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THE TOTAL GRAPH OF PATH RELATED GRAPHS AND THEIR EXTENSION

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Abstract. In the realm of abstract algebra, field extensions stand out as primary subjects of exploration within field theory. The fundamental concept revolves around initiating from a foundational field and then, through certain methods, constructing a broader field encompassing the base field while adhering to supplementary properties. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, the total graph $T(G)$ of G has vertex set $V(G) \cup E(G)$ and two vertices in $T(G)$ are adjacent if and only if they are adjacent or incident in G. In this paper, we try to study the structural properties of the total graph of path related graphs and their extensions.

1. Introduction

The problem of graph coloring have become a subject of great interest because of its diverse theoretical results and its numerous applications. The total coloring was introduced by M. Behzad [2]. In 1960s M. Behzad and Vizing independently conjectured that $\chi''(G) \leq \Delta(G) + 2$. where $\chi''(G)$ be the total chromatic number. In 1965 M. Behzad introduced a new graph called total graph. The total graph of a simple graph $G = (V, E)$ is a graph for which $V(T(G)) = V \cup E$ and such that two distinct vertices in $T(G)$ are adjacent if and only if they are adjacent vertices of G or adjacent edges of G or they are incident vertex and edge of G. Now, the total coloring of G can be simply viewed as the vertex coloring of $T(G)$.

In 1970, M. Behzad [3] characterize tatal graphs in terms of special points. In 1969 M. Behzad and Heydar Radjavi [4] gives the stricture of regular total graphs. In 2020 T.B. Athul and G.Suresh Singh [1] gave some characterization of total graphs of regular graphs.

In 2016,Suresh Singh G and Sunitha Grace Zacharia [6] introduce the concept of graph extension and they give some characterizations. Let G be a simple (p, q) graph. *Extension on* G is defined as the following; in the first extension, add one edge with G, denoted as G^1, $G^1 = G \cup \{e_1\}$. In the second extension add two edges with G^1 denoted by G^2, $G^2 = G \cup \{e_1, e_2, e_3\}$ and so on until no such an extension remain. If $G^n \cong K_p$, for some n, then G is said to be a *completely extendable* graph and n is known as the order of extension.

Consider the given graph G,

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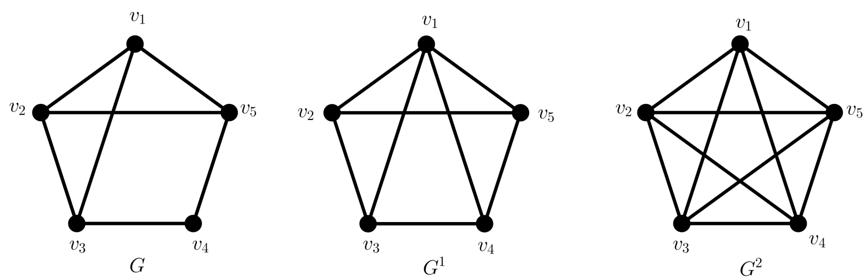


FIGURE 1. Graph G and its extentions

G^1 is obtained by adding one edge (v_1, v_4) with G. And G^2 is obtained by adding two edges (v_2, v_4) and (v_3, v_5) with G^1 and $G^2 \cong K_5$. Therefore G is completely extendable and the order of extension is 2.

Theorem 1.1. [6] *Let G be a (p, q) graph and let $G^k = G \cup \{e_1, e_2, \ldots, e_m\}$. If G^k is the extension of G, then*

$$m = \frac{k(k+1)}{2}$$

.

Let G be a (p, q) graph which is not completely extendable. Let r be the maximum possible number of extension in G such that G^r is not complete. Then the *deficiency number* of G is defined as the number of edges required to make G^r a complete graph. Consider the following graph G,

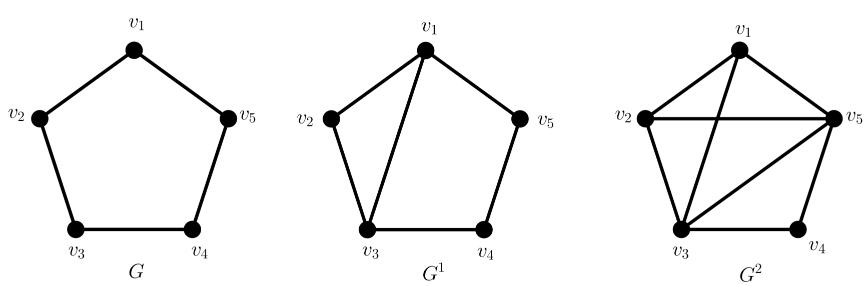


FIGURE 2

G^1 and G^2 are the possible extensions o f G but G^2 is not complete. Number of edges required to make G^2 as a complete graph is 2. Therefore 2 is the deficiency number of G.

In this paper we try to study the structural properties of the total graphs of path related graphs and their extensions. Throughout this paper we consider simple undirected graphs. For preliminary definitions and results we refer[5].

2. Total Graph of Path Related Graphs

In this section study some properties of total graph of path related graphs.

2.1. Total Graph of P_n. In this section we discuss the properties of $T(P_n)$ and its chromatic number. We have P_n has n vertices and $n-1$ edges. So $T(P_n)$ has $2n-1$ vertices. Since the line graph of P_n is a path graph with $n-1$ edges, so the number of edges in $T(P_n)$ is $4n-5$.

Example 2.1.

FIGURE 3. P_4 and its total graph $T(P_4)$

Theorem 2.2. *For $n > 2$ $T(P_n)$ can be decomposed in to $2n-3$ copies of K_3.*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of P_n and let $e_i = v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$. Then in $T(P_n)$, for each $i = 1, 2, \ldots, n-1$, $v_i e_i v_{i+1}$ forms a K_3. Hence there are $n-1$ K_3's are formed. For each $i = 1, 2, \ldots, n-2$, $e_i v_{i+1} e_{i+1}$ forms $n-3$ triangles in $T(P_n)$. So, there are $2n-3$ copies of K_3 in $T(P_n)$. □

Theorem 2.3. *For $n > 2$ $T(P_n)$ is planar.*

Proof. $T(P_n)$ don't contain $K_{3,3}$ or K_5 as an induced subgraph. □

Theorem 2.4. *For $n \geq 2$, $\omega(T(P_n)) = 3$.*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of P_n and let $e_i = v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$. Then $v_i e_i v_{i+1}$ forms a K_3 in $T(P_n)$. if we add any one vertex in $T(P_n)$ couldn't gives a K_4. Hence $\omega(T(P_n)) = 3$. □

Theorem 2.5. *$T(P_n)$ is 3-chromatic..*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of P_n and let $e_i = v_i v_{i+1}$ for $i = 1, 2, \ldots, n-1$.

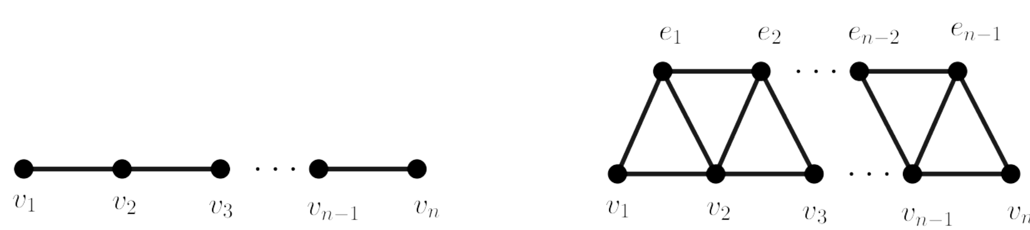


FIGURE 4. P_n and $T(P_n)$

Let u_1, u_2, u_3 be the vertices of K_3. Define a function $\phi : V(T(P_n)) \to V(K_3)$ by,

$$\phi(v_i) = \begin{cases} u_1 & \text{if } i \equiv 1(\mathrm{mod}\, 3) \\ u_2 & \text{if } i \equiv 2(\mathrm{mod}\, 3) \\ u_3 & \text{if } i \equiv 0(\mathrm{mod}\, 3) \end{cases}$$

and

$$\phi(e_i) = \begin{cases} u_1 & \text{if } i \equiv 1(\mathrm{mod}\, 3) \\ u_2 & \text{if } i \equiv 2(\mathrm{mod}\, 3) \\ u_3 & \text{if } i \equiv 0(\mathrm{mod}\, 3) \end{cases}$$

Then ϕ is a homomorphism and hence $T(P_n)$ is 3-chromatic. □

Corollary 2.6. *The total cromatic number of* $T(P_n)$ *is* 3.

Theorem 2.7. *For* $T(P_n)$, $\delta(T(P_n)) = \kappa(T(P_n)) = \lambda(T(P_n))$.

Proof. The removal of v_2 and v_3 will disconnect $T(P_n)$. Similarly, the removal of the edges v_1v_2 and v_1e_1 will disconnect $T(P_n)$.

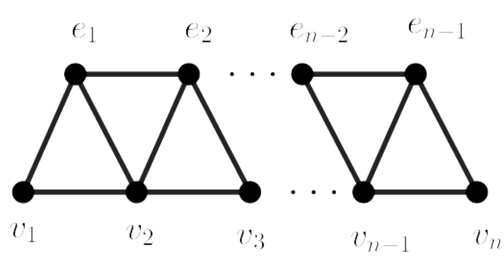


FIGURE 5. $T(P_n)$

Clearly, $\delta(T(P_n)) = 2$. □

2.2. Total Graph of Star Graph. The star graph $K_{1,n}$ has $n + 1$ vertices and n edges. So, $T(K_{1,n})$ has $2n + 1$ vertices.

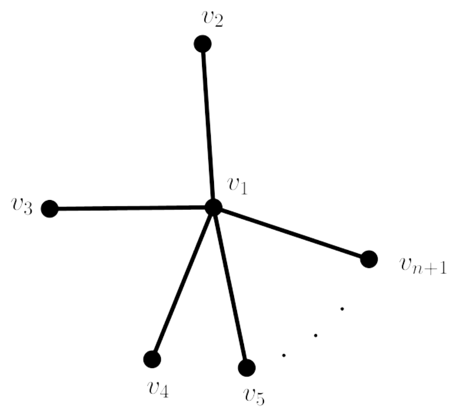


FIGURE 6. $K_{1,n}$

Since all the edges in $K_{1,n}$ are adjacent, $L(K_{1,n})$ has $\dfrac{n(n-1}{2}$ edges. Let v_1 be the maximum degree vertex in $K_{1,n}$, then v_1 is adjacent to every vertex of $L(K_{1,n})$ in $T(K_{1,n})$. This will contribute n edges in $T(K_{1,n})$. All other vertices is adjacent to exactly one vertex in $L(K_{1,n})$, so this will contribute n edges to $T(K_{1,n})$. Hence

$$\begin{aligned}\text{Total number of edges in } T(K_{1,n}) &= \frac{n(n-1}{2} + 3n \\ &= \frac{n^2 - n + 6n}{2} \\ &= \frac{n(n+5)}{2}\end{aligned}$$

,

In $T(K_{1,n})$, $\Delta = 2n$ and $\delta = 2$. The maximum degree vertex in $T(K_{1,n})$ is the vertex v_1.

Example 2.8.

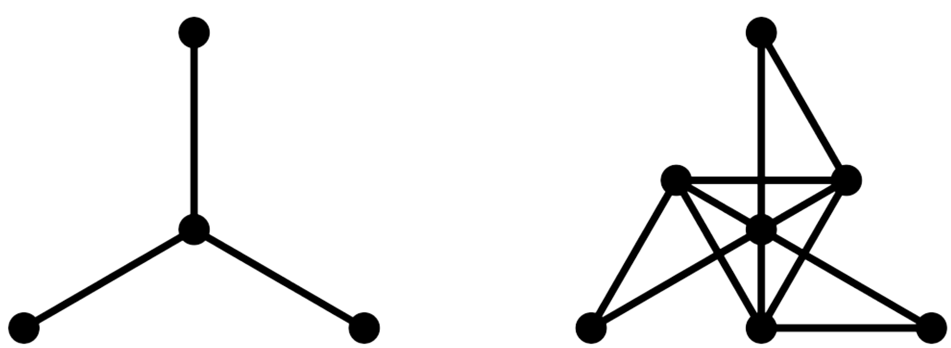
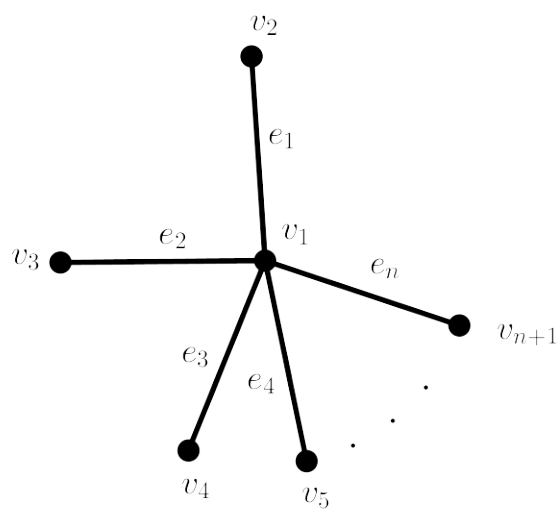


FIGURE 7. $K_{1,3}$ and $T(K_{1,3})$

Theorem 2.9. $\omega(K_{1,n}) = n + 1$

Proof. Let $\{v_1, v_2, \cdots, v_{n+1}\}$ be the vertices and $\{e_1, e_2, \ldots, e_n\}$ be the edges of $K_{1,n}$ with $\deg v_1 = n$.



Since every edges of $K_{1,n}$ are adjacent, so the induced subgraph of $\{e_1, e_2, \ldots, e_n\}$ in $T(K_{1,n})$ is a complete graph on n vertices. Hence $\omega(T(K_{1,n}) \geq n$. Since degv_i in $T(K_{1,n})$ is 2 for each $i = 1, 2, \ldots, n$, any of these vertices together with the induced graph of $\{e_1, e_2, \ldots, e_n\}$ does not form a complete graph. But v_1 is adjacent to each e_i in $T(K_{1,n})$ and the induced subgraph of $\{e_1, e_2, \ldots, e_n, v_1\}$ in $T(K_{1,n}))$ is a complete graph. Hence, $\omega(K_{1,n}) = n + 1$ □

3. Total Graphs and Complete Extension

In this section we discuss the complete extension of total graphs of some graph classes.

Theorem 3.1. *$T(p_n)$ is completely extendable and the order of extension is $2n-4$.*

Proof. We have $T(p_n)$ has $2n - 1$ vertices and $4n - 5$ edges. Suppose $T(p_n)^k \cong K_{2n-1}$, then,
Number of edges added with $T(p_n)$ to get K_{2n-1}

$$\begin{aligned}
&= \frac{(2n-1)(2n-2)}{2} - (4n-5) \\
&= (2n-1)(n-1) - (4n-5) \\
&= 2n^2 - 7n + 6 \\
&= (2n-3)(n-2) \\
&= \frac{(2n-3)(2n-4)}{2}
\end{aligned}$$

Hence by Theorem 1.1, $T(p_n)$ is completely extendable and the order of extension is $2n-4$. □

Theorem 3.2. *$T(C_n)$ is completely extendable if and only if $n=3$*

Proof. If $n=3$, Then the number of vertices of $T(C_3)$ is 6 and number of edges is 12.

$$\begin{aligned}\text{Number of edges added with } T(C_3) \text{ to get } K_6 &= 15-12\\ &= 3\\ &= \frac{3\times 4}{2}\end{aligned}$$

Hence $T(C_3)$ is completely extendable.

For $n>3$, $T(C_n)$ has $2n$ vertices and $4n$ edges.
Number of edges added with $T(C_n)$ to get K_{2n}

$$\begin{aligned}&= \frac{2n(2n-1)}{2}-4n\\ &= \frac{2n(2n-1)-8n}{2}\\ &= \frac{2n(2n-1-4)}{2}\\ &= \frac{2n(2n-5)}{2}\end{aligned}$$

By Theorem 1.1, $T(C_n)$ is not completely extendable. □

Theorem 3.3. *For $n>2$, $T(K_{1,n})$ is not completely extendable.*

Proof. The number of vertices and edges of $T(K_{1,n})$ is $2n+1$ and $2n+\dfrac{n(n+1)}{2}$.
Number of edges added with $T(K_{1,n})$ to get K_{2n+1}

$$\begin{aligned}&= \frac{(2n+1)2n}{2}-\left(2n+\frac{n(n+1)}{2}\right)\\ &= \frac{(2n+1)2n-4n-n(n+1)}{2}\\ &= \frac{4n^2+2n-4n-n^2-n}{2}\\ &= \frac{3n^2-3n}{2}\\ &= \frac{3n(n-1)}{2}\end{aligned}$$

Hence by Theorem 1.1, $T(K_{1,n})$ is not completely extendable. □

Theorem 3.4. *$T(W_n)$ is not completely extendable.*

Proof. The number of vertices and edges of $T(W_n)$ is $3n-2$ and $\dfrac{(n+16)(n-1)}{2}$.

Number of edges added with $T(W_n)$ to get K_{3n-2}

$$\begin{aligned}
&= \frac{(3n-2)(3n-3)}{2} - \frac{(n+16)(n-1)}{2} \\
&= \frac{n-1}{2}\left[3(3n-2) - (n+16)\right] \\
&= \frac{n-1}{2}\left[9n-6-n-16\right] \\
&== \frac{n-1}{2}\left[8n-22\right]
\end{aligned}$$

Hence by Theorem 1.1, $T(W_n)$ is not completely extendable. □

Theorem 3.5. *$T(K_n)$ is completely extendable and the order of the extension is $\frac{n(n-1)}{2}$.*

Proof. $T(K_n)$ has $\frac{n(n+1)}{2}$ vertices and $\frac{n(n+1)(n-1)}{2}$.

Number of edges added with $T(K_n)$ to get $K_{\frac{n(n+1)}{2}}$

$$\begin{aligned}
&= \frac{\left(\frac{n(n+1)}{2}\right)\left[\frac{n(n+1)}{2} - 1\right]}{2} - \frac{n(n+1)(n-1)}{2} \\
&= \frac{n(n+1)}{2}\left[\frac{n^2+n-2-4n+4}{4}\right] \\
&= \frac{n(n+1)}{2}\left[\frac{n^2-3n+2}{4}\right] \\
&= \frac{n(n+1)(n-1)(n-2)}{8} \\
&= \frac{\frac{n(n-1)}{2} - \left[\frac{n(n-1)}{2} - 1\right]}{2}
\end{aligned}$$

Hence by Theorem 1.1, $T(K_n)$ is completely extendable. and the order of the extension is $\frac{n(n-1)}{2}$. □

Theorem 3.6. *The total graph of double star graph $K_{1,n,n}$ is completely extendable if and only if $n = 1$ or 2.*

Proof. $T(K_{1,n,n})$ has $4n+1$ vertices and $\frac{n(n+13)}{2}$ edges.

Number of edges added with $T(K_{1,n,n})$ to get K_{4n+1}

$$\begin{aligned} &= \frac{(4n+1)4n}{2} - \frac{n(n+13)}{2} \\ &= \frac{n}{2}\left[16n+4-n-13\right] \\ &= \frac{n}{2}\left[15n-9\right] \\ &= \frac{3n(5n-3)}{2} \end{aligned}$$

If $T(K_{1,n,n})$ is completely extendable then $3n(5n-3)$ is a product of two consecutive positive integers. this gives either $3n+1 = 5n-3$ or $3n = 5n-3$. On solving these, we get either $n = 1$ or $n = 2$. Hence the proof. □

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References

1. Athul, T. B. and G. Suresh Singh, *Total Graph of Regular Graphs*, Advances in Mathematics: Scientific Journal, **9(6)** (2020),4213-4220. https://doi.org/10.37418/amsj.9.6.103.
2. M. Behzad, *Graphs and and their chromatic numbers*, Ph.D. thesis, Michigan State University, 1965.
3. M. Behzad, *Characterization of Total Graphs*, Proceedings of American Mathematical Society (1970), 383-389.
4. M. Behzad and H. Radjavi, *Structure of Regular Total Graphs*, Journal of London Mathematical Society, **44** (1969), 433-436.
5. Suresh Singh G, *Graph Theory*, PHI Learning Private Limited, 2010.
6. Suresh Singh G and Sunitha Grace Zacharia, *Some Resuts on Completely Extendable ans Weighted Extendable Graphs*, International Journal of Mathematical Archieve, **7(9)** (2016), 1-6.

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