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VERTEX COVERING TRANSVERSAL GEODOMATIC NUMBER OF A GRAPH

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Abstract. A geodetic set $S \subseteq V$ in a simple graph $G = (V, E)$, which intersects every minimum vertex covering set (α_0-set), is called a vertex covering transversal geodetic set [7]. The minimum cardinality of a vertex covering transversal geodetic set of G is called the vertex covering transversal geodetic number of G and is denoted by $g_{vct}(G)$ [7]. A partition $(S_1, S_2, ..., S_k)$ of V is called a vertex covering transversal geodomatic partition of G if each S_i is a vertex covering transversal geodetic set in G. The maximum cardinality of a vertex covering transversal geodomatic partition of G is called the vertex covering transversal geodomatic number of G and is denoted by $d_{g_{vct}}(G)$. In this paper, we investigate the parameter known as *the vertex covering transversal geodomatic number* for different types of graphs and examine its structural properties.

1. Introduction

Let $G = (V, E)$ be a finite, undirected, connected graph without loops or multiple edges, as considered throughout this study. The number of vertices in G, denoted by n, is referred to as the *order* of the graph [10, 11]. The *distance* between two vertices u and v in G, denoted by $d(u, v)$, is defined as the length of the shortest path connecting u and v [9, 10, 11].

For a vertex $v \in V(G)$, the *eccentricity* of v, denoted by $e(v)$, is the maximum distance from v to any other vertex in G [10], i.e.,

$$e(v) = \max\{d(v, u) : u \in V(G)\}.$$

The *radius* of the graph G, denoted by $\mathrm{rad}(G)$, is the minimum eccentricity among all vertices in G [10], i.e.,

$$\mathrm{rad}(G) = \min\{e(v) : v \in V(G)\}.$$

Similarly, the *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum eccentricity among all vertices in G [10], given by

$$\mathrm{diam}(G) = \max\{e(v) : v \in V(G)\}.$$

A vertex $v \in V(G)$ is called an *extreme vertex* if the subgraph induced by its neighbors forms a complete subgraph. All these definitions and related concepts are discussed in [10, 11].

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Let S be any non-empty set. An ordered k-tuple $(S_1, S_2, \ldots, S_k)$ is called a *partition* of S if each $S_i \subseteq S$, $S = S_1 \cup S_2 \cup \cdots \cup S_k$, and $S_i \cap S_j = \emptyset$ for all $i \neq j$.

A set $S \subseteq V(G)$ is called a *geodetic set* if every vertex of G lies on a shortest u–v path for some $u, v \in S$. The minimum cardinality among all geodetic sets is called the *geodetic number*, denoted by $g(G)$ [1, 2]. A geodetic set of minimum cardinality is referred to as a *g-set* [1, 2].

A geodetic set $S \subseteq V(G)$ that intersects every minimum vertex covering set (also known as an α_0-set) is called a *vertex covering transversal geodetic set* [7]. The minimum cardinality among all such sets is called the *vertex covering transversal geodetic number* of G, denoted by $g_{vct}(G)$ [7].

Geodetic sets and the geodetic number were introduced and studied in [1, 2]. Further investigations established connections between the geodetic number and the Steiner number of a graph [3]. The *vertex covering transversal domination number* was introduced in [4] and analyzed for regular graphs in [5], while the *vertex covering transversal domatic number* was introduced and studied in [6].

For standard graph-theoretic terminology, we refer to Harary [11].

In this paper, we introduce the parameter *vertex covering transversal geodomatic number* of a graph. We analyze this parameter for several standard graphs and explore its structural properties in detail.

Relevant existing theorems are invoked where appropriate throughout the discussion to support the development of our results.

Theorem 1.1. **[7]** If $K_{m,n}$ is a complete bipartite graph with $m, n \geq 2$, then

$$g_{vct}(K_{m,n}) = \begin{cases} 2 & if \quad m = 2 \text{ and } n > 2 \\ 3 & if \quad m = 2, n = 2 \text{ and if } m = 3\ ,\ n > 3 \\ 4 & if \quad m, n \geq 3 \end{cases}$$

Theorem 1.2. **[7]** If K_n is a complete graph on n vertices, then $g_{vct}(K_n) = n$.

Theorem 1.3. **[7]** If C_{2n+1} is an odd cycle with $n \geq 1$, then $g_{vct}(C_{2n+1}) = 3$.

Theorem 1.4. **[7]** If C_{2n} is a cycle with $n \geq 2$, then

$$g_{vct}(C_{2n}) = \begin{cases} 2 & if \quad n \quad is \quad odd \\ 3 & if \quad n \quad is \quad even \end{cases}$$

Theorem 1.5. **[7]** If $W_{1,n}$ is a wheel graph with $n \geq 3$, then

$$g_{vct}(W_{1,n}) = \begin{cases} 4 & if \quad n = 3 \\ \lceil \frac{n}{2} \rceil & if \quad n \quad is \quad odd \\ \frac{n}{2} + 1 & if \quad n \quad is \quad even \end{cases}$$

Theorem 1.6. **[7]** If Q_n is a hypercube on n vertices with $n \geq 3$, then

$$g_{vct}(Q_n) = \begin{cases} 2 & if \quad n \quad is \quad odd \\ 3 & if \quad n \quad is \quad even \end{cases}$$

Theorem 1.7. **[7]** Let G be a simple connected graph. Then every extreme vertex of G belongs to every vertex covering transversal geodetic set of G. In particular, every end vertex of G is contained in every vertex covering transversal geodetic set of G.

Proposition 1.1. **[8]** If G is a graph of order $n \geq 2$, then $1 \leq d_g(G) \leq \frac{n}{2}$.

Proposition 1.2. **[8]** If G contains a simplical vertex, then $d_g(G) = 1$.

Proposition 1.3. **[8]** If C_n is a cycle of length n, then

$$d_g(C_n) = \begin{cases} \frac{n}{2} & when \quad n \quad is \quad even \\ \lfloor \frac{n}{3} \rfloor & if \quad n \quad is \quad odd \end{cases}$$

2. Definition and Examples

Definition 2.1. Let $G = (V, E)$ be a simple connected graph with at least three vertices. A partition $(S_1, S_2, ..., S_k)$ of V is called a vertex covering transversal geodomatic partition of G if each S_i for $i = 1, 2, ..., k$ is a vertex covering transversal geodetic set in G. The maximum cardinality of a vertex covering transversal geodomatic partition of G is called the vertex covering transversal geodomatic number of G and is denoted by $d_{g_{vct}}(G)$.

Example 2.2. Let G denote the graph illustrated in Figure 1.
S_1=$\{v_1,\ v_3,\ v_6\}$ and $S_2 = \{v_2,\ v_4,\ v_5\}$ are vertex covering transversal geodetic

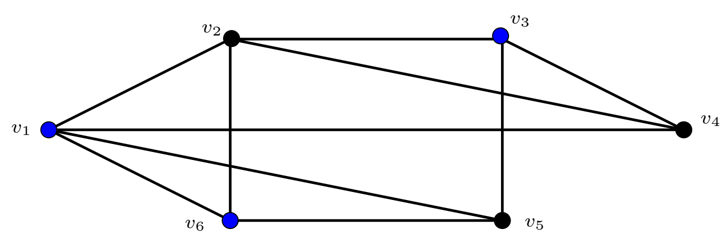

FIGURE 1. Graph G

sets and are also of minimum cardinality in G. That is, S_1 and S_2 are g_{vct}-sets in G. Then $V = (S_1, S_2)$ is a vertex covering transversal geodomatic partition of maximum cardinality in G and so $d_{g_{vct}}(G) = 2$.

Remark 2.3. In general, a vertex covering transversal geodomatic partition $V = (S_1, S_2, ..., S_k)$ is of maximum cardinality if each S_i for $i = 1, 2, ..., k$ is a g_{vct}-set. However, a vertex covering transversal geodomatic partition $V = (S_1, S_2, ..., S_k)$ of maximum cardinality does not necessarily require that all subsets S_i be g_{vct}-sets.This distinction is clearly demonstrated in Example 2.4.

Example 2.4. Consider the graph G presented in Figure 2.

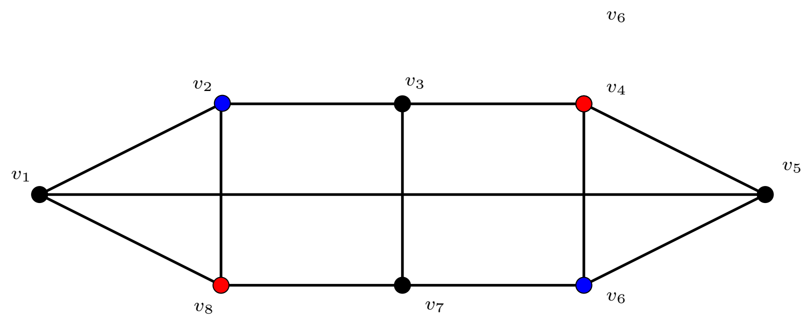

FIGURE 2. Graph G

S_1=$\{v_2, v_6\}$, S_2=$\{v_4, v_8\}$ and S_3=$\{v_1, v_3, v_5, v_7\}$ are vertex covering transversal geodetic sets in G. Of which S_1 and S_2 are g_{vct}-sets and S_3 is a vertex covering transversal geodetic set in G.

Then $V = (S_1, S_2, S_3)$ is a vertex covering transversal geodomatic partition of maximum cardinality in G and so $d_{g_{vct}}(G) = 3$.

Remark 2.5. In a graph $G = (V, E)$, if it is not possible to find disjoint vertex covering transversal geodetic sets in G which are subsets of V whose union is V, then the entire vertex set can be considered as a vertex covering transversal geodetic set and $V = (V)$ itself is the unique vertex covering transversal geodomatic partition in G. Hence in such cases, $d_{g_{vct}}(G) = 1$. This is best illustrated in the following Example 2.6.

Example 2.6. Consider the graph G shown in Figure 3.

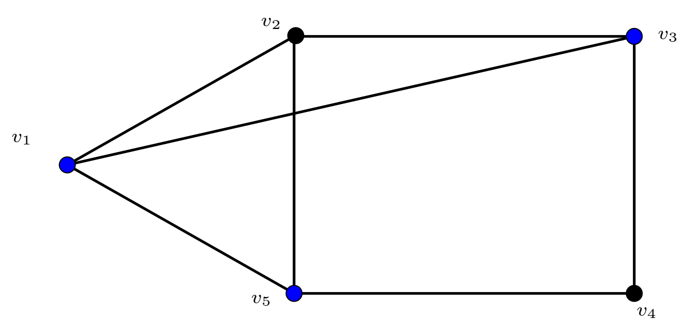

FIGURE 3. Graph G

It is clear that S=$\{v_1, v_3, v_5\}$ is a vertex covering transversal geodetic set in G. But its complement $\{v_2, v_4\}$ consisting of the remaining vertices in $V(G)$ is not a vertex covering transversal geodetic set.

Furthermore, if we consider any other vertex covering transversal geodetic set in G, its complement fails to be a vertex covering transversal geodetic set.
So we can consider the entire vertex set $V = \{v_1,\ v_2,\ v_3,\ v_4,\ v_5\}$ as the vertex covering transversal geodetic set. And so $V = (V)$ itself is the unique vertex covering transversal geodomatic partition in G.
Hence $d_{g_{vct}}(G) = 1$.

Remark 2.7. For the graph G considered in Example 2.6, S=$\{v_3,\ v_5\}$ is a geodetic set of minimum cardinality in G. And its complement $\{v_1,\ v_2,\ v_4\}$ is also a geodetic set in G. So the geodomatic number of G is $d_g(G) = 2$.

3. On the vertex covering transversal geodomatic number of graphs

In this section, we analyze bounds for the vertex covering transversal geodomatic number of a graph G. Additionally, we determine the vertex covering transversal geodomatic number for several standard graph classes.

Theorem 3.1. If $G = (V, E)$ is a graph on n vertices, then $1 \leq d_{g_{vct}}(G) \leq \lfloor \frac{n}{2} \rfloor$.

Proof: In any vertex covering transversal geodomatic partition of G, there exists at least one set V, the entire vertex set which is a vertex covering transversal geodetic set in G. This implies that the vertex covering transversal geodomatic number of G is at least 1. That is, $d_{g_{vct}}(G) \geq 1$.
It is obvious that any vertex covering transversal geodetic set contains at least two vertices. That is, the minimum cardinality of a vertex covering transversal geodetic set is 2. In other words, any g_{vct}-set of G contains at least two vertices. So if a vertex covering transversal geodomatic partition of G contains only g_{vct}-sets of G, then the vertex covering transversal geodomatic number of G is at most $\lfloor \frac{n}{2} \rfloor$.
That is, $d_{g_{vct}}(G) \leq \lfloor \frac{n}{2} \rfloor$.

□

Theorem 3.2. If a graph $G = (V, E)$ has at least one extreme vertex or end vertex, then $d_{g_{vct}}(G) = 1$.

Proof: By Theorem 1.7, each extreme vertex of a graph G belongs to every vertex covering transversal geodetic set of G. Also each end vertex of G belongs to every vertex covering transversal geodetic set of G.
Therefore V cannot be partitioned into disjoint subsets which are all vertex covering transversal geodetic sets in G.
Hence V itself is the unique vertex covering transversal geodomatic partition of G which implies that $d_{g_{vct}}(G) = 1$. □

Corollary 3.3. If T is a tree with $k \geq 2$ end vertices, then $d_{g_{vct}}(T) = 1$.

Corollary 3.4. If P_n (n $\geq$ 2) is a path on n vertices, then $d_{g_{vct}}(P_n) = 1$.

Result 3.5. If G is a star graph on $n+1$ vertices with vertex set $\{u,\ v_1,\ v_2,\ ...,\ v_n\}$, then $d_{g_{vct}}(G)$ is equal to 1.This is because the entire vertex set $V(G)$ forms a vertex covering transversal geodetic set which itself, constitutes a vertex covering transversal geodomatic partition of G.

Remark 3.6. It can be noted that the graph G considered in Result 3.5 is the complete bipartite graph $K_{1,n}$ which is also known as a star.

Theorem 3.7. If $K_{m,n}$ is a complete bipartite graph with $2 \leq m \leq n$, then

$$d_{g_{vct}}(K_{m,n}) = \begin{cases} 1 & if \quad m = 2,3 \quad and \quad n \geq 2 \\ \lfloor \frac{m}{2} \rfloor & if \quad 4 \leq m \leq n \end{cases}$$

Proof: $K_{m,n}$ is a complete bipartite graph with the vertex set V partitioned into V_1 and V_2 where $V_1 = \{u_1,\ u_2,\ ...,\ u_m\}$ and $V_2 = \{v_1,\ v_2,\ ...,\ v_n\}$. Also each vertex of V_1 is joined to every vertex of V_2 by an edge.

Case 1: $m = 2,3$ and $n \geq 2$.

If $m = n = 2$, then $V_1 = \{u_1,\ u_2\}$ and $V_2 = \{v_1,\ v_2\}$. Therefore $V = \{u_1,\ u_2,\ v_1,\ v_2\}$ is the unique vertex covering transversal geodetic set and so $V = (V)$ itself is the unique vertex covering transversal geodomatic partition in $K_{m,n}$.

Similarly, if $m = n = 3$, then $V_1 = \{u_1,\ u_2,\ u_3\}$ and $V_2 = \{v_1,\ v_2,\ v_3\}$. Therefore $V = \{u_1,\ u_2,\ u_3,\ v_1,\ v_2,\ v_3\}$ is the unique vertex covering transversal geodetic set and so $V = (V)$ itself is the unique vertex covering transversal geodomatic partition in $K_{m,n}$.

And if $m = 2,3$ & $n > 3$, then $V_1 = \{u_1,\ u_2\}$ or $\{u_1,\ u_2,\ u_3\}$ and $V_2 = \{v_1,\ v_2,\ ...,\ v_n\}$.

Here V_1 is the unique minimum vertex covering set of $K_{m,n}$ and hence V itself is the unique vertex covering transversal geodetic set of $K_{m,n}$. Therefore, $V = (V)$ itself is the unique vertex covering transversal geodomatic partition in $K_{m,n}$.

Thus $d_{g_{vct}}(K_{m,n}) = 1$ in this case.

Case 2: $4 \leq m \leq n$.

In this case, the vertex covering transversal geodetic number is 4 as it is clear that $S = \{u_i,\ u_j,\ v_k,\ v_l\}$ for any $i, j = 1, 2, ..., m$ & $k, l = 1, 2, ..., n$ and $i \neq j$ & $k \neq l$ is a vertex covering transversal geodetic set of minimum cardinality in $K_{m,n}$.

So if m is even, $V_1 = \{u_1,\ u_2,\ ...,\ u_m\}$ can be partitioned into $\frac{m}{2}$ subsets V_{1i}; $i = 1, 2, ..., \frac{m}{2}$, each containing two vertices from V_1. Similarly, since $n \geq m$, $V_2 = \{v_1,\ v_2,\ ...,\ v_m,\ v_{m+1},\ ...,\ v_n\}$ can be partitioned into $\frac{m}{2}$ subsets of which $\frac{m}{2} - 1$ subsets V_{2k}; $k = 1, 2, ..., \frac{m}{2} - 1$ contain two vertices and one subset $V_{2\frac{m}{2}}$ contains $n - m + 2$ vertices.

Then $S_{ik} = V_{1i} \cup V_{2k}$ for $i = k = 1, 2, ..., \frac{m}{2}$ forms a partition of V and also each S_{ik} for $i = k = 1, 2, ..., \frac{m}{2}$ is a vertex covering transversal geodetic set in $K_{m,n}$.

So $V = (S_{11}, S_{22}, ..., S_{\frac{m}{2}\frac{m}{2}})$ is a vertex covering transversal geodomatic partition of maximum cardinality in $K_{m,n}$.

Hence $d_{g_{vct}}(K_{m,n}) = \frac{m}{2}$ if m is even.

If m is odd, then $V_1 = \{u_1,\ u_2,\ ...,\ u_m\}$ can be partitioned into $\frac{m-1}{2}$ subsets, in which there are $\frac{m-1}{2} - 1$ subsets V_{1i}; $i = 1, 2, ..., \frac{m-1}{2} - 1$, each containing two

vertices from V_1 and one subset $V_{1\frac{m-1}{2}}$ containing three vertices. Similarly, since $n \geq m$, $V_2 = \{v_1,\ v_2,\ ...,\ v_{m-1},\ v_m,\ v_{m+1},\ ...,\ v_n\}$ can be partitioned into $\frac{m-1}{2}$ subsets out of which $\frac{m-1}{2} - 1$ subsets V_{2k}; $k = 1, 2, ..., \frac{m-1}{2} - 1$ containing two vertices and one subset $V_{2\frac{m-1}{2}}$ containing $n - m + 3$ vertices.
Then $S_{ik} = V_{1i} \cup V_{2k}$ for $i = k = 1, 2, ..., \frac{m-1}{2}$ forms a partition of V and also each S_{ik} for $i = k = 1, 2, ..., \frac{m-1}{2}$ is a vertex covering transversal geodetic set in $K_{m,n}$.
So $V = (S_{11}, S_{22}, ..., S_{\frac{m-1}{2}\frac{m-1}{2}})$ is a vertex covering transversal geodomatic partition of maximum cardinality in $K_{m,n}$.
Hence $d_{g_{vct}}(K_{m,n}) = \frac{m-1}{2}$ if m is odd.
Thus $d_{g_{vct}}(K_{m,n}) = \lfloor \frac{m}{2} \rfloor$ in this case. □

Theorem 3.8. If $W_{1,n}$ is a wheel graph with $n \geq 3$, then $d_{g_{vct}}(W_{1,n}) = 1$.

Proof: For any vertex covering transversal geodetic set S in $W_{1,n}$, its complement is not a vertex covering transversal geodetic set.
Hence the entire vertex set of $W_{1,n}$ can be considered as a vertex covering transversal geodetic set and so it forms a vertex covering transversal geodomatic partition also.
Hence $d_{g_{vct}}(W_{1,n}) = 1$.

□

4. Vertex covering transversal geodomatic number of Regular graphs

The vertex covering transversal geodomatic number is determined for standard 3-regular graphs, including the triangular prism graph and the Petersen graph. Furthermore, specific classes of regular graphs such as complete graphs, cycles, and hypercubes are analyzed, and their corresponding vertex covering transversal geodomatic numbers are established.

Example 4.1. Consider the Triangular prism graph G shown in Figure 4 which is 3-regular.

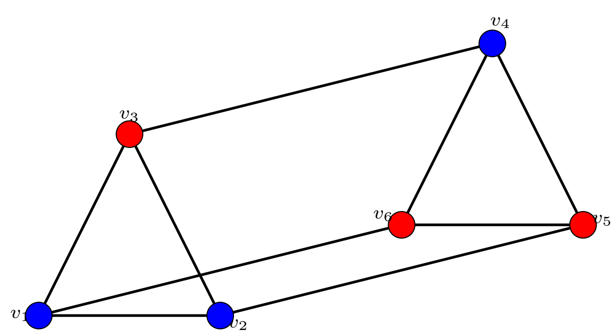

FIGURE 4. Triangular prism graph G

It is clear that S $=\{v_1,\ v_2,\ v_4\}$ is a vertex covering transversal geodetic set of minimum cardinality in G. Now S^c $=\{v_3,\ v_5,\ v_6\}$ is also a vertex covering

transversal geodetic set of minimum cardinality in G.
So $V(G) = (S, S^c)$ is a vertex covering transversal geodomatic partition of maximum cardinality since S and S^c are g_{vct}-sets in G.
Therefore $d_{g_{vct}}(G) = 2$.

Example 4.2. Consider Peterson graph G shown in Figure 5 which is 3-regular.
It is clear that S ={u_1, u_3, u_4, v_2, v_5} is a vertex covering transversal geodetic

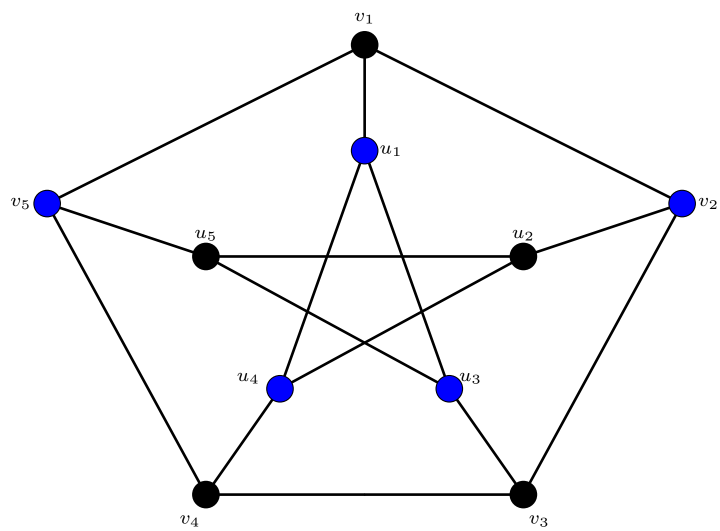

FIGURE 5. Peterson graph G

set of minimum cardinality in G. But its complement is not a vertex covering transversal geodetic set in G.
Similarly, if we consider any other vertex covering transversal geodetic set, its complement fails to be a vertex covering transversal geodetic set in G.
So V itself is the unique vertex covering transversal geodomatic partition of G.
Therefore $d_{g_{vct}}(G) = 1$.

Theorem 4.3. For the complete graph K_n on n vertices, $d_{g_{vct}}(K_n) = 1$.

Proof: Let V be the vertex set of K_n consisting of n vertices.
Since the entire vertex V is the unique vertex covering transversal geodetic set in K_n, it forms the unique vertex covering transversal geodomatic partition of K_n.
Therefore, $d_{g_{vct}}(K_n) = 1$.

□

Theorem 4.4. If C_{2n+1} is an odd cycle with $n \geq 1$, then

$$d_{g_{vct}}(C_{2n+1}) = \begin{cases} 1 & if \quad n = 1, 2 \\ \lfloor \frac{2n+1}{3} \rfloor & if \quad n \geq 3 \end{cases}$$

Proof: Let V ={1, 2, 3, ..., $2n+1$} be the vertex set of the odd cycle C_{2n+1}.
Case 1: $n = 1, 2$
When $n = 1$, we get the cycle graph C_3.
It is obvious that the entire vertex set of C_3 is a vertex covering transversal geodetic set and hence it forms the unique vertex covering transversal geodomatic partition in C_3.
When $n = 2$, we have C_5 and S_i ={i, $i+1$, $i+3$}, $i = 1, 2$ is a vertex covering transversal geodetic set of minimum cardinality in C_5. But its complement is not a vertex covering transversal geodetic set.
Similarly, if we consider any other vertex covering transversal geodetic set in C_5, its complement fails to be a vertex covering transversal geodetic set in C_5. Hence the entire vertex set of C_5 forms the unique vertex covering transversal geodomatic partition in C_5.
Therefore $d_{g_{vct}}(C_{2n+1}) = 1$ if $n = 1, 2$.
Case 2: $n \geq 3$.
For an odd cycle C_{2n+1}, S_i ={i, $(i+1)mod(2n+1)$, $(i+n+1)mod(2n+1)$} is a vertex covering transversal geodetic set of minimum cardinality for each $i = 1, 2, .., 2n+1$.
When $i = 1$, S_1 ={1, 2, $n+2$}
and when $i = 2$, S_2 ={2, 3, $n+3$}.
This shows that $S_1 \cap S_2$={2}$\neq \phi$.
This happens for all successive values of i. That is, $S_i \cap S_{i+1} \neq \phi$ for all $i = 1, 2, ..., 2n$.
So the vertex covering transversal geodetic sets S_i, $i = 1, 2, ..., 2n+1$ do not form a partition for $V(C_{2n+1})$.
Since any g_{vct}-set contains three vertices, there is a possibility of finding $\lfloor \frac{2n+1}{3} \rfloor$ vertex covering transversal geodetic sets in C_{2n+1} for $n \geq 3$.
This case can be further subdivided into three distinct subcases.
Subcase 2.1: $n \cong 0(mod\ 3)$
Here n is a multiple of 3.
So it is possible to find $\frac{2n+1}{3}$ g_{vct}-sets as any g_{vct}-set in C_{2n+1} contains three vertices.
Now let
S_1 ={1, $1+n$, $2+n$}, S_2 ={2, 3, $3+n$},
S_3 ={4, $4+n$, $5+n$}, S_4 ={5, 6, $6+n$},
$\ldots$,
$S_{\frac{2n+1}{3}}$ ={n, $2n$, $2n+1$}.
Then S_1, S_2, S_3, S_4, ..., $S_{\frac{2n+1}{3}}$ are g_{vct}-sets in C_{2n+1} and are disjoint with each other.
Hence $V(C_{2n+1}) = (S_1, S_2, S_3, S_4, ..., S_{\frac{2n+1}{3}})$ is a vertex covering transversal geodomatic partition of maximum cardinality in C_{2n+1}.
Therefore $d_{g_{vct}}(C_{2n+1}) = \frac{2n+1}{3}$ if $n \cong 0(mod\ 3)$.
Subcase 2.2: $n \cong 1(mod\ 3)$
Here n is not a multiple of 3.
So it is possible to find $\lfloor \frac{2n+1}{3} \rfloor$ vertex covering transversal geodetic sets, of which $\lfloor \frac{2n+1}{3} \rfloor - 1$ are g_{vct}-sets containing three vertices and one is a vertex covering

transversal geodetic set containing four vertices from C_{2n+1}.
Now let
S_1 ={1, $1+n$, $2+n$}, S_2 ={2, 3, $3+n$},
S_3 ={4, $4+n$, $5+n$}, S_4 ={5, 6, $6+n$},
$\ldots,$
$S_{\lfloor \frac{2n+1}{3} \rfloor -1}$ ={$n-2$, $2n-2$, $2n-1$},
$S_{\lfloor \frac{2n+1}{3} \rfloor}$ ={$n-1$, n, $2n$, $2n+1$}.
Then S_1, S_2, S_3, S_4, ..., $S_{\lfloor \frac{2n+1}{3} \rfloor -1}$ are g_{vct}-sets in C_{2n+1} and $S_{\lfloor \frac{2n+1}{3} \rfloor}$ is a vertex covering transversal geodetic set in C_{2n+1}. Also they are all disjoint with each other.
Hence $V(C_{2n+1}) = (S_1, S_2, S_3, S_4, ..., S_{\lfloor \frac{2n+1}{3} \rfloor})$ is a vertex covering transversal geodomatic partition of maximum cardinality in C_{2n+1}.
Therefore $d_{g_{vct}}(C_{2n+1}) = \lfloor \frac{2n+1}{3} \rfloor$ if $n \cong 1(mod\ 3)$.
Subcase 2.3: $n \cong 2(mod\ 3)$
Here n is not a multiple of 3.
So it is possible to find $\lfloor \frac{2n+1}{3} \rfloor$ vertex covering transversal geodetic sets, of which $\lfloor \frac{2n+1}{3} \rfloor - 1$ are g_{vct}-sets containing three vertices and one is a vertex covering transversal geodetic set containing five vertices from C_{2n+1}.
Now let
S_1 ={1, $1+n$, $2+n1$}, S_2 ={2, 3, $3+n$},
S_3 ={4, $4+n$, $5+n$}, S_4 ={5, 6, $6+n$},
$\ldots,$
$S_{\lfloor \frac{2n+1}{3} \rfloor -1}$ ={$n-3$, $n-2$, $2n-2$},
$S_{\lfloor \frac{2n+1}{3} \rfloor}$ = {$n-1$, n, $2n-1$, $2n$, $2n+1$}.
Then S_1, S_2, S_3, S_4, ..., $S_{\lfloor \frac{2n+1}{3} \rfloor -1}$ are g_{vct}-sets and $S_{\lfloor \frac{2n+1}{3} \rfloor}$ is a vertex covering transversal geodetic set in C_{2n+1}. Also they are all disjoint with each other.
Hence $V(C_{2n+1}) = (S_1, S_2, S_3, S_4, ..., S_{\lfloor \frac{2n+1}{3} \rfloor})$ is a vertex covering transversal geodomatic partition of maximum cardinality in C_{2n+1}.
Therefore $d_{g_{vct}}(C_{2n+1}) = \lfloor \frac{2n+1}{3} \rfloor$ if $n \cong 2(mod\ 3)$.

Thus $d_{g_{vct}}(C_{2n+1}) = \lfloor \frac{2n+1}{3} \rfloor$ if $n \geq 3$.

□

Theorem 4.5. If C_{2n} is an even cycle with $n \geq 2$, then

$$d_{g_{vct}}(C_{2n}) = \begin{cases} 1 & if \quad n = 2 \\ n & if \quad n \quad is \quad odd \\ \lfloor \frac{2n}{3} \rfloor & if \quad n \quad is \quad even \end{cases}$$

Proof: Let $V(C_{2n})$ = {1, 2, 3, ..., $2n$}.
Obviously, the sets S_1 ={1, 3, 5,..., $2n-1$} and S_2 ={2, 4, 6, ..., $2n$} are the only two α_0-sets of C_{2n}.

Case 1: $n = 2$.
By Theorem 1.4, any g_{vct}-set of C_{2n} contains three vertices.
For the cycle graph C_4, the entire vertex set itself is a vertex covering transversal geodetic set. Consequently, $V(C_4)$ also forms a vertex covering transversal geodomatic partition of maximum cardinality in C_4.
Therefore $d_{g_{vct}}(C_{2n}) = 1$ if $n = 2$.

Case 2: n is odd.
Let S_i =$\{i,\, i+n\}$, $i = 1, 2, ..., n$
Then each S_i is a vertex covering transversal geodetic set of minimum cardinality in C_{2n}.
Also it is clear that $S_i \cap S_j = \phi$ for all $i \neq j$ with $i, j = 1, 2, ..., n$.
Also $V(C_{2n}) = S_1 \cup S_2 ... \cup S_n$.
This shows that $V(C_{2n}) = (S_1, S_2, ..., S_n)$ is a vertex covering transversal geodomatic partition of maximum cardinality in C_{2n}.
Therefore $d_{g_{vct}}(C_{2n}) = n$ if n is odd.

Case 3: n is even.
It is clear that S_i =$\{i,\, (i+1)mod(2n),\, (i+n)mod(2n)\}$, $i = 1, 2, ..., 2n$ is a vertex covering transversal geodetic set of minimum cardinality in C_{2n}.
When $i = 1$, $S_1 = \{1,\, 2,\, n+1\}$
and when $i = 2$, S_2 =$\{2,\, 3,\, n+2\}$.
This shows that $S_1 \cap S_2$=$\{2\} \neq \phi$.
This happens for all successive values of i.
That is, $S_i \cap S_{i+1} \neq \phi$ for all $i = 1, 2, ..., 2n-1$.
So the vertex covering transversal geodetic sets S_i, $i = 1, 2, ..., 2n$ do not form a partition for $V(C_{2n})$.
Since any g_{vct}-set contains three vertices, there is a possibility of finding $\lfloor \frac{2n}{3} \rfloor$ vertex covering transversal geodetic sets in C_{2n} when n is even.
So there are three sub cases in this case.

Subcase 3.1: $n \cong 0(mod\ 3)$
Here n is a multiple of 3.
So it is possible to find $\frac{2n}{3}$ g_{vct}-sets as any g_{vct}-set in C_{2n} contains three vertices.
Now let
S_1 =$\{1,\, 1+n,\, 2+n\}$, $S_2 = \{2,\, 3,\, 3+n\}$,
$S_3 = \{4,\, 4+n,\, 5+n\}$, $S_4 = \{5,\, 6,\, 6+n\}$,
$\ldots$,
$S_{\frac{2n}{3}} = \{n-1,\, n,\, 2n\}$.
Then S_1, S_2, S_3, S_4, ..., $S_{\frac{2n}{3}}$ are g_{vct}-sets in C_{2n} and are disjoint with each other.
Hence $V(C_{2n}) = (S_1, S_2, S_3, S_4, ..., S_{\frac{2n}{3}})$ is a vertex covering transversal geodomatic partition of maximum cardinality in C_{2n}.
Therefore $d_{g_{vct}}(C_{2n}) = \frac{2n}{3}$ if $n \cong 0(mod\ 3)$.

Subcase 3.2: $n \cong 1(mod\ 3)$
Here n is not a multiple of 3.
So it is possible to find $\lfloor \frac{2n}{3} \rfloor$ vertex covering transversal geodetic sets, of which $\lfloor \frac{2n}{3} \rfloor - 1$ are g_{vct}-sets containing three vertices and one is a vertex covering transversal geodetic set containing four vertices from C_{2n}.
Now let
S_1 =$\{1,\, 1+n,\, 2+n\}$, $S_2 = \{2,\, 3,\, 3+n\}$,
S_3 =$\{4,\, 4+n,\, 5+n\}$, S_4 =$\{5,\, 6,\, 6+n\}$,
$\ldots$,
$S_{\lfloor \frac{2n}{3} \rfloor - 1}$ =$\{n-3,\, n-2,\, 2n-2\}$,
$S_{\lfloor \frac{2n}{3} \rfloor}$ =$\{n-1,\, n,\, 2n-1,\, 2n\}$.

Then S_1, S_2, S_3, S_4, ..., $S_{\lfloor \frac{2n}{3} \rfloor - 1}$ are g_{vct}-sets in C_{2n} and $S_{\lfloor \frac{2n}{3} \rfloor}$ is a vertex covering transversal geodetic set in C_{2n}. Also they are all disjoint with each other.
Hence $V(C_{2n}) = (S_1, S_2, S_3, S_4, ..., S_{\lfloor \frac{2n}{3} \rfloor})$ is a vertex covering transversal geodomatic partition of maximum cardinality in C_{2n}.
Therefore $d_{g_{vct}}(C_{2n+1}) = \lfloor \frac{2n}{3} \rfloor$ if $n \cong 1(mod\ 3)$.
Subcase 3.3: $n \cong 2(mod\ 3)$
Here n is not a multiple of 3.
So it is possible to find $\lfloor \frac{2n}{3} \rfloor$ vertex covering transversal geodetic sets, of which $\lfloor \frac{2n}{3} \rfloor - 1$ are g_{vct}-sets containing three vertices and one is a vertex covering transversal geodetic set containing five vertices from C_{2n}.
Now let
$S_1 = \{1,\ 1+n,\ 2+n1\}$, $S_2 = \{2,\ 3,\ 3+n\}$,
$S_3 = \{4,\ 4+n,\ 5+n\}$, $S_4 = \{5,\ 6,\ 6+n\}$,
$\ldots,$
$S_{\lfloor \frac{2n}{3} \rfloor - 1} = \{n-3,\ 2n-3,\ 2n-2\}$,
$S_{\lfloor \frac{2n}{3} \rfloor} = \{n-2,\ n-1,\ n,\ 2n-1,\ 2n\}$.
Then S_1, S_2, S_3, S_4, ..., $S_{\lfloor \frac{2n}{3} \rfloor - 1}$ are g_{vct}-sets and $S_{\lfloor \frac{2n}{3} \rfloor}$ is a vertex covering transversal geodetic set in C_{2n}. Also they are all disjoint with each other.
Hence $V(C_{2n}) = (S_1, S_2, S_3, S_4, ..., S_{\lfloor \frac{2n}{3} \rfloor})$ is a vertex covering transversal geodomatic partition of maximum cardinality in C_{2n}.
Therefore $d_{g_{vct}}(C_{2n}) = \lfloor \frac{2n}{3} \rfloor$ if $n \cong 2(mod\ 3)$.

Thus $d_{g_{vct}}(C_{2n}) = \lfloor \frac{2n}{3} \rfloor$ if n is even.

□

Definition 4.6. Hypercube
For n $\geq$ 2, the hypercube or n-dimensional cube Q_n is defined as the graph containing 2^n vertices whose vertex set is the set of ordered n-tuples of 0's and 1's in which two vertices are adjacent if their ordered n-tuples differ in exactly one position [4].

Theorem 4.7. If Q_n is a hypercube on n vertices with $n \geq 3$, then

$$d_{g_{vct}}(Q_n) = \begin{cases} 2^{n-1} & if \quad n \quad is \quad odd \\ \lfloor \frac{2^n}{3} \rfloor & if \quad n \quad is \quad even \end{cases}$$

Proof: The hypercube Q_n contains 2^n vertices and is n-regular.
Any vertex $v \in Q_n$ is the n-tuple binary number and its complement v^c is also an n-tuple binary number obtained by replacing 0 by 1 and 1 by 0 in v.
Case 1: n is odd.
Obviously, $S = \{v, v^c\}$ where v is any vertex in Q_n is a vertex covering transversal geodetic set of minimum cardinality. That is, S is a g_{vct}-set in Q_n.

Now let $S_i = \{v_i, v_i^c\}$, $i = 1, 2, ..., 2^{n-1}$ for any $v_i \in Q_n$.
Then $S_i \cap S_j = \phi$ for all $i \neq j$; $i, j = 1, 2, ..., 2^{n-1}$ and $V(Q_n) = S_1 \cup S_2 \cup ... \cup S_{2^{n-1}}$.
And each S_i is a g_{vct}-set in Q_n.
Thus $V(Q_n) = (S_1, S_2, ..., S_{2^{n-1}})$ is a vertex covering transversal geodomatic partition of maximum cardinality in Q_n.

$\therefore d_{g_{vct}}(Q_n)= 2^{n-1}$ if n is odd.
Case 2: n is even.
It is obvious that $S = \{u, v, u^c\}$ where u and v are any two adjacent vertices in Q_n is a g_{vct}-set.
Now let $S_i = \{u_i, v_i, u_i^c\}$, $i = 1, 2, ..., \lfloor \frac{2^n}{3} \rfloor - 1$.
Here v_i is chosen to be adjacent to u_i for any i. Also the selection of v_i should be such that $S_i \cap S_j = \phi$ for all $i \neq j$; $i, j = 1, 2, ..., \lfloor \frac{2^n}{3} \rfloor - 1$.
Now let $S_{\lfloor \frac{2^n}{3} \rfloor} = V(Q_n) - (S_1 \cup S_2 \cup ... \cup S_{\lfloor \frac{2^n}{3} \rfloor - 1})$.
Then $V(Q_n) = (S_1, S_2, ..., S_{\lfloor \frac{2^n}{3} \rfloor})$ is a vertex covering transversal geodomatic partition of maximum cardinality in Q_n.
So $d_{g_{vct}}(Q_n)= \lfloor \frac{2^n}{3} \rfloor$ if n is even.

□

5. Relation between $d_g(G)$ and $d_{g_{vct}}(G)$

In this section, we investigate the relationship between the geodomatic number $d_g(G)$ and vertex covering transversal geodomatic number $d_{g_{vct}}(G)$ of a graph G. Based on the results and theorems established in the preceding sections, we formulate necessary and sufficient conditions on G that characterize when $d_g(G) = d_{g_{vct}}(G)$ as well as when $d_g(G) \neq d_{g_{vct}}(G)$.
These conditions provide deeper insight into the structural properties of graphs that influence the interplay between these two parameters.

Proposition 5.1. For any simple connected graph G, $d_g(G) \geq d_{g_{vct}}(G)$.

The following proposition follows immediately from Proposition **[8]1.2** and Theorem 3.2.

Proposition 5.2. If G is a simple connected graph with at least one end vertex or extreme vertex, then $d_g(G) = d_{g_{vct}}(G) = 1$.

Proposition 5.3. If G is a simple connected graph in which every g-set is a g_{vct}-set, then $d_g(G) = d_{g_{vct}}(G)$.

Proposition 5.4. If G is a simple connected graph in which $g(G) \neq g_{vct}(G)$, then $d_g(G) > d_{g_{vct}}(G)$.

6. Scope

This study opens avenues for further analysis of the vertex covering transversal geodomatic number in graph classes not addressed herein, particularly in r-regular graphs. Moreover, a detailed examination of the relationship between the geodomatic number $d_g(G)$ and and the vertex covering transversal geodomatic number $d_{g_{vct}}(G)$ is warranted. Investigating the interplay between these parameters may provide deeper insights into their structural correlations and potential applications.

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