$|T(\varsigma)(t) - T(\vartheta)(t)| \leq \sqrt{\alpha}|\mathcal{M}(\varsigma,\vartheta)|$$

or

$$d(T\varsigma, Tv) \leq \sqrt{\alpha}|\mathcal{M}(\varsigma,\vartheta)|.$$

All the assumptions of result 2.10 are met for $\phi(\varsigma) = \varsigma^2$. Hence, integral equation 3.1 has a solution, which is a unique. □

References

1. Banach, S.: Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, *Fundam. Math.* **3**(1922) 133-181.
2. Boyd, DW, Wong, JSW.: On nonlinear contractions, *Proc. Am. Math. Soc.* **20**(1969) 458-464.
3. Ciric, L.: A generalization of Banachs contraction principle, *Proc. Am. Math. Soc.* **45**(1974) 267-273.
4. Kirk, W.A.: Fixed points of asymptotic contractions, *J. Math. Anal. Appl.* **277**(2003), 645-650.
5. Meir, A, Keeler, E.: A theorem on contraction mappings, *J. Math. Anal. Appl.* **28**(1969) 326-329.
6. Rakotch, E.: A note on contractive mappings, *Proc. Am. Math. Soc.* **13**(1962) 459-465.
7. Reich, S.: Fixed points of contractive functions, *Boll. Unione Mat. Ital.* **5**(1972) 26-42.
8. Suzuki, T.: Fixed point theorem for asymptotic contractions of Meir-Keeler type in complete metric spaces, *Nonlinear Anal.* **64**(2006) 971-978.
9. Wardowski, D.: Fixed points of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.* **2012**(2012) 94-100.
10. Singh, B., Singh, V.: Fixed point theorems for new type of mappings in metric spaces and application, *J. Int. Acad. Phys. Sci.* **25**(2021) 11-24.
11. Berinde, V.: *Iterative approximation of fixed points*, Springer-Verlag, Berlin Heidelberg, 2007.
12. Berinde, V.: Approximation fixed points of weak ϕ-contractions using the Picard iteration, *Fixed point theory*, **4**(2004) 43-53.

13. Bari, CD, Vetro, P.: Common fixed points in generalized metric spaces, *Appl. Math. Comput.* **218**(2012) 7322-7325.
14. Branciari, A.: A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, *Publ. Math. (Debr.)* **57**(2000) 31-37.
15. Cherichi, M, Samet, B.: Fixed point theorems on ordered gauge spaces with applications to nonlinear integral equations, *Fixed Point Theory Appl.* **2012**(2012) 13-32.
16. Das, P.: A fixed point theorem on a class of generalized metric spaces, *Korean J. Math. Sci.* **9**(2002) 29-33.
17. Frigon, M.: Fixed point results for generalized contractions in gauge spaces and applications, *Proc. Am. Math. Soc.* **128** 2957-2965.
18. Jankovic, S, Kadelburg, Z, Radenovic, S.: On cone metric spaces: a survey, *Nonlinear Anal.* **74**(2011) 2591-2601.
19. Khamsi, MA, Kozlowski, WM, Reich, S.: Fixed point theory in modular function spaces, *Nonlinear Anal.* **14**(1990) 935-953.
20. Kirk, WA, Shahzad, N.: Generalized metrics and Caristis theorem, *Fixed Point Theory Appl.* **2013**(2013) 129-138.
21. Lakzian, H, Samet, B.: Fixed points for (ψ, ϕ)-weakly contractive mappings in generalized metric spaces, *Appl. Math. Lett.* **25**(2012) 902-906.
22. Samet, B.: Discussion on A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces by A. Branciari, *Publ. Math. (Debr.)* **76**(2010) 493-494.
23. Sarama, IR, Rao, JM, Rao, SS.: Contractions over generalized metric spaces, *J. Nonlinear Sci. Appl.* **2**(2009) 180-182.
24. Tarafdar, E.: An approach to fixed point theorems on uniform spaces, *Trans. Am. Math. Soc.* **191**(1974) 209-225.
25. Turinici, M.: Functional contractions in local Branciari metric spaces (2012). arXiv:1208.4610v1 [math.GN]
26. Vetro, C.: On Branciaris theorem for weakly compatible mappings, *Appl. Math. Lett.* **23**(2010) 700-705.
27. Nadler, SB Jr.: Multi-valued contraction mappings, *Pac. J. Math.* **30**(1969) 475-488.
28. Berinde, M, Berinde, V.: On a general class of multi-valued weakly Picard mappings, *J. Math. Anal. Appl.* **326**(2007) 772-782.
29. Ciric, L.: Multi-valued nonlinear contraction mappings, *Nonlinear Anal.* **71**(2009) 2716-2723.
30. Markin, JT.: A fixed point theorem for set-valued mappings, *Bull. Am. Math. Soc.* **74**(1968) 639-640.
31. Mizoguchi, N, Takahashi, W.: Fixed point theorems for multi-valued mappings on complete metric spaces, *J. Math. Anal. Appl.* **141**(1989) 177-188.
32. Naidu, SVR.: Fixed point theorems for a broad class of multimaps, *Nonlinear Anal.* **52**(2003) 961-969.
33. Zhang, CK, Zhu, J, Zhao, PH.: An extension of multi-valued contraction mappings and fixed points, *Proc. Am. Math. Soc.* **128**(2000) 2439-2444.

Sanehlata[1]: Research Scholar, School of Basic and Applied Science, K.R. Manglam University, Sohna, Gurugram-122001, Haryana, India
E-mail address: snehboora21@gmail.com

Pooja Vats[2]: Assistant Professor, School of Basic and Applied Science,, K.R. Manglam University, Sohna, Gurugram-122001, Haryana, India
E-mail address: pooja.vats@krmanglam.edu.in

Vizender Singh[3]: Assistant Professor in Mathematics, Department of Mathematics, G.J.U.S.T, Hisar-125001, Haryana, India
E-mail address: vsihag3@gmail.com