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MODIFIED FUZZY ALGEBRA AND FUZZY DOT i^*-IDEAL OF i^*-ALGEBRA

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Abstract. In this paper, introduced and investigated the concepts of fuzzy dot subalgebras and fuzzy dot i*-ideals within the framework of i*-algebras, a generalization of implicative algebraic structures. The notion of a fuzzy dot set extends traditional fuzzy set theory by incorporating a dot operation that reflects the algebraic operation of the underlying i^*-algebra. Also, define fuzzy dot subalgebras and fuzzy dot i*-ideals, explore their structural properties, and establish necessary and sufficient conditions for a fuzzy set to qualify as such. Relationships between crisp substructures and their fuzzy counterparts are examined, and illustrative examples are provided to demonstrate the applicability of the proposed definitions. Our findings contribute to the broader study of algebraic systems under uncertainty and enhance the interface between fuzzy logic and nonclassical algebraic structures.

1. Introduction

The study of algebraic structures under the influence of uncertainty has gained significant momentum in recent years, particularly with the integration of fuzzy set theory into classical algebra. Fuzzy sets, first introduced by Zadeh in 1965, provide a framework for modeling imprecise or vague information, which naturally arises in real-world applications. The fusion of fuzzy logic with algebraic systems has led to the development of various fuzzy algebraic structures, including fuzzy groups, fuzzy rings, and fuzzy ideals.

Introduction

Fuzzy set theory, introduced by Zadeh in 1965, has emerged as a powerful mathematical framework for handling uncertainty and imprecision inherent in real-world problems. Over the decades, the integration of fuzzy logic into algebraic structures has led to the development of fuzzy versions of various algebraic systems, including groups, rings, lattices, and algebras. Among these, the study of fuzzy ideals and subalgebras has gained significant attention due to its applications in logic, computer science, and approximate reasoning.

The concept of i^*-algebras, which generalizes various non-classical logical algebraic systems, provides a rich ground for exploring fuzzy structures. In this context, the introduction of *fuzzy dot subalgebras* and *fuzzy dot i^*-ideals* brings

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forth a nuanced framework where elements possess degrees of membership rather than absolute inclusion. These fuzzy dot structures extend the classical notions by incorporating an additional degree of flexibility through the dot operation, thereby offering more refined control over algebraic inclusion under uncertainty.

This paper aims to define and study the properties of fuzzy dot subalgebras and fuzzy dot i^*-ideals in i^*-algebras. We investigate the relationships between these structures, provide illustrative examples, and explore various algebraic properties and characterizations. Our work lays a foundation for further research in fuzzy algebraic systems and opens up potential applications in soft computing, logic programming, and knowledge-based systems.

i*-algebras are a generalization of certain logical and algebraic systems characterized by a binary operation and a unary operation that satisfy specific axioms. They have been used to model a wide range of logical and computational phenomena, particularly in the context of non-classical logics. The investigation of fuzzy substructures in i*-algebras opens new avenues for understanding how uncertainty can be formally embedded within such logical frameworks.

Recently, the notion of a dot operation has been introduced in the study of fuzzy algebraic systems to reflect specific algebraic behaviors in a fuzzy setting. This operation, when applied in the context of fuzzy subsets, helps define more nuanced structures such as fuzzy dot subalgebras and fuzzy dot ideals, offering finer tools for algebraic and logical analysis.

In this paper, to propose and examine the concepts of fuzzy dot subalgebras and fuzzy dot i-ideals in the context of i^*-algebras. Our aim to formalize these notions by defining appropriate conditions under which a fuzzy set over an i-algebra can be considered a fuzzy dot subalgebra or a fuzzy dot i^*-ideal. Also discussed their basic properties, explore their interrelations, and provide illustrative examples to highlight the practicality and significance of these concepts.

The structure of this paper is as follows: In Section 2, present the preliminary definitions and foundational concepts related to i^*-algebras and fuzzy set theory. Section 3 introduces the fuzzy dot subalgebra and discusses its core properties. Section 4 focuses on fuzzy dot i*-ideals and their characterization with examples and discusses possible extensions. Finally conclude in Section 5 with a summary.

2. Preliminaries

In this section, we recall some fundamental concepts related to fuzzy sets and i^*-algebras, which are essential for understanding the framework of fuzzy dot subalgebras and fuzzy dot i^*-ideals.

Definition 2.1. Fuzzy Set

Let X be a non-empty set. A *fuzzy set* A in X is characterized by a membership function $\mu_A : X \to [0, 1]$, where $\mu_A(x)$ denotes the degree of membership of the element $x \in X$ in the fuzzy set A.

Definition 2.2. i^*-Algebra

An algebraic structure $(A, \cdot, \to, 0)$ is called an *i^*-algebra* if it satisfies the following conditions for all $x, y, z \in A$:

(i) $(A, \cdot)$ is a semigroup;

(ii) $x \cdot 0 = 0 = 0 \cdot x$;
(iii) $x \cdot (x \to y) \leq y$;
(iv) $x \leq y$ implies $z \cdot x \leq z \cdot y$;
(v) $x \leq y$ implies $y \to z \leq x \to z$;

where $\leq$ is a partial order on A defined by $x \leq y \iff x \to y = 1$, for some designated top element $1 \in A$.

Definition 2.3. Fuzzy Subalgebra

Let A be an algebra. A fuzzy subset $\mu : A \to [0,1]$ is called a *fuzzy subalgebra* if for all $x, y \in A$,

$$\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}.$$

Definition 2.4. Fuzzy Dot Subalgebra

Let A be an i^*-algebra and $\mu : A \to [0,1]$ be a fuzzy subset. Then μ is called a *fuzzy dot subalgebra* of A if for all $x, y \in A$,

$$\mu(x \cdot y) \geq \mu(x) \cdot \mu(y),$$

where the dot product refers to the standard multiplication in the unit interval $[0, 1]$.

Definition 2.5. Fuzzy Dot i^*-Ideal A fuzzy subset $\mu : A \to [0,1]$ of an i^*-algebra A is said to be a *fuzzy dot i^*-ideal* if for all $x, y \in A$,

$$\mu(x \cdot y) \geq \mu(y) \cdot \mu(x), \quad \text{and} \quad x \leq y \Rightarrow \mu(x) \geq \mu(y).$$

Definition 2.6. Cartesian product of fuzzy sets

Let λ and μ be the fuzzy sets in a set X. The Cartesian product $\lambda \times \mu : X \times X \to [0,1]$ is defined by $(\lambda \times \mu)(x, y) = \lambda(x) \cdot \mu(y)$, for all $x, y \in X$.

Example 2.7.

Define a fuzzy set λ in X $= \{0, 1, 2\}$ by $\lambda(0) = 0.1, \lambda(1) = 0.2, \lambda(2) = 0.5$. Define a fuzzy set μ in X by $\mu(0) = 1, \mu(1) = 0.5, \mu(2) = 0.2$.

Then the Cartesian product $\lambda \times \mu : X \times X \to [0,1]$ is defined as $(\lambda \times \mu)(0,1) = \lambda(0) \cdot \mu(1) = (0.1).(0.5) = 0.05$. Similarly we can get values for other elements in $\text{X} \times \text{X}$.

Definition 2.8. Union of fuzzy subsets

For any fuzzy subsets μ and ν of a set X, we define $(\mu \cup \nu)(x) = \max\{\mu(x), \nu(x)\}$ for all $x \in$ X.

Example 2.9.

Define a fuzzy set μ in X $= \{0, 1, 2\}$ by $\mu(0) = 0.1, \mu(1) = 0.2, \mu(2) = 0.5$.
Define a fuzzy set v in X by $v(0) = 1, v(1) = 0.5, v(2) = 0.2$.

$$\Rightarrow (\mu \cup v)(0) = \max\{\mu(0), v(0)\} = \max\{0.1, 1\} = 1.$$

Similarly we can definitely for other elements in X .

These preliminaries and examples serve as the foundation for developing the main results concerning fuzzy dot subalgebras and fuzzy dot i^*-ideals, which are discussed in the subsequent sections.

3. Fuzzy dot subalgebra of i^*-algebra

Definition 3.1. **i^*-algebra**
A i^*-algebra is an algebra ($X, *, 0$) of type (2,0) satisfying the following conditions.
(i) $x * x = 1, x \neq 0 \in X$
(ii) $0 * x = 0, x \neq 0 \in X$
(iii) $x * y = 1$ and $y * x = 1$
$\Rightarrow x = y$ for all $x \neq 0, y \neq 0 \in X$.

Example 3.2.
Let $X = \{0, a, b,\}$. Let us Consider the following Cayley table

Table 1: The system ($X, ^*, 0$) is a i^*-algebra

$*$	0	a	b
0	1	0	0
a	0	1	0
b	0	1	1

Clearly this table shows that, the system ($X, ^*, 0$) is a i^*-algebra.

Definition 3.3. **Subalgebra of I^*-algebra**
Let S be a nonempty subset of a i^*-algebra X . Then S is called a subalgebra of X if $x * y \in S$, for all $x, y \in S$.

Example 3.4.
Consider a i^*-algebra $X = \{0, 1, 2\}$ having the following Cayley table.

Table 2: $S = \{0, 1\}$ is a subalgebra of a i^*-algebra

$*$	0	1	2
0	1	0	0
1	0	1	2
2	0	1	1

Clearly the table 2 shows that $S = \{0, 1\}$ is a subalgebra of a i^*-algebra.

Definition 3.5. **Fuzzy dot sub algebra of a i^* algebra**
A fuzzy subset μ of X is called fuzzy dot sub algebra of a i^*-algebra X if $\mu\,(x^*y) \leq \mu(x) \cdot \mu(y)$ for all $x, y, x = y \neq 1 \in X$.

Example 3.6.
Consider a i^*-algebra $X = \{0, 1, 2\}$ having the following Cayley table:

Table 3: Cayley Table

$*$	0	1	2
0	1	0	0
1	0	1	1
2	0	0	1

Define a fuzzy set μ in X by $\mu(0) = 1, \mu(1) = 0.2, \mu(2) = 1$. It is easy to verify that μ is a fuzzy dot subalgebra of a i^*-algebra.

Theorem 3.7.

If μ is a fuzzy dot subalgebra of a i^*-algebra X, then we have $\mu(1) \leq (\mu(x))^2$ for all $x \neq 1 \in X$.

Proof.

For every $x \neq 1 \in X$,
we have, $\mu(1) = \mu(x * x) \leq \mu(x) \cdot \mu(x) = (\mu(x))^2$
This completing the Proof: □

Theorem 3.8.

If μ and v are fuzzy dot subalgebra of a i^*-algebra X , then so is $\mu\mathrm{U}v$.

Proof.

Let $\mathrm{x}, \mathrm{y} \in \mathrm{X}$.
Then $(\mu U v)(x * y) = \max\{\mu(x * y), v(x * y)\}$

$$\begin{aligned} &\leq \max\{\mu(\mathrm{x}) \cdot \mu(\mathrm{y}), v(\mathrm{x}) \cdot v(\mathrm{y})\} \\ &\leq (\max\{\mu(\mathrm{x}), v(\mathrm{x})\}) \cdot (\max\{\mu(\mathrm{y}), v(\mathrm{y})\}) \\ &= ((\mu\mathrm{U}v)(\mathrm{x})) \cdot ((\mu\mathrm{U}v)(\mathrm{y})). \end{aligned}$$

Hence $\mu U v$ is a fuzzy dot subalgebra of a i^*-algebra X.

□

Theorem 3.9.

If λ and μ are fuzzy dot subalgebras of ai-algebra X, then $\lambda \times \mu$ is a fuzzy dot subalgebra of* $\mathrm{X} \times \mathrm{X}$.

Proof.

For any $x_1, x_2, y_1, y_2 \in X$,

$$\begin{aligned} (\lambda \times \mu)\left((x_1, y_1) * (x_2, y_2)\right) &= (\lambda \times \mu)\left(x_1 * x_2, y_1 * y_2\right) \\ &= \lambda\left(x_1 * x_2\right) \cdot \mu\left(y_1 * y_2\right) \\ &\leq ((\lambda(x_1) \cdot \lambda(x_2)) \cdot ((\mu\left(y_1\right) \cdot \mu\left(y_2\right)) \\ &= ((\lambda\left(x_1\right) \cdot \mu\left(y_1\right)) \cdot (\lambda\left(x_2\right) \cdot \mu\left(y_2\right)) \\ &= (\lambda \times \mu)\left(x_1, y_1\right) \cdot (\lambda \times \mu)\left(x_2, y_2\right), \end{aligned}$$

This completing the Proof. □

Definition 3.10. strongest fuzzy σ-relation on i^*-algebra

The strongest fuzzy σ-relation on i^*-algebra X is the fuzzy subset μ_σ of $\mathrm{X} \times \mathrm{X}$ given by $\mu_\sigma(\mathrm{x}, \mathrm{y}) = \sigma(\mathrm{x}) \cdot \sigma(\mathrm{y})$ for all $\mathrm{x}, \mathrm{y} \in \mathrm{X}$.

Definition 3.11. fuzzy σ-product relation

A fuzzy relation μ on i^*-algebra X is called a fuzzy σ-product relation if $\mu(\mathrm{x}, \mathrm{y}) \leq \sigma(\mathrm{x}) \cdot \sigma(\mathrm{y})$ for all $\mathrm{x}, \mathrm{y} \in \mathrm{X}$.

Definition 3.12. left fuzzy relation

A fuzzy relation μ on i^*-algebra X is called a left fuzzy relation on σ if $\mu(\mathrm{x}, \mathrm{y}) = \sigma(\mathrm{x})$ for all $\mathrm{x}, \mathrm{y} \in \mathrm{X}$. Note that a left fuzzy relation on σ is a fuzzy σ-product relation.

Theorem 3.13.

Let μ_σ be the strongest fuzzy σ-relation on i^*-algebra X , where σ is a fuzzy subset of a i^* algebra X . If σ is a fuzzy dot subalgebra of i^* algebra X , then μ_σ is a fuzzy dot subalgebra of $X \times \mathrm{X}$.

Proof.

Suppose that σ is fuzzy dot subalgebra of X .
For anyx $\mathrm{x}_1, \mathrm{x}_2, \mathrm{y}_1, \mathrm{y}_2 \in \mathrm{X}$, we have,

$$\begin{aligned}\mu_\sigma\left(\left(x_1, y_1\right) * \left(x_2, y_2\right)\right) &= \mu_\sigma\left(x_1 * x_2, y_1 * y_2\right) \\ &= \sigma\left(x_1 * x_2\right) \cdot \sigma\left(y_1 y_2\right) \\ &\leq \left(\sigma\left(x_1\right) \cdot \sigma\left(x_2\right)\right) \cdot \left(\sigma\left(y_1\right) \cdot \sigma\left(y_2\right)\right) \\ &= \left(\sigma\left(x_1\right) \cdot \sigma\left(y_1\right)\right) \cdot \left(\sigma\left(x_2\right) \cdot \sigma\left(y_2\right)\right) \\ &= \mu_\sigma\left(x_1, y_1\right) \cdot \mu_\sigma\left(x_2, y_2\right)\end{aligned}$$

so μ_σ is a fuzzy dot subalgebra of $X \times X$. □

Theorem 3.14.

Let μ be a left fuzzy relation on a fuzzy subset σ of a i^*-algebra X. If μ is a fuzzy dot subalgebra of $\mathrm{X} \times \mathrm{X}$*, then σ is a fuzzy dot subalgebra of a i^*-algebra X .*

Proof.

Suppose that a left fuzzy relation μ on σ is fuzzy dot subalgebra of $X \times X$.
Then $\sigma\left(x_1 * x_2\right) = \mu\left(x_1 * x_2, y_1 * y_2\right)$

$$\begin{aligned}&= \mu\left(\left(x_1, y_1\right) * \left(x_2, y_2\right)\right) \\ &\leq \mu\left(x_1, y_1\right) \cdot \mu\left(x_2, y_2\right) \\ &= \sigma\left(x_1\right) \cdot \sigma\left(x_2\right) \text{ for all } x_1, x_2, y_1, y_2 \in X.\end{aligned}$$

Hence σ is a fuzzy dot subalgebra of a i^*-algebra X. □

4. fuzzy dot i^*-ideal of i^*-algebra

Definition 4.1.

i^*-ideal of i^*-algebra Let X be a i*-algebra and I be a subset of X , then I is called i^*-ideal of X if it satisfies the following conditions.
(i) $1 \in I$
(ii) $x^* y \in I$ and $y \in I \Rightarrow x \in I$
(iii) $x \in I, y \in X$

$$\Rightarrow \mathrm{x} * \mathrm{y} \in \mathrm{I}.$$

Example 4.2.
Consider a i^*-algebra X = $\{0,1,2,3\}$ having the following Cayley table:

Table 4: $I=\{0,1,2\}$ **is** i^***-ideal of** i^***-algebra**

$*$	0	1	2	3
0	0	0	0	0
1	1	1	2	0
2	2	1	1	0
3	1	0	0	1

Clearly this table shows that I = $\{0,1,2\}$ is i^*-ideal of i^*-algebra.
A fuzzy subset μ of X is called a fuzzy i^*-ideal of X if it satisfies the following conditions for all $x, y \in X$:
(i) $\mu(1) \leq \mu(x)$
(ii) $\mu(\mathrm{x}) \leq \max\{\mu(\mathrm{x}^*\mathrm{y}), \mu(\mathrm{y})\}$
(iii) $\mu(\mathrm{x} * \mathrm{y}) \leq \max\{\mu(\mathrm{x}), \mu(\mathrm{y})\}$

Example 4.3. For the table 3 , Define a fuzzy set μ in X by $\mu(0)=0.6, \mu(1)=0.5, \mu(2)=0.8, \mu(3)=0.9$. Then it is easy to verify that μ is a fuzzy i^*-ideal algebra of i^*-algebra X.

Definition 4.4. Fuzzy Dot i^*-ideals of i^*-algebras A fuzzy subset μ of X is called a fuzzy dot i^*-ideal of X if it satisfies the following conditions for all $x, y, x=y \neq 1 \in X$.
(i) $\mu(1) \leq \mu(x)$
(ii) $\mu(x) \leq \mu(x^*y) \cdot \mu(y)$
(iii) $\mu(x^*y) \leq \mu(x) \cdot \mu(y)$

Example 4.5.
Let X = $\{0,1,\mathrm{a},\mathrm{b}\}$. Consider the following Cayley table.

($X, ^*, 0$) **is a** i^***-algebra.** $I=\{0,a,1\}$ **is a** i^***-ideal**

$*$	0	a	b	1
0	0	0	0	0
a	a	1	0	a
b	b	0	1	b
1	1	0	0	1

Clearly this table shows that, the system ($\mathrm{X}, ^*, 0$) is a i^*-algebra. I = $\{0,\mathrm{a},1\}$ is a i^*-ideal.
Define a fuzzy set μ in X by $\mu(0)=1, \mu(1)=0.5, \mu(\mathrm{a})=\mu(\mathrm{b})=1$. Then it is easy to verify that μ is a fuzzy dot subalgebra of a i^*-algebra. Also it is fuzzy dot i^*-ideal of i^*-algebra X .

Theorem 4.6.
Every fuzzy dot i^*-ideal of a i^*-algebra X is a fuzzy dot subalgebra of X .

Proof.
By the definition of fuzzy dot i^*-ideal of a i^*-algebra X , it is clearly true that every fuzzy dot i^*-ideal of a i^*-algebra X is a fuzzy dot subalgebra of X .

□

Remark 4.7.
The converse of Proposition is not true.

Theorem 4.8.
If μ and v are fuzzy dot i^*-ideals of a i^*-algebra X , then so is $\mu \cup v$.

Proof.
Let $\mathrm{x}, \mathrm{y} \in \mathrm{X}$.
(I) Then $(\mu \cup v)(1) = \max\{\mu(1), v(1)\}$

$$\begin{aligned} &\leq \max\{\mu(x), v(x)\} \\ &= (\mu \cup v)(x). \end{aligned}$$

(ii) Also, $(\mu \cup v)(\mathrm{x}) = \max\{\mu(\mathrm{x}), v(\mathrm{x})\}$

$$\begin{aligned} &\leq \max\left\{\mu\left(x^* y\right) \cdot \mu(y), v\left(x^* y\right) \cdot v(y)\right\} \\ &\leq \left(\max\left\{\mu\left(x^* y\right), v\left(x^* y\right)\right\}\right) \cdot \left(\max\{\mu(x), v(x)\}\right) \\ &= \left((\mu \cup v)\left(x^* y\right)\right) \cdot \left((\mu \cup v)(x)\right). \end{aligned}$$

(iii) And, $(\mu \cup v)(\mathrm{x} * \mathrm{y}) = \max\{\mu(\mathrm{x} * \mathrm{y}), v(\mathrm{x} * \mathrm{y})\}$

$$\begin{aligned} &\leq \max\{\mu(\mathrm{x}) \cdot \mu(\mathrm{y}), v(\mathrm{x}) \cdot v(\mathrm{y})\} \\ &\leq (\max\{\mu(\mathrm{x}), v(\mathrm{x})\}) \cdot (\max\{\mu(\mathrm{y}), v(\mathrm{y})\}) \\ &= ((\mu \cup v)(\mathrm{x})) \cdot ((\mu \cup v)(\mathrm{y})). \end{aligned}$$

Hence $\mu \cup v$ is a fuzzy dot i^*-ideal of a i^*-algebra X.

Theorem 4.9.
If λ and μ are fuzzy dot i^*-ideal of a d-algebra X , then $\lambda \times \mu$ is a fuzzy dot i^*-ideal of $\mathrm{X} \times \mathrm{X}$.

Proof.
Let $\mathrm{x}, \mathrm{y} \in \mathrm{X}$.
(i) $(\lambda \times \mu)(1,1) = \lambda(1) \cdot \mu(1) \leq \lambda(\mathrm{x}) \cdot \mu(\mathrm{y})$
For any $\mathrm{x}, \mathrm{x}_1, \mathrm{y}, \mathrm{y}_1 \in \mathrm{X}$, we have

(ii)

$$(\lambda \times \mu)(\mathrm{x}, \mathrm{y}) = \lambda(\mathrm{x}) \cdot \mu(\mathrm{y}) \qquad \text{(ii)}$$
$$\leq (\lambda(\mathrm{x}^* \mathrm{x}_1) \cdot \lambda(\mathrm{x}_1)) \cdot (\mu(\mathrm{y}^* \mathrm{y}_1) \cdot \mu(\mathrm{y}_1))$$
$$= (\lambda(\mathrm{x}^* \mathrm{x}_1) \cdot \mu(\mathrm{y}^* \mathrm{y}_1)) \cdot (\lambda(\mathrm{y}_1) \cdot \mu(\mathrm{y}_1))$$
$$= (\lambda \times \mu)(\mathrm{x}, \mathrm{y}) \cdot (\lambda \times \mu)(\mathrm{x}_1, \mathrm{y}_1)$$
$$= (\lambda \times \mu)((\mathrm{x}, \mathrm{y}) * (\mathrm{x}_1, \mathrm{y}_1)) \cdot (\lambda \times \mu)(\mathrm{x}_1, \mathrm{y}_1)$$
$$(\lambda \times \mu)((\mathrm{x}, \mathrm{y}) * (\mathrm{x}_1, \mathrm{y}_1)) = (\lambda \times \mu)((\mathrm{x}^* \mathrm{x}_1), (\mathrm{y}_1, \mathrm{y}_1))$$
$$= \lambda(\mathrm{x}^* \mathrm{x}_1) \cdot \mu(\mathrm{y}^* \mathrm{y}_1)$$
$$\leq ((\lambda(\mathrm{x}) \cdot \lambda(\mathrm{x}_1)) \cdot (\mu(\mathrm{y}) \cdot \mu(\mathrm{y}_1))$$
$$= (\lambda(\mathrm{x}) \cdot \mu(\mathrm{y})) \cdot (\lambda(\mathrm{x}_1) \cdot \mu(\mathrm{y}_1))$$
$$= (\lambda \times \mu)(\mathrm{x}, \mathrm{y}) * (\lambda \times \mu)(\mathrm{x}_1, \mathrm{y}_1)$$

Hence $\lambda \times \mu$ is a fuzzy dot d-ideal of $\mathrm{X} \times \mathrm{X}$.

Theorem 4.10.

Let σ be a fuzzy subset of a i^*-algebra X and σ_μ be the strongest fuzzy σ-relation on i^* algebra X . Then σ is a fuzzy dot i^*-ideal of X if and only if σ_μ is a fuzzy dot i^*-ideal of $X \times X$.

Proof.

Assume that σ is a fuzzy dot d-ideal of X .
For any $\mathrm{x}, \mathrm{y} \in \mathrm{X}$ we have $\mu_\sigma(1,1) = \sigma(1) \cdot \sigma(1)$

$$\leq \sigma(\mathrm{x}) \cdot \sigma(\mathrm{y})$$
$$= \mu_\sigma(\mathrm{x}, \mathrm{y})$$

Let $\mathrm{x}, \mathrm{x}', \mathrm{y}, \mathrm{y}' \in \mathrm{X}$.
Then $\mu_\sigma((\mathrm{x}, \mathrm{x}') * (\mathrm{y}, \mathrm{y}')) \cdot \mu_\sigma(\mathrm{y}, \mathrm{y}') = \mu_\sigma(\mathrm{x} * \mathrm{y}, \mathrm{x}' * \mathrm{y}') \cdot \mu_\sigma(\mathrm{y}, \mathrm{y}')$

$$= (\sigma(\mathrm{x} * \mathrm{y}) \cdot \sigma(\mathrm{x}' * \mathrm{y}')) \cdot (\sigma(\mathrm{y}) \cdot \sigma(\mathrm{y}'))$$
$$= (\sigma(\mathrm{x} * \mathrm{y}) \cdot \sigma(\mathrm{y})) \cdot (\sigma(\mathrm{x}' * \mathrm{y}') \cdot \sigma(\mathrm{y}'))$$
$$\geq \sigma(\mathrm{x}) \cdot \sigma(\mathrm{x}')$$
$$= \mu_\sigma(\mathrm{x}, \mathrm{x}')$$

and, $\mu_\sigma(\mathrm{x}, \mathrm{x}') \cdot \mu_\sigma(\mathrm{y}, \mathrm{y}') = (\sigma(\mathrm{x}) \cdot \sigma(\mathrm{x}')) \cdot (\sigma(\mathrm{y}) \cdot \sigma(\mathrm{y}'))$

$$= (\sigma(\mathrm{x}) \cdot \sigma(\mathrm{y})) \cdot (\sigma(\mathrm{x}') \cdot \sigma(\mathrm{y}'))$$
$$\geq \sigma(\mathrm{x} * \mathrm{y}) \cdot \sigma(\mathrm{x}' * \mathrm{y}')$$
$$= \mu_\sigma(\mathrm{x} * \mathrm{y}, \mathrm{x}' * \mathrm{y}')$$
$$= \mu_\sigma((\mathrm{x}, \mathrm{x}') * (\mathrm{y}, \mathrm{y}'))$$

Thus μ_σ is a fuzzy dot i^*-ideal of $\mathrm{X} \times \mathrm{X}$.
Conversely suppose that μ_σ is a fuzzy dot i^*-ideal of $\mathrm{X} \times \mathrm{X}$.

$$\begin{aligned}(\sigma(1))^2 = \sigma(1)\cdot\sigma(1) = \mu_\sigma(1,1) &\le \mu\sigma(x,x)\\ &= \mu(x)\cdot\mu(x) = (\mu(x))^2\end{aligned}$$

and so $\sigma(1) \le \sigma(x)$ for all $x \in X$. Also we have

$$\begin{aligned}(\sigma(x))^2 &= \mu_\sigma(x,\mathrm{x})\\ &\le \mu_\sigma((x,x)*(y,y))\cdot\mu_\sigma(y,y)\\ &= \mu_\sigma((x*y),(x*y))\cdot\mu_\sigma(y,y)\\ &= \sigma((x*y)\cdot\sigma(y))^2\end{aligned}$$

which implies that $\sigma(x) \le \sigma(x*y)\cdot\sigma(y)$ for all $x, y \in X$. Also we have

$$\begin{aligned}(\sigma(x*y))^2 &= \mu_\sigma(x*y, x*y)\\ &= \mu_\sigma((x,x)*(y,y))\\ &\le \mu_\sigma(x,x)\cdot\mu_\sigma(y,y)\\ &= (\sigma(x)\cdot\sigma(y))^2\end{aligned}$$

So $\sigma\,(x^*y) \le \sigma(x)\cdot\sigma(y)$ for all $x, y \in X$. Therefore σ is a fuzzy dot i^*-ideal of X.

Theorem 4.11.

Let μ be a left fuzzy relation on a fuzzy subset σ of a i^*-algebra X. If μ is a fuzzy dot d-ideal of $X \times X$, then σ is a fuzzy dot i^*-ideal of a i^*-algebra X.

Proof.

Suppose that a left fuzzy relation μ on σ is a fuzzy dot i^*-ideal of $X \times X$.
Then $\sigma(1) = \mu(1,z), \quad \forall z \in X$
By putting $z = 1$

$$\sigma(1) = \mu(1,1) \le \mu(x,y) = \sigma(x), \text{ for all } x \in X.$$

For any $x, x', y, y' \in X$

$$\begin{aligned}\sigma(x) = \mu(x,y) &\le \mu\left((x,y)*(x',y')\right)\cdot\mu\left(x',y'\right)\\ &= \mu\left(\left(x*x'\right),\left(y*y'\right)\right)\cdot\mu\left(x',y'\right)\\ &= \sigma\left(x*x'\right)\cdot\sigma\left(x'\right).\end{aligned}$$

$$\text{Also,}\qquad \begin{aligned}\sigma\left(x*x'\right) =&\mu\left(x*x', y*y'\right)\\ &= \mu\left((x,y)*(x',y')\right)\\ &\le \mu(x,y)\cdot\mu\left(x',y'\right)\\ &= \sigma(x)\cdot\sigma\left(x'\right)\end{aligned}$$

Thus σ is a fuzzy dot i^*-ideal of a i^*-algebra X.

5. Conclusion

In this paper, we have introduced and investigated the concepts of *fuzzy dot subalgebra* and *fuzzy dot i^*-ideal* within the framework of i^*-algebras. These notions serve as meaningful generalizations of classical subalgebras and ideals in the fuzzy setting, incorporating uncertainty and graded membership. The fuzzy dot structures provide a flexible algebraic framework that accommodates partial belonging, making them highly applicable in areas where imprecise information is inherent.

We have explored their fundamental properties, characterized their behaviors through illustrative examples, and established several inclusion and intersection results. Furthermore, we demonstrated that the structure of a fuzzy dot subalgebra and fuzzy dot i^*-ideal can significantly influence the lattice structure and functional operations within an i^*-algebra.

Our findings open avenues for further exploration in fuzzy algebraic systems, particularly in approximate reasoning, fuzzy logic-based control systems, and soft computing. Future work may involve extending these notions to other generalized algebras, such as Γ-algebras, BCI-algebras, and τ^*-algebras, and exploring their categorical and topological aspects.

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