Global and Stochastic Analysis
Vol. 12 No. 4 (July, 2025)

Received: 11th March 2025 *Revised: 10th June 2025* *Accepted: 12th June 2025*

PRICE BOUNDS FOR GUARANTEED ANNUITY OPTIONS IN A MODEL-INDEPENDENT SETTING

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Abstract. In this paper, we consider the valuation of Guaranteed Annuity Options (GAOs) in a very generalised modelling framework where both interest rate and mortality risk are stochastic and correlated. It is extremely hard to price these type of options in the correlated environment and as a result there is an absence of a closed form solution in the literature. We utilise doubly stochastic stopping times to incorporate the randomness about the time of death and employ a suitable change of measure to make the valuation of survival benefit possible, there by adapting the pay-off of the GAO in terms of the pay-off of a basket call option. We derive general price bounds for GAOs by employing the theory of comonotonicity, the Rogers-Shi ([72]) approach and the general closed form basket option pricing bounds as discussed in [16]. The theory derived is then applied to affine models to generate some very interesting formulae for the bounds under the affine set up. Numerical examples are furnished and benchmarked against Monte Carlo simulations to estimate the price of a GAO for the well known Vasicek model.

1. Introduction

The present times have witnessed a huge leap in life expectancy sending ripples across financial institutions who face unexpected challenges in the pricing of key longevity linked products such as 'Guaranteed Annuity Options' sparking a lot of interest in this area. A sneak peek into the history of mankind reveals that in the twentieth century, being a centenarian was considered to be a matter of great pride and almost an impossible feat to achieve. So much so that about a century ago, the British monarch started sending anniversary messages to "current citizens of [the monarch's] realms or UK Overseas Territories" who reached the age of 100. In 1917, King George V sent a total of 24 celebratory messages to centenarians. By 1952 this had increased more than 10-fold to 255, and in 2016, it has exploded to nearly 60-fold to 14500 (c.f. [66]). The million dollar question is: Where will it end?

In a recent study based on data from Office of National Statistics, UK (c.f. [67]), [3] concluded that, the straight line increase in the numbers of UK citizens reaching an age of 100 years seems set to continue. According to the latest posting on the Official Statistics website[1], one in three babies born in the year 2016 will

2000 *Mathematics Subject Classification.* 91G20, 60J25.
Key words and phrases. Guaranteed Annuity Option (GAO), model-independent bounds, Affine Processes, interest rate risk, mortality risk, change of measure, Basket option.

[1]Accessed on 24th June 2017

live to see their 100th birthday. Interestingly, female life expectancy scores over their male counterparts. Around 13% of girls born in 1951 are expected to be alive in 2051. For girls born in 2016 the figure is estimated to be 35% and around 60% of girls born in 2060, might expect to live long enough to receive a message from the reigning monarch. At this rate, the number of centenarians is also projected to continue rising – reaching a mammoth 83,300 in 2039 which is more than enough to keep any future monarch busy!

This interesting excerpt highlights the gravity of the problem that is looming large over financial institutions today viz. longevity risk - the risk that people outlive their expected lifetimes. Longevity risk is a considerable risk that affects adversely both the willingness and ability of financial institutions to supply retired households with financial products to deal with wealth de-cumulation in retirement. Depending on the scenario and need, longevity risk can be defined in a variety of ways. A statistical perspective of the definition is furnished by [20] who provide the following concise yet complete definition: "*It is a combination of*

- *uncertainty surrounding the trend increases in life expectancy*
  and
- *variations around this uncertain trend that is the real problem.*

This is what is meant by longevity risk and it arises as a result of unanticipated changes in mortality rates". Longevity risk is borne by every institution making payments that depend on the life span of individuals. These include Defined Benefit (DB) pension plan sponsors, insurance companies selling life annuities, and governments through the social security pension system and the salary-related pension plans of public-sector employees. The present scenario is particularly acute for insurance companies operating in the European Union (EU) where a new regulatory regime, Solvency II, was introduced in 2016. This requires insurers to possess a pool of significant additional capital to back their annuity liabilities if longevity risk cannot be hedged effectively or marked to market. In the next couple of sections we throw light on the causes of longevity risk and see what possible solutions can be proposed.

Interestingly, the product that was responsible for bringing longevity risk into limelight was Guaranteed Annuity Option (GAO) through the closure of the world's oldest life office, the Equitable Life Assurance Society (ELAS) in December 2000. Between 1957 and 1988, ELAS had sold a type of pension annuities with the so-called "Guaranteed Annuity Options (GAOs)" as an embedded feature of the contracts. A guaranteed annuity option (GAO) gives the policy holder a right to convert his accumulated fund at retirement at a guaranteed rate rather than at market annuity rate. At the time of issuance, the value of these GAOs was considered worthless, but they became very valuable at the time of maturity, due to two factors:

- *Reductions in market interest rates*
  and
- *Unanticipated falls in mortality rates at the oldest ages.*

The resultant liability obligations from the guarantees, resulted in serious solvency concerns for ELAS, requiring the setting up of extra reserves, and finally lead to unforeseen financial crisis for the firm (c.f. [7]). Although it appears that the reason behind the problem was poor risk management of the company, and that the problems could be avoided if ELAS had hedged its exposure to both interest rate risk and longevity risk. However, [13] have clearly pointed out that, even if ELAS had anticipated the problem, it still lacked good instruments to hedge its exposure to both risks, particularly longevity risk, back to that time. Therefore, this is in fact not only the problem of ELAS. During the late 1970s and 1980s, guaranteed annuity rate between cash and pension was a common feature of individual pension policies in the UK and was sold by more than 40 companies in the market.

The flourishing market of sophisticated insurance products with benefits linked to financial variables along with various guarantees has given impetus to the active use of stochastic modeling of both interest and mortality rates in the valuation of annuity-related products. In this sub-section, we present a brief recap of the research carried out in the last two decades in context of GAOs. A good reference in this regard is [42].

In the present era, considering mortality to be independent of financial markets appears to be a far fetched assumption and a more realistic belief is that the two underlying risks are correlated. This belief is supported by researchers and practitioners. For example, [39] examine the likelihood that the slowly evolving mean in the log dividend-price ratio is related to demographic trends. [64] investigates how demographic changes affect the value of financial assets. He experiments with a continuous time overlapping generations model having stochastic birth and mortality rates. His model suggests that demographic transitions have an important role to play in explaining parts of the time variation in the real interest rate, equity premium and conditional stock price volatility. Moreover, he provides adequate conditions for the interest rate to be decreasing in the birth rate and increasing in the death rate. In [25], the authors furnish some empirical evidence of a changing behaviour of the economy and the financial markets during periods of extreme mortality. Further, [24] explore existence of this dependence within the Feller process framework. To take care of this scenario, EU's Solvency II Directive has laid out new insurance risk management practices for capital adequacy requirements based on the assumption of dependence between financial markets and life/health insurance markets including the correlation between the two underpinning risks viz. interest rate and mortality (c.f. Quantitative Impact Study 5:Technical Specifications [71]). [51] introduced a pricing framework in which the dependence between the mortality and the interest rates is explicitly modeled. In their methodology, the mortality rate was modeled as an affine-type diffusion process just like the short rate process. They derived analytic expressions for mortality-linked insurance products employing the change of measure technique. Their approach paved the path for new perspectives and methodology in the valuation of other insurance products under a more reasonable assumption that risk factors are dependent.

This line of research has triggered research on evaluation of GAOs under the assumption of correlation between mortality and financial risks. [62, 63], were the pioneers to consider the correlated framework for valuing a GAO and they developed a pricing formula where the interest rate and mortality processes follow bivariate Gaussian dynamics. In their setting, the dependence between mortality and interest rates is described by one constant, namely the pairwise linear correlation coefficient. In fact in [62] they use the theory of comonotonic bounds in approximating the sums of lognormal random variables to obtain convex price bounds for GAOs in the Gaussian setting. [43] propose a modeling framework, where the interest and mortality rates are correlated and the dynamics of each risk factor possess regime-switching affine structures, to facilitate the GAO valuation. The correlation introduced through the diffusion components of the risk factors and the underlying Markov chain driving the switching of regimes provides an explanation of the the rates' relation and dynamics. A different measure called endowment-risk-adjusted measure, which first appeared in [62] and was subsequently used in [45] under several competing models, is employed to price the GAO.

More recently, [28] scrutinize the consequences of the dependence assumption on the pricing of a GAO. They assume that mortality and interest rates are driven by systematic and idiosyncratic factors, modelled by affine models which remain positive such as the multi-CIR and the Wishart models. They employ the above mentioned change of measure to value the GAO using Monte Carlo methodology. Their investigation reveals that for an advanced affine model (such as the Wishart one) that permits a non-trivial dependence between the mortality and the interest rates, the value of a GAO cannot be explained only in terms of the initial pairwise linear correlation and this fact plays an important role in risk management in the presence of an unknown dependence. Finally, [44] address the problem of setting capital reserves for a guaranteed annuity option (GAO). They formulate the modeling framework for the loss function of a GAO. A one-decrement actuarial model having death as the only decrement is employed. Once again, the interest and mortality risk factors follow correlated affine structures. Risk measures are calculated using moment-based density method and compared with the Monte-Carlo simulation. Bootstrap technique is used to assess the variability of risk measure estimates. The authors also establish the relation between a desired level of risk measure accuracy and required sample size under the constraints of computing time and memory. A sensitivity analysis of parameters is also conducted. Their numerical investigations furnish practical considerations for insurers to abide by certain regulatory requirements. Thus, dealing with Guaranteed Annuity Options under the correlation assumption of mortality and financial risks offers a fertile ground for future research.

The existing literature in the direction of pricing of GAO's under the correlation assumption is rather scarce and only Monte Carlo estimation of the GAO price is available for sophisticated models (c.f. [28]). But Monte Carlo method is generally time consuming for complex models (c.f. [40]). This article is a concrete step in the direction of pricing of GAOs under the correlation direction. It investigates the designing of price bounds for GAO's under the assumption of dependence between mortality and interest rate risks and provides a much needed confidence interval for the pricing of these options. In a set up similar to [8], we advocate the use of doubly stochastic stopping times to incorporate the randomness about the time of death. Moreover the proposed bounds are model-free or general in the sense they are applicable for all kinds of models and in particular suitable for the affine set up. Keeping pace with the relevant literature (c.f. [62], [28]), we applied a change of probability measure with the 'Survival Zero Coupon Bond' as numéraire for the valuation of the GAO. This change of measure facilitates computation and enhances efficiency (c.f. [63]). The organization of the paper follows. In Section 2 we introduce the market framework with the necessary notations. In Section 3 we define GAOs and show that their pay-off is similar to that of a basket option. This is followed by Section 4 which highlights the technicalities of affine processes. Sections 5 and 6 are the core sections which present details on finding lower and upper bounds for GAOs. In Section 7 we present examples while numerical investigations in support of the developed theory appear in Section 8. Section 9 then concludes the paper.

2. The Market Framework

In this section, we introduce the necessary set up required to construct the mathematical interplay between financial market and the mortality model. Quite clearly, due to the presence of both mortality and interest rate risk, we are handling a pricing problem in an incomplete market so that a unique pricing measure does not exist. We therefore utilize the fact that in the absence of arbitrage, at least one equivalent martingale measure (EMM) $\mathbb{Q}$ exists that can then be used to find fair prices of mortality contingent securities. We exploit this fact and refrain from assuming that the mortality evolution process behaves according to a given model, but aim to draw conclusions that hold under any model. This is in contrast to the standard approach to pricing mortality contingent products which is to postulate a model and to determine the price of the underlying as the suitably discounted risk neutral expectation of the payoff under that model. A major problem with this approach is that no model can capture the real world behaviour of mortality linked securities fully, thus exposing the entire procedure to model risk.

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ where $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ such that the filtration is large enough to support a process X in $\mathbb{R}^k$, representing the evolution of financial variables and a process Y in $\mathbb{R}^d$, representing the evolution of mortality. We concentrate on an insured life aged x at time 0, with random residual lifetime denoted by τ_x which is an $\mathbb{F}$-stopping time.

The filtration $\mathbb{F}$ includes knowledge of the evolution of all state variables up to each time t and of whether the policyholder has died by that time. More explicitly, we have:

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$$

where

$$\mathcal{G}_t \vee \mathcal{H}_t = \sigma\left(\mathcal{G}_t \cup \mathcal{H}_t\right)$$

with

$$\mathcal{G}_t = \sigma\left(Z_s : \ 0 \leq s \leq t\right), \quad \mathcal{H}_t = \sigma\left(\mathbb{1}_{\{\tau \leq s\}} : \ 0 \leq s \leq t\right)$$

and where $Z = (X, Y)$ is the joint state variables process in $\mathbb{R}^{k+d}$. Thus we have

$$\mathcal{G}_t = \mathcal{G}_t^X \vee \mathcal{G}_t^Y .$$

In fact $\mathbb{H} = \{\mathcal{H}_t\}_{t\geq 0}$ is the smallest filtration with respect to which τ is a stopping time. In other words $\mathbb{H}$ makes $\mathbb{F}$ the smallest enlargement of $\mathbb{G} = \{\mathcal{G}_t\}_{t\geq 0}$ with respect to which τ is a stopping time, i.e.,

$$\mathcal{F}_t = \cap_{s>t}\mathcal{G}_s \vee \sigma\left(\tau \wedge s\right), \ \forall t.$$

We may think of $\mathcal{G}_t$ as carrying information captured from medical/demographical data collected at population/ industry level and of $\mathcal{H}_t$ as recording the actual occurrence of death in an insurance portfolio. Further, we take as given a predictable short rate process $r = \{r_t\}_{t\geq 0}$ such that it satisfies the technical condition $\int_0^t r_s ds < \infty$ a.s. for all $t \geq 0$. The short rate process r represents the continuously compounded rate of interest of a risk-less security and it is $\mathbb{G}$-predictable.

Finally, we assume that the stopping time τ_x is governed by an intensity μ_x such that μ_x is a non-negative $\mathcal{G}_t$-predictable process satisfying $\int_0^t \mu_x\left(s\right) ds < \infty$ a.s. for all $t \geq 0$.

One can refer to [8] to compute the 'probability of survival' up to time $T \geq t$, on the set $\{\tau > t\}$, i.e.

$$\mathbb{Q}\left(\tau > T|\mathcal{F}_t\right) = \mathbb{E}\left[e^{-\int_t^T \mu_s ds}|\mathcal{F}_t\right], \tag{2.1}$$

where $\mathbb{E}$ denotes the usual expectation w.r.t the EMM $\mathbb{Q}$.

In fact, we characterize the conditional law of τ in several steps. Given that the non-negative $\mathcal{G}_t$-predictable process μ is satisfying $\int_0^t \mu_x\left(s\right) ds < \infty$ a.s. for all $t > 0$, we consider an exponential random variable Φ with parameter 1, independent of $\mathcal{G}_\infty$ and define the random time of death τ as the first time when the process $\int_0^t \mu_s ds$ is above the random threshold Φ, i.e.,

$$\tau \doteq \{t \in \mathbb{R}^+ : \int_0^t \mu_s\left(s\right) ds \geq \Phi\}. \tag{2.2}$$

It is evident from (2.2) that $\{\tau > T\} = \{\int_0^T \mu_s ds < \Phi\}$, for $T \geq 0$. Next, we work out $\mathbb{Q}\left(\tau > T|\mathcal{G}_t\right)$ for $T \geq t \geq 0$ by using tower property of conditional expectation, independence of Φ and $\mathcal{G}_\infty$ and facts that μ is a $\mathcal{G}_t$-predictable process and $\Phi \sim \mathit{Exponential}\left(1\right)$, i.e.,

$$\mathbb{Q}\left(\tau > T|\mathcal{G}_t\right) = \mathbb{E}\left[e^{-\int_0^T \mu_s ds}|\mathcal{G}_t\right]. \tag{2.3}$$

In fact, the same result holds for $0 \leq T < t$. Further, we observe that $\{\tau > t\}$ is an atom of $\mathcal{H}_t$. As a result, in a manner similar to [8], we have constructed a doubly stochastic $\mathcal{F}_t$-stopping time driven by $\mathcal{G}_t \subset \mathcal{F}_t$ in the following way (c.f. [12], ex 34.4, p.455):

$$\begin{aligned}\mathbb{Q}\left(\tau > T|\mathcal{G}_T \vee \mathcal{F}_t\right) &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[\mathbb{1}_{\{\tau>T\}}|\mathcal{G}_T \vee \mathcal{H}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}}e^{-\int_t^T \mu_s ds}. \end{aligned} \tag{2.4}$$

Next, the conditioning on $\mathcal{F}_t$ can be replaced by conditioning on $\mathcal{G}_t$ as shown in the Appendix C of [8].

We remark that, we do not take $\mathcal{G}_t \vee \sigma(\Phi)$ as our filtration $\mathcal{G}_t$ because, in that case, the stopping time τ would be predictable and would not admit an intensity. The construction portrayed here guarantees that τ is a totally inaccessible stopping time, a concept intuitively meaning that the insured's death arrives as a total surprise to the insurer (see [70], Chapter III.2, for details). With this, we move to the focal point of this paper viz. GAOs.

3. Guaranteed Annuity Options

3.1. Introduction. A Guaranteed Annuity Option(GAO) is a contract that gives the policyholder the flexibility to convert his/her survival benefit into an annuity at a pre-specified conversion rate. The guaranteed conversion rate denoted by g, can be quoted as an annuity/cash value ratio. According to [14], the most popular choice for the guaranteed conversion rate g for males aged 65 in the UK in the 1980s was $g = \frac{1}{9}$, which means that per £1000 cash value can be converted into an annuity of £111 per annum. The GAO would have a positive value if the guaranteed conversion rate is higher than the available conversion rate; otherwise the GAO is worthless since the policyholder could use the cash to obtain higher value of annuity from the primary market. As a result, the moneyness of the GAO at maturity depends on the price of annuity at that time in the market and this in turn is calculated using the prevailing interest and mortality rates.

3.2. Mathematical Formulation. Consider an x year old policyholder at time 0 who has an access to a unity amount at his retirement age R_x. Then, a GAO gives the policyholder a choice to choose at time $T = R_x - x$ between an annual payment of g or a cash payment of 1. Let $\ddot{a}_x(T)$ denote the value at time T of a whole life annuity due for a person aged x at time 0, which gives an annual payment of one unit amount at the start of each year, this payment beginning from time T and conditional on survival. If w is the largest possible survival age then the annuity price (which is truncated in a way as the largest survival age is assumed to be w) is given by

$$\begin{aligned} \ddot{a}_x(T) &= \sum_{j=0}^{w-(T+x)-1} \mathbb{E}\left[e^{-\int_T^{T+j}(r_s+\mu_s)ds} | \mathcal{G}_T\right] \\ &= \sum_{j=0}^{w-(T+x)-1} SZ(T, T+j) \end{aligned} \tag{3.1}$$

and

$$SZ(t,T) = \mathbb{E}\left[e^{-\int_t^T(r_s+\mu_s)ds} | \mathcal{G}_t\right] \tag{3.2}$$

denotes the price at time t of a pure endowment insurance with maturity T for an insured of age x at time 0 who is still alive at time t. This insurance instrument is nomenclated as a *survival zero-coupon bond* abbreviated as SZCB by [28] and the authors remark that it can be used as a numèraire because it can be replicated by a strategy that involves longevity bonds (c.f. [60]) assuming a liquid market for these instruments. This is in analogy with the usual bootstrapping methodology used to find the zero rate curve starting by coupon bonds. This insurance instrument pays one unit of money at time T upon the survival of the insured at that time.

In fact $r+\mu$ can be viewed as a fictitious short rate or yield to compare these instruments with their financial counterparts.

At time T, the value of the contract having the above embedded GAO can be described by the following decomposition

$$\begin{aligned} V(T) &= \max(g\ddot{a}_x(T), 1) \\ &= 1 + g\max\left(\ddot{a}_x(T) - \frac{1}{g}, 0\right). \end{aligned} \tag{3.3}$$

In order to apply risk neutral evaluation, we state a result from [8] to compute the fair values of a basic pay-off involved by standard insurance contracts. These are benefits, of amount possibly linked to other security prices, contingent on survival over a given time period. We require the short rate process r and the intensity of mortality μ to satisfy the technical conditions stated in Section 2.

Proposition 3.1. *(Survival benefit). Let C be a bounded $\mathcal{G}_t$-adapted process. Then under the EMM $\mathbb{Q}$, the time-t fair value $SB_t(C_T;T)$ of the time-T survival benefit of amount C_T, with $0 \le t \le T$, is given by:*

$$SB_t(C_T;T) = \mathbb{E}\left[e^{-\int_t^T r_s ds}\mathbb{1}_{\{\tau>T\}}C_T|\mathcal{F}_t\right] = \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[e^{-\int_t^T (r_s+\mu_s)ds}C_T|\mathcal{G}_t\right] \tag{3.4}$$

In particular, if C is $\mathcal{G}_t^X$-adapted and X and Y are independent, then, the following holds

$$SB_t(C_T;T) = \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[e^{-\int_t^T r_s ds}C_T|\mathcal{G}_t^X\right]\mathbb{E}\left[e^{-\int_t^T \mu_s ds}|\mathcal{G}_t^Y\right] \tag{3.5}$$

Proof. A comprehensive proof can be found in [8]. □

Thus, we have the value at time $t=0$ of the second term in (3.3), which is called the GAO option price entered by an x-year policyholder at time $t=0$ as

$$C(0,x,T) = \mathbb{E}\left[e^{-\int_0^T (r_s+\mu_s)ds}g\left(\ddot{a}_x(T) - \frac{1}{g}\right)^+\right]. \tag{3.6}$$

In order to facilitate calculation, we adopt the following change of measure.

3.3. Change of Measure. We advocate a change of measure similar to the one adopted in [28]. We define a new probability measure $\tilde{Q}$ with the Radon-Nikodym derivative of $\tilde{Q}$ w.r.t $\mathbb{Q}$ as:

$$\frac{d\tilde{Q}}{d\mathbb{Q}} := \eta_T = \frac{e^{-\int_0^T (r_s+\mu_s)ds}}{\mathbb{E}\left[e^{-\int_0^T (r_s+\mu_s)ds}\right]}. \tag{3.7}$$

We will use $\tilde{E}$ to denote the expectation w.r.t the new probability measure $\tilde{Q}$. An important point here is that the measure $\tilde{Q}$ preserves the doubly stochastic nature of the death time, as condition (A4) in [9] is satisfied. The setting considered here is considerably more powerful than plain vanilla market consistent valuation. For example, it could apply to situations in which calibration of the measures $\tilde{Q}$ and $\mathbb{Q}$ is informed by empirical evidence on policyholder behaviour, as long as the latter is captured by a suitable EMM preserving the main features of the framework (c.f. [4], section 4.5). In this case, the approach could take into account the "private valuation" of the annuity cashflows in addition to the case of a benchmark market annuity rate.

Further on using Bayes' Rule for conditional expectation, the survival benefit in (3.4) can be rewritten as

$$SB_t\left(C_T;T\right)=\mathbb{1}_{\{\tau>t\}}SZ\left(t,T\right)\tilde{E}\left[C_T|\mathcal{G}_t\right] \tag{3.8}$$

The advantage of the change of measure approach is that the complex expectation appearing in the survival benefit given in (3.4) has been decomposed into two simpler expectations: the first one corresponds to the price of the SZCB given in (3.2) and the second one is connected to the expected value of the survival benefit C_T under the new probability measure $\tilde{Q}$ which needs to be determined. In the passing, one notes that in (3.8) if $C_T=1$, we get an interesting relationship

$$SB_t\left(1;T\right)=\mathbb{1}_{\{\tau>t\}}SZ\left(t,T\right). \tag{3.9}$$

where $SZ\left(t,T\right)$ has been defined in(3.2). Clearly $SZ\left(t,T\right)$ is the pre-death counterpart of process $SB_t\left(1;T\right)$ (c.f.[10], section 2). Another good reference is [52]. In particular

$$SB_0\left(1;T\right)=\mathbb{1}_{\{\tau>t\}}SZ\left(0,T\right). \tag{3.10}$$

A similar change of measure has been employed by [63] and [62] with the only difference that they use the unitary survival benefit given in (3.9) as the numèraire. On the contrary, [51] have used a twin change of measure to compute value of a GAO.

3.4. Pay-off. Under the new probability measure $\tilde{Q}$ defined in (3.7), the GAO option price decomposes into the following product

$$C(0,x,T)=gSZ\left(0,T\right)\tilde{E}\left[\left(\ddot{a}_x\left(T\right)-\frac{1}{g}\right)^+\right] \tag{3.11}$$

where $SZ\left(0,T\right)$ is defined in (3.2). To develop ideas further, we express the pay-off in a more appealing form as follows:

$$C(0,x,T)=gSZ\left(0,T\right)\tilde{E}\left[\left(\sum_{i=1}^{n-1}S_T^{(i)}-(K-1)\right)^+\right] \tag{3.12}$$

where we utilize the fact that $SZ\left(T,T\right)=1$ and define $n=w-(T+x)$; $K=\frac{1}{g}$ and

$$S_T^{(i)}=SZ\left(T,T+i\right);\ i=1,2,...,n-1. \tag{3.13}$$

The last term on the R.H.S in the payoff of the GAO resembles the pay-off of a basket option having unit weights and the SZCBs, maturing at times $T+1,T+2,...,w-x-1$ acting as the underlying assets. We seek to evaluate tight model-independent bounds for the GAOs in the ensuing sections. To the best of our knowledge, the equations (3.6) and (3.11) have only been valued by Monte Carlo simulations for specific choice of models. In [63], numerical experiments in the Gaussian setting have shown that (3.11) is a little bit more precise and in particular it is less time consuming than the implementation of (3.6). [28] have investigated these calculations for different affine models such as the multi-CIR and the Wishart cases. [62] have computed very specific comonotonic bounds for GAOs in the Gaussian framework.

It is important to point out here that the more generalized version of GAOs include what is called the "quanto" or "equity" component. Assume that we have a single premium equity-linked policy. The contract is assumed to mature at time T. The premium is invested into an account with market value $SF(t)$ at time t, where $SF(t)$ is a random process. At maturity, the proceeds of the policy are $SF(T)$. As a result, the full-fledged version of GAO pay-off in this case on lines of (3.11) is

$$C(0,x,T) = gSZ(0,T)\,\tilde{E}\left[SF(T)\left(\ddot{a}_x(T) - \frac{1}{g}\right)^{+}\right] \tag{3.14}$$

The simplest framework for valuing books of equity linked GAO policies in run off, for example, typically relies on a two factor affine model for the short rate and a geometric Brownian motion for the equity/quanto component (c.f. [6]). The correlation structure of the three-dimensional Brownian motion is a very important dimension of the model when it comes to GAO valuation. Although the extension to a quanto pay-off component is outside the scope of the present work, we plan to undertake the same in near future. We now discuss affine processes and utilize these processes to test our bounds.

4. Affine Processes

Affine processes are essentially Markov processes with conditional characteristic function of the exponentially affine form. A thorough discussion of these processes on canonical state space appears in [36] and [41]. More recently the development of multivariate stochastic volatility models has lead to the evolution of applications of affine processes on non-canonical state spaces, in particular on the cone of positive semi-definite matrices. A plethora of research papers are available to explore and interested readers can refer to [23] for details. A unified approach on affine processes is presented in [57] and following this approach we recall the details of the affine processes in the Appendix A. In regards to the evolution of interest rates and the force of mortality we consider a set up similar to [28], i.e., we present the affine setting within the time homogeneous subclass of models. However extension to the time inhomogeneous case is possible and interested readers can refer to [41] and [58].

Suppose we have a time-homogeneous affine Markov process X taking values in a non-empty convex subset E of $\mathbb{R}^d$, $(d \geq 1)$ equipped with the inner product $\langle \cdot, \cdot \rangle$. We then assume that the dynamics of the interest rate and force of mortality are given respectively as follows.

$$r_t = \bar{r} + \langle R, X_t \rangle \tag{4.1}$$

and

$$\mu_t = \bar{\mu} + \langle M, X_t \rangle \tag{4.2}$$

where $\bar{r}, \bar{\mu} \in \mathbb{R}$, $M, R \in \mathbb{R}_d$ or M_d where M_d is the set of real square matrices of order d.

This means that the interest rate and mortality are linear projections of the common stochastic factor X along constant directions given by the parameter R and M respectively. We will be interested in the cases where the X is a classical affine process on the state space $\mathbb{R}_+^m \times \mathbb{R}^n$ or an affine Wishart process on the state space S_d^+, which is the set of $d \times d$ symmetric positive definite matrices. The inner product possesses the flexibility to condense into scalar product or trace depending on the nature of R and M being respectively vectors or matrices. In the case of Vasicek model (c.f. [74]), the affine set up is uni-dimensional. A very good reference to show that the stochastic processes underlying the Vasicek model fall under the affine set up is [56].

In the passing it is important to note that the affineness of the underlying model is preserved as we move from the physical world to the the risk neutral environment, although new affine dynamics emerge (c.f. [11] and [37]). In fact the behaviour of affine processes under changes of measure depends on the risk premia associated with the underlying measures. More recently [32] examine the conditions under which it is possible or not to translate the independence assumption from the physical world to the pricing world.

We now state without proof the following proposition which indeed presents the methodology to value SZCBs and in turn GAOs. A detailed proof appears in [46] and the necessary notations are defined in the Appendix A.

Proposition 4.1. *Let X be a conservative affine process on S_d^+ under the risk neutral measure $\mathbb{Q}$. Let the short rate be given in accordance with* (4.1)*. Let $\tau^{'} = T - t$, then the price of a zero-coupon bond is given by*

$$\begin{aligned} SZ(t,T) &= \mathbb{E}\left[e^{-\int_t^T (\bar{r}+\bar{\mu}+\langle R+M, X_u\rangle)du} | \mathcal{F}_t\right] \\ &= e^{-(\bar{r}+\bar{\mu})\tau} e^{-\tilde{\phi}\left(\tau^{'},R+M\right) - \langle \tilde{\psi}\left(\tau^{'},R+M\right), X_t\rangle}, \end{aligned} \tag{4.3}$$

where $\tilde{\phi}$ and $\tilde{\psi}$ satisfy the following Ordinary Differential Equations (ODEs) which are known also as Riccati ODE's.

$$\frac{\partial \tilde{\phi}}{\partial \tau'} = \tilde{\Im}\left(\tilde{\psi}\left(\tau^{'}, R+M\right)\right), \quad \tilde{\phi}(0, R+M) = 0, \tag{4.4}$$

$$\frac{\partial \tilde{\psi}}{\partial \tau'} = \tilde{\Re}\left(\tilde{\psi}\left(\tau^{'}, R+M\right)\right), \quad \tilde{\psi}(0, R+M) = 0, \tag{4.5}$$

with

$$\tilde{\Im}\left(\tilde{\psi}\left(\tau^{'}, R+M\right)\right) = \langle b, \tilde{\psi}\left(\tau^{'}, R+M\right)\rangle - \int_{S_d^+\backslash\{0\}} \left(e^{-\langle \tilde{\psi}\left(\tau^{'},R+M\right),\xi\rangle} - 1\right) m(d\xi) \tag{4.6}$$

and

$$\begin{aligned} \tilde{\Re}\left(\tilde{\psi}\left(\tau^{'}, R+M\right)\right) &= -2\tilde{\psi}\left(\tau^{'}, R+M\right)\alpha\tilde{\psi}\left(\tau^{'}, R+M\right) + B^T\left(\tilde{\psi}\left(\tau^{'}, R+M\right)\right) \\ &- \int_{S_d^+\backslash\{0\}} \left(\frac{e^{-\langle \tilde{\psi}\left(\tau^{'},R+M\right),\xi\rangle} - 1 + \langle \chi(\xi), \tilde{\psi}\left(\tau^{'}, R+M\right)\rangle}{\parallel \xi \parallel^2 \wedge 1}\right) \mu(d\xi) + R + M. \end{aligned} \tag{4.7}$$

In fact it is interesting to note that assuming this kind of affine structure means that our fictitious yield model is "affine" in the sense that there is, for each maturity T, an affine mapping $Z_T : \mathbb{R}^n \to \mathbb{R}$ such that, at any time t, the yield of any SZCB of maturity T is $Z_T(X_t)$ echoing the results obtained in the seminal paper of [35].

As a result we have for $i = 1, 2, ..., n-1$,

$$S_T^{(i)} = e^{-(\bar{r}+\bar{\mu})i} e^{-\tilde{\phi}(i,R+M) - \langle \tilde{\psi}(i,R+M), X_T \rangle}, \tag{4.8}$$

where $\tilde{\phi}(i, R+M)$ and $\tilde{\psi}(i, R+M)$ satisfy the equations (4.4) and (4.5) with $\tau^{'} = i$. Alternatively, one may write

$$S_T^{(i)} = S_0^{(i)} e^{X_T^{(i)}};\ i = 1, 2, ..., n-1, \tag{4.9}$$

with

$$S_0^{(i)} = e^{-\left((\bar{r}+\bar{\mu})i + \tilde{\phi}(i,R+M)\right)} \tag{4.10}$$

and

$$X_T^{(i)} = -\langle \tilde{\psi}(i, R+M), X_T \rangle. \tag{4.11}$$

As a result in the affine case, by using equation (4.8) in (3.12) the formula for GAO pay-off can be written in a very compact form as shown below

$$\begin{aligned} C(0,x,T) &= gSZ(0,T) \\ &\quad \times \tilde{E}\left[\left(\sum_{i=1}^{n-1} e^{-(\bar{r}+\bar{\mu})i} e^{-\tilde{\phi}(i,R+M) - \langle \tilde{\psi}(i,R+M), X_T \rangle} - (K-1)\right)^{+}\right] \end{aligned} \tag{4.12}$$

where $SZ(0,T)$ is given by equation (4.3) with $\tau^{'} = T$. As a result in the affine case, our quest of bounds for the GAO becomes simplified as we are dealing only with X_T.

The analytical tractability of affine processes is essentially linked to generalized Riccati equations as given above which can be in general solved by numerical methods although explicit solutions are available in the Vasicek (c.f. [74]) model without jumps.

5. Lower Bounds for Guaranteed Annuity Options

We now proceed to work out appropriate lower bounds for the payoff of the GAO as given in (3.12). Invoking Jensen's inequality , we have

$$\tilde{E}\left[\left(\sum_{i=1}^{n-1} S_T^{(i)} - (K-1)\right)^{+}\right] \geq \tilde{E}\left[\left(\sum_{i=1}^{n-1} \tilde{E}\left(S_T^{(i)} | \Lambda\right) - (K-1)\right)^{+}\right]. \tag{5.1}$$

The general derivation concerning lower bounds for stop loss premium of a sum of random variables based on Jensen's inequality can be found in [73] and for its application to Asian basket options, one can refer to [27]. [1] utilize comonotonicity theory to find a price range for Asian options. Define

$$S = \sum_{i=1}^{n-1} S_T^{(i)} \tag{5.2}$$

and

$$S^l = \sum_{i=1}^{n-1} \tilde{E}\left(S_T^{(i)}|\Lambda\right) \tag{5.3}$$

Thus, we have obtained

$$S \geq_{cx} S^l. \tag{5.4}$$

where *cx* denotes convex ordering (see for example in [30]). Now, suitably tailoring the inequality (5.1), we obtain

$$C(0,x,T) \geq gSZ\,(0,T)\,\tilde{E}\left[\left(\sum_{i=1}^{n-1}\tilde{E}\left(S_T^{(i)}|\Lambda\right) - (K-1)\right)^+\right]. \tag{5.5}$$

5.1. A First Lower Bound. In case, if the random variable Λ is independent of the prices of pure endowments having term periods $1,2,...,n-1$ at the time T, i.e., of $S_T^{(i)}$; $i=1,2,...,n-1$, respectively, the bound in (5.5) simply reduces to:

$$C(0,x,T) \geq gSZ\,(0,T)\,\tilde{E}\left[\left(\sum_{i=1}^{n-1}\tilde{E}\left(S_T^{(i)}\right) - (K-1)\right)^+\right]. \tag{5.6}$$

or even more precisely as the outer expectation is redundant, we obtain a very trivial bound for GAO expressed in terms of expectation of S_T^i, i.e.,

$$C(0,x,T) \geq gSZ\,(0,T)\left(\sum_{i=1}^{n-1}\tilde{E}\left(S_T^{(i)}\right) - (K-1)\right)^+ =: \text{ GAOLB}. \tag{5.7}$$

5.1.1. *The Lower Bound under the Affine Set Up.* Under the affine set up of Section 4 (c.f. equation (4.8)), the lower bound given in equation (5.7) reduces to

$$\begin{aligned}\text{GAOLB}^{\mathit{aff}} &= gSZ\,(0,T)\\ &\quad\times\left(\sum_{i=1}^{n-1}\left(e^{-\left((\bar{r}+\bar{\mu})i+\tilde{\phi}(i,R+M)\right)}\mathcal{L}\left(\tilde{\psi}\,(i,R+M)\right)\right) - (K-1)\right)^+\end{aligned} \tag{5.8}$$

where $\mathcal{L}$ denotes the Laplace transform of X_T with parameter $\tilde{\psi}\,(i,R+M)$ under the transformed measure $\tilde{Q}$. This means that if one can lay hands on the distribution of X_T, this bound has a very compact form.

5.2. The Comonotonic Lower Bound. As the next step, we obtain a tighter lower bound by assuming that the endowment products S_i have an asset price process given in terms of exponential Lévy model as follows:

$$S_T^{(i)} = S_0^{(i)}\exp\left(X_T^{(i)}\right);\ i=1,2,...,n-1, \tag{5.9}$$

where $X_T^{(i)}$ is a Lévy process observed at time T and $S_0^{(i)}$ is the price of pure endowment of term i years at time 0. For each i, let μ_i and σ_i^2 represent the expectation and variance of X_i respectively. Further, let ρ_{ij} denote the correlation of $X_T^{(i)}$ and $X_T^{(j)}$ and assume that, for all $i,\ j$, this is non-negative. Again using Jensen's inequality, one may write

$$\tilde{E}\left[S_T^{(i)}|S_T^{(j)}=s\right]\geq S_0^{(i)}\exp\left(\tilde{E}\left[X_T^{(i)}|X_T^{(j)}=\log_e\left(\frac{s}{S_0^{(j)}}\right)\right]\right) \tag{5.10}$$

Further, we assume that the Lévy process has no jumps so that

$$\tilde{E}\left[X_T^{(i)}|X_T^{(j)}=x_j\right]=\mu^{(i)}+\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}\left(x_j-\mu^{(j)}\right) \tag{5.11}$$

where

$$\mu^{(i)}=\tilde{E}\left[X_T^{(i)}\right]=\tilde{E}\left[\log_e\left(\frac{S_T^{(i)}}{S_0^{(i)}}\right)\right];\ i=1,2,...,n-1 \tag{5.12}$$

$$\left(\sigma^{(i)}\right)^2=\operatorname{Var}\left[X_T^{(i)}\right]=\operatorname{Var}\left[\log_e\left(\frac{S_T^{(i)}}{S_0^{(i)}}\right)\right];\ i=1,2,...,n-1 \tag{5.13}$$

Further for $i\neq j=1,2,...,n-1$, we have

$$\rho^{(ij)}=\operatorname{Corr}\left[X_T^{(i)},X_T^{(j)}\right]=\operatorname{Corr}\left[\log_e\left(\frac{S_T^{(i)}}{S_0^{(i)}}\right),\log_e\left(\frac{S_T^{(j)}}{S_0^{(j)}}\right)\right] \tag{5.14}$$

Also from [30], we know that

$$S\geq_{cx}\sum_{i=1}^{n-1}\tilde{E}\left(S_T^{(i)}|S_T^{(j)}\right). \tag{5.15}$$

Combining (5.10), (5.11) and (5.15), we get that

$$S\geq_{sl}\sum_{i=1}^{n-1}S_0^{(i)}\left(\frac{S_T^{(j)}}{S_0^{(j)}}\right)^{\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}}\exp\left(\mu^{(i)}-\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}\mu^{(j)}\right). \tag{5.16}$$

On comparing (5.15) with (5.3) and (5.4), we see that $S_T^{(j)}$ is in fact playing the role of Λ. Further, let $Y_T^{(ij)}$ denote the individual components of the sum on the right hand side of equation (5.16). Since we have assumed that $\rho^{(ij)}\geq 0\ \forall i,j$, it follows that the vector $\left(Y_T^{(1j)},Y_T^{(2j)},...,Y_T^{((n-1)j)}\right)$ is comonotonic since its components are strictly increasing functions of a single variable $S_T^{(j)}$ and so we define

$$S_j^{l_2}=\sum_{i=1}^{n-1}Y_T^{(ij)} \tag{5.17}$$

and from (5.16) and (5.17), it is evident that

$$S\geq_{sl}S_j^{l_2}. \tag{5.18}$$

Further, the stop-loss transform of $S_j^{l_2}$ can be written as the sum of stop-loss transform of its components (see for example in [30]), i.e.,

$$\tilde{E}\left[\left(S_j^{l_2}-(K-1)\right)^+\right]=\sum_{i=1}^{n-1}\tilde{E}\left[\left(Y_T^{(ij)}-F^{-1}_{Y_T^{(ij)}}\left(F_{S_j^{l_2}}(K-1)\right)\right)^+\right]-K_2 \tag{5.19}$$

where

$$K_2 = \left((K-1) - F^{-1}_{S_j^{l_2}} \left(F_{S_j^{l_2}} (K-1) \right) \right) \left(1 - F_{S_j^{l_2}} (K-1) \right) \tag{5.20}$$

and $(K-1) \in \left(F^{-1+}_{S_j^{l_2}} (0), F^{-1}_{Sj^{l_2}} (1) \right)$. Further, $F_{S_j^{l_2}} (K-1)$ is the distribution function of S^{l_2} evaluated at $K-1$ so that we have:

$$F_{S_j^{l_2}} (K-1) = \mathbf{P} \left[S_j^{l_2} \leq (K-1) \right]$$

$$= \mathbf{P} \left[\sum_{i=1}^{n-1} S_0^{(i)} \left(\frac{S_T^{(j)}}{S_0^{(j)}} \right)^{\rho^{(ij)} \frac{\sigma_i}{\sigma^{(j)}}} \exp \left(\mu^{(i)} - \rho^{(ij)} \frac{\sigma^{(i)}}{\sigma^{(j)}} \mu^{(j)} \right) \leq (K-1) \right] \tag{5.21}$$

In fact $S_j^{l_2} \leq (K-1)$ if and only if $S_T^{(j)} \leq x' S_0^{(j)}$ provided that $\rho^{(ij)} \geq 0 \; \forall \; i,j$, where we substitute x' for $S_j / S_0^{(j)}$ in the above expression and obtain its value by solving the following equation

$$\sum_{i=1}^{n-1} S_0^{(i)} \left(x' \right)^{\rho^{(ij)} \frac{\sigma^{(i)}}{\sigma^{(j)}}} \exp \left(\mu^{(i)} - \rho^{(ij)} \frac{\sigma^{(i)}}{\sigma^{(j)}} \mu^{(j)} \right) - (K-1) = 0. \tag{5.22}$$

As a result, we have:

$$\begin{aligned} F_{S_j^{l_2}} (K-1) &= F_{S_T^{(j)}} \left(x' S_0^{(j)} \right) \\ &= F_{Y_T^{(ij)}} \left(S_0^{(i)} \left(x' \right)^{\rho^{(ij)} \frac{\sigma^{(i)}}{\sigma^{(j)}}} \exp \left(\mu^{(i)} - \rho^{(ij)} \frac{\sigma^{(i)}}{\sigma^{(j)}} \mu^{(j)} \right) \right) \end{aligned} \tag{5.23}$$

Using this result in (5.19) along with the stop-loss order relationship between S and $S_j^{l_2}$ as given by equation (5.18), we obtain

$$\begin{aligned} C(0,x,T) \;&\geq\; gSZ(0,T) \left(\sum_{i=1}^{n-1} S_0^{(i)} \left(S_0^{(j)} \right)^{-\rho^{(ij)} \frac{\sigma^{(i)}}{\sigma^{(j)}}} \exp \left(\mu^{(i)} - \rho^{(ij)} \frac{\sigma^{(i)}}{\sigma^{(j)}} \mu^{(j)} \right) \times \right. \\ &\qquad \left. P \left(x' S_0^{(j)}, T, \rho^{(ij)} \frac{\sigma^{(i)}}{\sigma^{(j)}}, j \right) - K_2 \right) \\ &=: \; \mathrm{GAOLB}_j^{(2)}, \end{aligned} \tag{5.24}$$

where K_2 is defined in (5.20). Further, $\mu^{(i)}$, $\left(\sigma^{(i)}\right)^2$ and $\rho^{(ij)}$ are given respectively in (5.12)-(5.14) and P is defined as the *asymmetric power expectation* function given by

$$P \left(x', t, z, j \right) = \tilde{E} \left[\left(\left(S_t^{(j)} \right)^z - \left(x' \right)^z \right)^+ \right], \tag{5.25}$$

where in our case $t = T$ and we are using $S_T^{(j)}$ in place of $S_t^{(j)}$. Since the above lower bound is a lower bound for every j, we can maximise this for $j \in \{1, 2, ..., n-1\}$ to obtain an optimal lower bound for GAO.

We have derived this lower bound under the assumption of positive correlation between the objects viz. pure endowments in the basket. Although from the point of view of stochastic processes, this assumption may be restrictive, in reality, it is quite a reasonable assumption. This is because we talking about SZCBs or pure endowments of different duration issued to the same set of lives at the same time T.

5.3. The General Lower Bound. To obtain a more general bound, we now relax the assumption of positive correlation between the pure endowments. We adapt the approach undertaken by [27] for Asian basket options for GAOs. This approach considers a non-comonotonic sum based on the methodology of [72] for Asian options.

Let us define $X_T^{(i)}$ in the same way as we have done in the comonotonic case. Next, we choose a single random variable Λ such that $\left(X_T^{(i)}, \Lambda\right)$ for every $i \in \{1, 2, ..., n-1\}$ is Bivariate Normally Distributed (BVN) with correlation coefficient given by $\rho^{(i\Lambda)}$. Clearly a simple application of Jensen's inequality yields the following convex order lower bound for S_i given any random variable Λ.

$$S_T^{(i)} \geq_{cx} \tilde{E}\left(S_T^{(i)}|\Lambda\right). \tag{5.26}$$

As a result

$$S \geq_{cx} S^l := \sum_{i=1}^{n-1} \tilde{E}\left(S_T^{(i)}|\Lambda\right). \tag{5.27}$$

We know that if $(X, Y) \sim \text{BVN}\left(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho\right)$, the conditional distribution of the lognormal random variable e^X, given the event $Y = y$ is given as

$$F_{e^X|Y=y}(x) = \Phi\left(\frac{\log_e x - \left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y - \mu_Y)\right)}{\sigma_X\sqrt{1-\rho^2}}\right). \tag{5.28}$$

where Φ denotes the c.d.f. of standard normal distribution. In our case by assumption, we have $(X_i, \Lambda) \sim \text{BVN}\left(\mu^{(i)}, \mu_\Lambda, \left(\sigma^{(i)}\right)^2, \sigma_\Lambda^2, \rho^{(i\Lambda)}\right)$. As a result, the distribution function of $S_T^{(i)}$ conditional on the event $\Lambda = \lambda$ is given as

$$F_{S_T^{(i)}|\Lambda=\lambda}\left(x^{'}\right) = \Phi\left(a\left(x^{'}\right)\right) \tag{5.29}$$

where $a(x)$ is given by

$$a\left(x^{'}\right) = \frac{\log_e x^{'} - \left(\log\left(S_0^{(i)}\right) + \mu^{(i)} + \rho^{(i\Lambda)}\frac{\sigma^{(i)}}{\sigma_\Lambda}(\lambda - \mu_\Lambda)\right)}{\sigma^{(i)}\sqrt{\left(1 - \left(\rho^{(i\Lambda)}\right)^2\right)}}. \tag{5.30}$$

As the differentiation of c.d.f. yields the p.d.f., therefore the conditional density function of S_i given $\Lambda = \lambda$ satisfies the following equation:

$$f_{S_T^{(i)}|\Lambda=\lambda}\left(x^{'}\right) = \frac{1}{x^{'}\sigma^{(i)}\sqrt{\left(1 - \left(\rho^{(i\Lambda)^2}\right)\right)}}\phi\left(a\left(x^{'}\right)\right), \tag{5.31}$$

where ϕ denotes the p.d.f. of standard normal distribution. As a result, the conditional expectation of $S_T^{(i)}$ given $\Lambda = \lambda$ is given by the expression

$$\tilde{E}\left(S_T^{(i)}|\Lambda = \lambda\right) = S_0^{(i)} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2\left(1-\left(\rho^{(i\Lambda)}\right)^2\right)}{2} + \rho^{(i\Lambda)}\sigma^{(i)}\frac{(\lambda-\mu_\Lambda)}{\sigma_\Lambda}}. \tag{5.32}$$

We utilize this expression to obtain a lower bound for Guaranteed Annuity Option under the above setting. Clearly, using (5.5) and (5.27), we have

$$C(0,x,T) \geq gSZ(0,T)\,\tilde{E}\left[\left(\sum_{i=1}^{n-1} S_0^{(i)} e^{\mu_i + \frac{\sigma_i^2\left(1-\rho_{i\Lambda}^2\right)}{2} + \rho_{i\Lambda}\sigma_i\frac{(\lambda-\mu_\Lambda)}{\sigma_\Lambda}} - (K-1)\right)^+\right]. \tag{5.33}$$

To obtain the lower bound in a more compact form, we define

$$f(v) = \sum_{i=1}^{n-1} S_0^{(i)} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2\left(1-\rho_{i\Lambda}^2\right)}{2} + \rho^{(i\Lambda)}\sigma^{(i)}\Phi^{-1}(v)} - (K-1) \tag{5.34}$$

where

$$v = \Phi\left(\frac{\Lambda - \mu_\Lambda}{\sigma_\Lambda}\right) \tag{5.35}$$

Then

$$\tilde{E}\left[\left(S^l - (K-1)\right)^+\right] = \tilde{E}\left[\left(f(V)\right)^+\right] \tag{5.36}$$

with V being uniformly distributed on $(0,1)$. An important consideration in the valuation of $\tilde{E}\left[\left(f(V)\right)^+\right]$ will be the interval upon which f is positive. This can be obtained by using the following result. Clearly, $f(v)$ is no longer a monotone function of v as in the comonotonic case when not all $\rho^{(i\Lambda)}$ have the same sign.

Proposition 5.1. *If $\rho^{(i\Lambda)} \geq 0$ for every i, then f has a unique root in $(0,1)$. Otherwise, $f(v)$ has two solutions if and only if* $\inf_{v\in(0,1)} f(v) < 0$.

Proof. Let us first assume that $\rho_{i\Lambda} \geq 0$ for every i. Then, f is a continuous, strictly increasing function of v. Furthermore, we see that f tends to $-(K-1) < 0$ as $v \downarrow 0$ and ∞ as $v \uparrow 1$. Therefore, by applying the Intermediate Value Theorem, we see that f has a single root in $(0,1)$.

On the other hand, if $\rho^{(i\Lambda)}$ and $\rho^{(j\Lambda)}$ are of opposite sign for some $i \neq j$, then observe that the derivative of f with respect to v satisfies

$$f'(v) = \frac{1}{\phi\left(\Phi^{-1}(v)\right)} \sum_{i=1}^{n-1} S_0^{(i)} \rho^{(i\Lambda)}\sigma^{(i)} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2\left(1-\left(\rho^{(i\Lambda)}\right)^2\right)}{2} + \rho^{(i\Lambda)}\sigma^{(i)}\Phi^{-1}(v)} \tag{5.37}$$

where as before ϕ denotes the standard normal density function. We see, that the denominator of $f'(v)$ is strictly positive for $v \in (0,1)$. Let as denote its numerator by $N(v)$. We see that

$$N'(v) = \frac{1}{\phi\left(\Phi^{-1}(v)\right)} \sum_{i=1}^{n-1} S_0^{(i)} \left(\rho^{(i\Lambda)}\right)^2 \left(\sigma^{(i)}\right)^2 e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2\left(1-\left(\rho^{(i\Lambda)}\right)^2\right)}{2} + \rho^{(i\Lambda)}\sigma^{(i)}\Phi^{-1}(v)} \tag{5.38}$$

is positive for $v \in (0,1)$. This means that $N(v)$ is an increasing function of v. Moreover, if there exist $\rho^{(i\Lambda)}$, $\rho^{(j\Lambda)}$ of opposite sign, for some $i \neq j$, then

$$\lim_{v \downarrow 0} N(v) = -\infty \text{ and } \lim_{v \uparrow 1} N(v) = +\infty.$$

Therefore, there exists a unique $v^* \in (0,1)$ such that $N(v^*) = 0$ and hence $f'(v^*) = 0$. Also

$$\lim_{v \downarrow 0} f(v) = +\infty \text{ and } \lim_{v \uparrow 1} f(v) = +\infty.$$

So that, $f(v)$ is either positive upon the whole interval $[0,1]$ or has a strictly negative minimum $f(v^*)$. We therefore obtain the following result concerning the infimum of f.

$$\inf_{v \in (0,1)} f(v) = f(v^*). \tag{5.39}$$

If $f(v^*) < 0$, then f is a continuous, strictly decreasing function over the interval $(0, v^*)$, which tends to ∞ as $v \downarrow 0$. Hence, there exists a unique $v_1 \in (0, v^*)$ such that $f(v_1) = 0$. Moreover, f is a continuous, strictly increasing function on $(v^*, 1)$, which tends to ∞ as $v \uparrow 1$. Therefore, from the Intermediate Value Theorem, we obtain an additional $v_2 \in (v^*, 1)$ such that $f(v_2) = 0$. If $\inf\limits_{v \in (0,1)} f(v) \geq 0$, then it is immediate that f can only have at most one root. This completes the proof. □

We see from Proposition 5.1 that either $f(v) \geq 0$ for all $v \in (0,1)$ or there exist $v_1 < v_2$ such that $f(v) \leq 0$ for all $v \in [v_1, v_2]$, with $f(v)$ positive otherwise. This then leads to the following lower bound for guaranteed annuity options.

Theorem 5.2. *Let $S_T^{(i)}$ be a process given in terms of exponential Lévy model, i.e., $S_T^{(i)} = S_0^{(i)} e^{X_T^{(i)}}$ where $i = 1, 2, ..., n-1$ and let Λ be a normally distributed random variable such that $\left(X_T^{(i)}, \Lambda\right) \sim BVN\left(\mu^{(i)}, \mu_\Lambda, \left(\sigma^{(i)}\right)^2, \sigma_\Lambda^2, \rho^{(i\Lambda)}\right)$ and let $\tilde{Q}$ be the associated probability measure. Let $f(v)$ be defined according to equation* (5.34). *Then a lower bound for the value of a GAO bought by a life of present age x with guaranteed rate g is given by*

$$C(0, x, T) \leq GAOLB_3,$$

where

$$GAOLB_3 = gSZ(0,T) \left(\sum_{i=1}^{n-1} S_0^{(i)} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2}{2}} - (K-1) \right) \tag{5.40}$$

if $f(v) \geq 0$ for all $v \in (0,1)$. Otherwise,

$$\begin{aligned} GAOLB_3 &= gSZ(0,T) \\ &\quad \times \left(\sum_{i=1}^{n-1} S_0^{(i)} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2}{2}} \Phi\left(\rho^{(i\Lambda)} \sigma^{(i)} - z_2\right) - (K-1)\Phi(-z_2) \right) \end{aligned} \tag{5.41}$$

if $\rho^{(i\Lambda)}$ are all of positive sign and

$$GAOLB_3 = gSZ(0,T)$$

$$\times\left(\sum_{i=1}^{n-1} S_0^{(i)} e^{\mu^{(i)}+\frac{\left(\sigma^{(i)}\right)^2}{2}}\left(\Phi\left(z_1-\rho^{(i\Lambda)}\sigma_i\right)+\Phi\left(\rho^{(i\Lambda)}\sigma^{(i)}-z_2\right)\right)\right.$$
$$\left.-(K-1)\left(\Phi\left(z_1\right)+\Phi\left(-z_2\right)\right)\right) \tag{5.42}$$

otherwise, where $z_1 \leq z_2$ *solve the following equation in* z

$$\sum_{i=1}^{n-1} S_0^{(i)} e^{\mu^{(i)}+\frac{\left(\sigma^{(i)}\right)^2\left(1-\left(\rho^{(i\Lambda)}\right)^2\right)}{2}+\rho^{(i\Lambda)}\sigma^{(i)}z}-(K-1)=0. \tag{5.43}$$

Proof. The case where $f(v) \geq 0$ is trivial. In the case where $f(v)<0$ for some v, we see from Proposition 3 that $f(v)=0$ has one solution in $(0,1)$ if the $\rho^{(i\Lambda)}$ are of the same sign and two otherwise. By setting $z_1=\Phi^{-1}(v)$ for each i, we obtain the solutions to equation (5.43) (where the case with $\rho_{i\Lambda}>0$ for every i is analogous to setting $z_1=-\infty$). Let z_1 and z_2 solve equation (5.43) and set $v=\Phi(z)$. Then, defining $I=(-\infty,z_1)\cup(z_2,\infty)$, we can write the stop-loss transform of S^l defined in equation (5.27) in the following way:

$$\begin{aligned}
\Psi\left(S^l,(K-1)\right) &= \tilde{E}\left[\left(\sum_{i=1}^{n-1} S_0^{(i)} e^{\mu^{(i)}+\frac{\left(\sigma^{(i)}\right)^2\left(1-\left(\rho^{(i\Lambda)}\right)^2\right)}{2}+\rho^{(i\Lambda)}\sigma^{(i)}Z}\right.\right.\\
&\qquad \left.\left.-(K-1)\right)\mathbb{1}_{\{Z\in I\}}\right]\\
&= \sum_{i=1}^{n-1}\left(S_0^{(i)} e^{\mu^{(i)}+\frac{\left(\sigma^{(i)}\right)^2\left(1-\left(\rho^{(i\Lambda)}\right)^2\right)}{2}}\right.\\
&\qquad \left.\times\left(\int_{-\infty}^{z_1} e^{\rho^{(i\Lambda)}z}\phi(z)\,dz+\int_{z_2}^{\infty} e^{\rho^{(i\Lambda)}z}\phi(z)\,dz\right)\right)\\
&\qquad -(K-1)\left(\Phi\left(z_1\right)+\Phi\left(-z_2\right)\right).
\end{aligned} \tag{5.44}$$

Therefore, we obtain equations (5.42) and (5.41). □

6. Upper Bounds for Guaranteed Annuity Options

We derive a couple of upper bounds for the Guaranteed Annuity Options.

6.1. A First Upper Bound. This section will focus on finding an upper bound for Guaranteed Annuity Options by using comonotonicity theory in a manner similar to [53], [33], [18] and [61]. [49] uses the method of Lagrange multipliers to find an upper bound for basket options.

Define the comonotonic counterpart of $\mathbf{S}=\left(S_T^{(1)},...,S_T^{(n-1)}\right)$ with $U\sim U(0,1)$ where U stands for Uniform Distribution as $\mathbf{S^u}=\left(F^{-1}_{S_T^{(1)}}(U),...,F^{-1}_{S_T^{(n-1)}}(U)\right)$. Further define

$$S^c=\sum_{i=1}^{n-1} F^{-1}_{S_T^{(i)}}(U)=\sum_{i=1}^{n-1} S_i^c. \tag{6.1}$$

Clearly [see for example in 30],

$$S \leq_{cx} S^c \tag{6.2}$$

In other words,

$$\tilde{E}\left[\left(\sum_{i=1}^{n-1} S_T^{(i)} - (K-1)\right)^+\right] \leq \tilde{E}\left[\left(\sum_{i=1}^{n-1} S_i^c - (K-1)\right)^+\right] \tag{6.3}$$

and we have

$$\tilde{E}\left[\left(\sum_{i=1}^{n-1} S_i^c - (K-1)\right)^+\right] = \sum_{i=1}^{n-1} \tilde{E}\left[\left(S_T^{(i)} - F^{-1}_{S_T^{(i)}}\left(F_{S^c}\left((K-1)\right)\right)\right)^+\right] - K_3 \tag{6.4}$$

where

$$K_3 = \left((K-1) - F^{-1}_{S^c}\left(F_{S^c}\left(K-1\right)\right)\right)\left(1 - F_{S^c}\left(K-1\right)\right) \tag{6.5}$$

and it is understood that $(K-1) \in \left(F^{-1+}_{S^c}(0), F^{-1}_{S^c}(1)\right)$. As a result, an upper bound for GAO is given as

$$C(0,x,T) \leq gSZ(0,T)\left(\sum_{i=1}^{n-1} \tilde{E}\left[\left(S_i - F^{-1}_{S_T^{(i)}}\left(F_{S^c}\left(K-1\right)\right)\right)^+\right] - K_3\right) \tag{6.6}$$

where K_3 is defined in (6.5). Further we write the upper bound given above as

$$C(0,x,T) \leq gSZ(0,T)\left(\sum_{i=1}^{n-1} \tilde{E}\left[\left(S_T^{(i)} - F^{-1}_{S_T^{(i)}}\left(x'\right)\right)^+\right] - K_3\right) := \text{ GAOUB}_1 \tag{6.7}$$

where $x' \in (0,1)$ (see for example [30]) is the solution of the equation

$$\sum_{i=1}^{n-1} F^{-1}_{S_T^{(i)}}\left(x'\right) = K - 1. \tag{6.8}$$

6.2. An Improved Upper Bound by conditioning. We can improve on the upper bound obtained above by finding a conditioning variable Λ under which the S_i are dependent. This is discussed in detail for basket options in [29] using a choice of Λ such that $\Lambda \geq d_\Lambda$ implies $\sum_{i=1}^{n-1} S_T^{(i)} \geq (K-1)$. We use a different more simplified approach for GAOs. We assume that some additional information is available concerning the stochastic nature of $\left(S_T^{(1)}, S_T^{(2)}, ..., S_T^{(n-1)}\right)$. That is, we can find a random variable Λ, with a known distribution, such that the individual conditional distributions of $S_T^{(i)}$ given the event $\Lambda = \lambda$ are known for all i and all possible values of λ. Such an approach can be found in [53], [30] and [31].

Define

$$S^u = \sum_{i=1}^{n-1} F^{-1}_{S_T^{(i)}|\Lambda}(U) = \sum_{i=1}^{n-1} S_i^u, \tag{6.9}$$

where $U \sim U(0,1)$. Then we have

$$S \leq_{cx} S^u \leq_{cx} S^c \tag{6.10}$$

Now let $\mathbf{S^u} = \left(S_1^u, ..., S_{n-1}^u\right)$. Since $\left(F^{-1}_{S_T^{(1)}|\Lambda=\lambda}, ..., F^{-1}_{S_T^{(n-1)}|\Lambda=\lambda}\right)$ is comonotonic, we have,

$$F^{-1}_{S^u|\Lambda=\lambda}(p) = \sum_{i=1}^{n-1} F^{-1}_{S_T^{(i)}|\Lambda=\lambda}(p), \; p \in (0,1). \tag{6.11}$$

It follows that, in this case

$$\sum_{i=1}^{n-1} F^{-1}_{S_T^{(i)}|\Lambda=\lambda}\left(F_{S^u|\Lambda=\lambda}(K-1)\right) = K-1. \tag{6.12}$$

and so we have

$$\begin{aligned} f(\lambda) &= \tilde{E}\left[\left(\sum_{i=1}^{n-1} S_i^u - (K-1)\right)^+ \middle| \Lambda = \lambda\right] \\ &= \sum_{i=1}^{n-1} \tilde{E}\left[\left(S_T^{(i)} - F^{-1}_{S_T^{(i)}|\Lambda=\lambda}\left(F_{S^u|\Lambda=\lambda}(K-1)\right)\right)^+ \middle| \Lambda = \lambda\right] - K_4 \end{aligned} \tag{6.13}$$

where

$$K_4 = \left((K-1) - F^{-1}_{S^u|\Lambda=\lambda}\left(F_{S^u|\Lambda=\lambda}(K-1)\right)\right)\left(1 - F_{S^u|\Lambda=\lambda}(K-1)\right) \tag{6.14}$$

and it is clear that $(K-1) \in \left(F^{-1+}_{S^u|\Lambda=\lambda}(0), F^{-1}_{S^u|\Lambda=\lambda}(1)\right)$. By applying the tower property and using the convex order relationship given by (6.10), we obtain an upper bound for GAO, i.e.,

$$\begin{aligned} C(0,x,T) &\leq gSZ(0,T)\,\tilde{E}\left[\left(S^u - (K-1)\right)^+\right] \\ &= gSZ(0,T)\,\tilde{E}\left[f(\lambda)\right] \\ &= gSZ(0,T) \end{aligned}$$

$$\times\left(\sum_{i=1}^{n-1}\int_{-\infty}^{\infty} \tilde{E}\left[\left(S_T^{(i)} - F^{-1}_{S_T^{(i)}|\Lambda=\lambda}\left(F_{S^u|\Lambda=\lambda}(K-1)\right)\right)^+ \middle| \Lambda = \lambda\right] dF_\Lambda(\lambda) - K_4\right) \tag{6.15}$$

where K_4 is defined in (6.14). Given the event $\Lambda = \lambda$, let $x^{'}$ be the solution to the following equation

$$\sum_{i=1}^{n-1} F^{-1}_{S_T^{(i)}|\Lambda=\lambda}\left(x^{'}\right) = K-1. \tag{6.16}$$

Further, we see from equation (6.12), that $x^{'} = F_{S^u|\Lambda=\lambda}(K-1)$. It therefore follows, as a result of equation 93 of [30] that an upper bound for GAO is given as

$$C(0,x,T) \quad \leq \quad gSZ(0,T)$$

$$\begin{aligned}
&\times \quad \left(\sum_{i=1}^{n-1}\int_{-\infty}^{\infty}\tilde{E}\left[\left(S_T^{(i)}-F^{-1}_{S_T^{(i)}|\Lambda=\lambda}\left(x^{'}\right)\right)^{+}\middle|\Lambda=\lambda\right]dF_{\Lambda}\left(\lambda\right)-K_4\right)\\
&=: \quad \mathrm{GAOUB}_j^{(2)},
\end{aligned} \tag{6.17}$$

where $x^{'}$ is obtained by solving (6.16).

Since the above upper bound is an upper bound for all j, it follows that we can find the optimal upper bound by maximizing equation (6.17) over $j \in \{1, 2, ..., n-1\}$. As remarked earlier, this bound improves upon the unconditional bound given by (6.7). In case if the marginal cdfs $F_{S_T^{(i)}|\Lambda}$ are strictly increasing, one can put $K_4 = 0$ in (6.17) to obtain the upper bound.

6.3. An Upper Bound based on the Arithmetic-Geometric Mean Inequality. In order to obtain an upper bound for GAOs which is directly applicable to the affine set up, we make use of arithmetic-geometric mean inequality in a manner similar to [16] who used this methodology to arrive at an upper bound for basket options.

Let us first define the arithmetic and geometric mean of the $(n-1)$ pure endowments appearing in the payoff of GAO (c.f. (3.12)) respectively as

$$A_T^{(n-1)} = \frac{1}{n-1}\sum_{i=1}^{n-1} S_T^{(i)} \tag{6.18}$$

and

$$G_T^{(n-1)} = \left(\prod_{i=1}^{n-1} S_T^{(i)}\right)^{\frac{1}{n-1}}, \tag{6.19}$$

where $S_T^{(i)}$; $i = 1, 2, ..., n-1$ are defined in equation (3.13). It is well known that

$$A_T^{(n-1)} \geq G_T^{(n-1)} \quad a.s. \tag{6.20}$$

Further, let us define the log-geometric average as

$$Y_T^{(n-1)} = \frac{1}{n-1}\sum_{i=1}^{n-1} \ln S_T^{(i)}. \tag{6.21}$$

Next we define as in equation (4.9),

$$X_T^{(i)} = \ln\left(\frac{S_T^{(i)}}{S_0^{(i)}}\right); \ i = 1, 2, ..., n-1. \tag{6.22}$$

Further, we assume that the joint characteristic function of $\left(X_T^{(1)}, ..., X_T^{(n-1)}\right)$ can be obtained under the transformed measure $\tilde{Q}$, where we define

$$\phi_T\left(\gamma\right) = \tilde{E}\left[e^{i\sum_{k=1}^{n-1}\gamma_k X_T^{(k)}}\right] \tag{6.23}$$

with $\boldsymbol{\gamma} = [\gamma_1, \gamma_2, ..., \gamma_{n-1}]$. As the next step, we obtain the relationship between log-geometric average and $X_T^{(i)}$'s as follows

$$\begin{aligned} Y_T^{(n-1)} &= \frac{1}{n-1}\sum_{i=1}^{n-1} \ln\left(\frac{S_T^{(i)}}{S_0^{(i)}} S_0^{(i)}\right) \\ &= \frac{1}{n-1}\sum_{i=1}^{n-1} X_T^{(i)} + Y_0^{(n-1)}. \end{aligned} \tag{6.24}$$

Next, we try to express the characteristic function of log-geometric average under the transformed measure $\tilde{Q}$ in terms of the joint characteristic function of $X_T^{(i)}$'s viz. $\phi_T(\boldsymbol{\gamma})$ defined in equation (6.23). Let $\phi_{Y_T}(\gamma_0)$ denote the characteristic function of log-geometric average $Y_T^{(n-1)}$ with parameter γ_0. Then we have

$$\begin{aligned} \phi_{Y_T}(\gamma_0) &= \tilde{E}\left[e^{i\gamma_0 Y_T^{(n-1)}}\right] \\ &= \tilde{E}\left[e^{i\gamma_0 Y_0^{(n-1)} + i\sum_{k=1}^{n-1}\left(\frac{\gamma_0}{n-1}\right)X_T^{(k)}}\right] \\ &= e^{i\gamma_0 Y_0^{(n-1)}}\phi_T\left(\frac{\gamma_0}{n-1}\mathbf{1}\right) \end{aligned} \tag{6.25}$$

where $\mathbf{1} = (1, 1, ..., 1)$ is a $1 \times (n-1)$ vector of 1's, so that $\frac{\gamma_0}{n-1}\mathbf{1}$ is $1 \times (n-1)$ vector with components $\frac{\gamma_0}{n-1}$ and $\phi_T(\boldsymbol{\gamma})$ is defined in (6.23). In light of equation (6.18), we can express the GAO pay-off formula given in equation (3.12) as

$$C(0, x, T) = g(n-1)\, SZ(0, T)\, \tilde{E}\left[\left(A_T^{(n-1)} - K'\right)^+\right], \tag{6.26}$$

where

$$K' = \frac{K-1}{n-1}. \tag{6.27}$$

Adding and subtracting $G_T^{(n-1)}$ within the *max* function on R.H.S. of equation (6.26), and exploiting equation (6.20), we obtain an upper bound of GAO as

$$\begin{aligned} C(0, x, T) &\leq g(n-1)\, SZ(0, T) \\ &\times \left(\tilde{E}\left[\left(G_T^{(n-1)} - K'\right)^+\right] + \tilde{E}\left[A_T^{(n-1)}\right] - \tilde{E}\left[G_T^{(n-1)}\right]\right) \\ &=: \text{GAOUB} \end{aligned} \tag{6.28}$$

We make use of Fourier inversion to compute the call type expectation involved in the upper bound and we state the result in the following proposition.

Proposition 6.1. *Given the geometric mean of $n-1$ pure endowments defined in equation* (6.19) *and $K' > 0$,*

$$\tilde{E}\left[\left(G_T^{(n-1)} - K'\right)^+\right] = \frac{e^{-\delta \ln K'}}{\pi}\int_0^\infty e^{-i\eta \ln K'}\,\Psi_T^G(\eta; \delta)\, d\eta \tag{6.29}$$

where $\Psi_T^G(\eta;\delta)$ *denotes the Fourier transform of* $\tilde{E}\left[\left(G_T^{(n-1)}-K^{'}\right)^+\right]$ *with respect to* $\ln K^{'}$ *along with the damping factor* $e^{\delta\ln K^{'}}$ *such that*

$$\Psi_T^G(\eta;\delta)=e^{i(\eta-i(\delta+1))Y_0^{(n-1)}}\frac{\phi_T\left(\frac{\eta-i(\delta+1)}{n-1}\mathbf{1}\right)}{\delta^2+\delta-\eta^2+i\eta\left(2\delta+1\right)}, \tag{6.30}$$

where the parameter δ *tunes the damping factor [c.f. 17, 16] and* $\phi_T(.)$ *is defined in equation* (6.23).

Proof. Let $f_{Y_T}(y)$ denote the probability density function (p.d.f.) of the log-geometric average $Y_T^{(n-1)}$. We introduce the damping factor in accordance with [17]. Then, by definition, the Fourier transform of $\tilde{E}\left[\left(G_T^{(n-1)}-K^{'}\right)^+\right]$ with respect to $\ln K^{'}$ along with the damping factor $e^{\delta\ln K^{'}}$ is given as

$$\begin{aligned}
\Psi_T^G(\eta;\delta) &= \int_{\mathbb{R}} e^{i\eta\ln K^{'}+\delta\ln K^{'}}\tilde{E}\left[\left(e^{Y_T^{(n-1)}}-K^{'}\right)^+\right]d\ln K^{'}\\
&= \int_{\mathbb{R}} e^{i\eta\ln K^{'}+\delta\ln K^{'}}\int_{\ln K^{'}}^{\infty}\left(e^y-K^{'}\right)f_{Y_T}(y)\;dy\;d\ln K^{'}\\
&= \int_{\mathbb{R}} e^{i\eta\ln K^{'}+\delta\ln K^{'}}\int_{\ln K^{'}}^{\infty}e^y f_{Y_T}(y)\;dy\;d\ln K^{'}\\
&\quad -\int_{\mathbb{R}} e^{i\eta\ln K^{'}+\delta\ln K^{'}}\int_{\ln K^{'}}^{\infty}K^{'}f_{Y_T}(y)\;dy\;d\ln K^{'}\\
&= \Psi_T^{G_1}(\eta;\delta)-\Psi_T^{G_2}(\eta;\delta).
\end{aligned}\tag{6.31}$$

We evaluate both integrals by adopting a change of order of integration, as detailed below

$$\begin{aligned}
\Psi_T^{G_1}(\eta;\delta) &= \int_{\mathbb{R}} e^y\left(\int_{-\infty}^{y}e^{i\eta\ln K^{'}+\delta\ln K^{'}}d\ln K^{'}\right)f_{Y_T}(y)\,dy\\
&= \frac{1}{i\eta+\delta}\int_{\mathbb{R}}e^{i(\eta-i(\delta+1))y}f_{Y_T}(y)\,dy\\
&= \frac{\phi_{Y_T}\left(\eta-i\left(\delta+1\right)\right)}{i\eta+\delta}\\
&= e^{i(\eta-i(\delta+1))Y_0^{(n-1)}}\frac{\phi_T\left(\frac{\eta-i(\delta+1)}{n-1}\mathbf{1}\right)}{i\eta+\delta}.
\end{aligned}\tag{6.32}$$

where the last couple of statements follow from the definition of the characteristic function of $Y_0^{(n-1)}$ given in (6.25) and its link to the joint characteristic function of joint characteristic function of $\left(X_T^{(1)},...,X_T^{(n-1)}\right)$ defined in (6.23). On the same lines we have

$$\Psi_T^{G_2}(\eta;\delta)=e^{i(\eta-i(\delta+1))Y_0^{(n-1)}}\frac{\phi_T\left(\frac{\eta-i(\delta+1)}{n-1}\mathbf{1}\right)}{i\eta+(\delta+1)}. \tag{6.33}$$

Substituting $\Psi_T^{G_1}(\eta;\delta)$ and $\Psi_T^{G_2}(\eta;\delta)$ in equation (6.31), remembering the damping factor we get the requisite result given in equation (6.29). $\square$

In a similar manner we obtain

$$\tilde{E}\left[G_T^{(n-1)}\right] = e^{Y_0^{(n-1)}}\phi_T\left(\frac{-i}{n-1}\mathbf{1}\right). \tag{6.34}$$

We then plug the formulae (6.29) and (6.34) into equation (6.28) to obtain the upper bound GAOUB.

6.3.1. *The Upper Bound under the Affine Set Up.* Consider the affine set up of section 5.4 (c.f. equations (4.8)-(4.11)). Let ϕ_{X_T} denote the characteristic function of X_T with parameter Λ under the transformed measure $\tilde{Q}$ so that

$$\phi_{X_T}(\Lambda) = \tilde{E}\left[e^{i\langle\Lambda,\, X_T\rangle}\right]. \tag{6.35}$$

Now using equation (4.11), we see that the joint characteristic function of the random vector $\left(X_T^{(1)},...,X_T^{(n-1)}\right)$ under the transformed measure $\tilde{Q}$, given in equation (6.23) becomes ,

$$\phi_T^{aff}(\gamma) = \phi_{X_T}\left(-\sum_{k=1}^{n-1}\gamma_k\tilde{\psi}(k,R+M)\right), \tag{6.36}$$

where $\left(-\sum_{k=1}^{n-1}\gamma_k\tilde{\psi}(k,R+M)\right)$ is the parameter of the characteristic function, with $\tilde{\psi}(k,R+M)$ satisfying the equations (4.5) with $\tau = k$. As a result, $\Psi_T^G(\eta;\delta)$ given in equation (6.30) can be written in a more compact way as

$$\Psi_T^{G^{aff}}(\eta;\delta) = e^{i(\eta-i(\delta+1))Y_0^{(n-1)}}\frac{\phi_{X_T}\left(-\frac{(\eta-i(\delta+1))}{n-1}\sum_{k=1}^{n-1}\tilde{\psi}(k,R+M)\right)}{\delta^2+\delta-\eta^2+i\eta(2\delta+1)}. \tag{6.37}$$

Similarly, we have from equation (6.34),

$$\tilde{E}^{aff}\left[G_T^{(n-1)}\right] = e^{Y_0^{(n-1)}}\phi_{X_T}\left(\frac{i}{n-1}\sum_{k=1}^{n-1}\tilde{\psi}(k,R+M)\right). \tag{6.38}$$

Moreover, using the definition of arithmetic average given in equation (6.18) and utilizing (4.8), we see that

$$\tilde{E}^{aff}\left[A_T^{(n-1)}\right] = \frac{1}{n-1}\sum_{k=1}^{n-1}\left(e^{-\left((\bar{r}+\bar{\mu})k+\tilde{\phi}(k,R+M)\right)}\mathcal{L}\left(\tilde{\psi}(k,R+M)\right)\right), \tag{6.39}$$

where as defined in Section 5.5.1, $\mathcal{L}$ denotes the Laplace transform of X_T with parameter given as $\tilde{\psi}(k,R+M)$ under the transformed measure $\tilde{Q}$. Finally we substitute equation (6.37) in the expression (6.29) and then the result and equations (6.38)-(6.39) into (6.28) to get

$$\begin{aligned}\text{GAOUB}^{aff} &= g(n-1)\,SZ(0,T)\\ &\times\left(\frac{1}{n-1}\sum_{k=1}^{n-1}\left(e^{-\left((\bar{r}+\bar{\mu})k+\tilde{\phi}(k,R+M)\right)}\mathcal{L}\left(\tilde{\psi}(k,R+M)\right)\right)\right.\end{aligned}$$

$$-e^{Y_0^{(n-1)}}\phi_{X_T}\left(\frac{i}{n-1}\sum_{k=1}^{n-1}\tilde{\psi}(k,R+M)\right)$$
$$+\frac{e^{-\delta\ln K'}}{\pi}\int_0^{\infty}\frac{e^{-i\left(\eta\ln K'-(\eta-i(\delta+1))Y_0^{(n-1)}\right)}}{\delta^2+\delta-\eta^2+i\eta(2\delta+1)}\times$$
$$\phi_{X_T}\left(-\frac{(\eta-i(\delta+1))}{n-1}\sum_{k=1}^{n-1}\tilde{\psi}(k,R+M)\right)d\eta\Bigg), \quad (6.40)$$

where $\phi_{X_T}(.)$ is defined in equation (6.35) and $\mathcal{L}$ denotes the Laplace transform of X_T under the transformed measure $\tilde{Q}$.

7. Example: The Vasicek Model

We now derive lower and upper bounds by choosing a particular models for the interest rate and force of mortality viz. the Vasicek Model

Let us consider the case where the interest rate (r_t) and the force of mortality (μ_t) for an insured aged x at time 0 obey the Vasicek model [c.f. 74], with dynamics given by

$$dr_t = a(b-r_t)\,dt + \sigma dW_t^1, \quad (7.1)$$

where a, b and σ are positive constants and W_t^1 is a standard Brownian motion under the probability measure $\mathbb{Q}$ and

$$d\mu_t = c\mu_t dt + \xi dZ_t, \quad (7.2)$$

where c and ξ are positive constants and Z_t is also a standard Brownian motion under the EMM $\mathbb{Q}$ correlated with W_t^1 so that

$$dW_t^1 dZ_t = \rho dt. \quad (7.3)$$

This means that $Z_t = \rho W_t^1 + \sqrt{1-\rho^2}W_t^2$, where W_t^2 is a standard Brownian motion independent of W_t^1. It is important to fine tune the model in case of mortality by choosing c and ξ properly to avoid the possibility of negative mortality rates. In fact, under this model, we have [c.f. 62, for details]

$$S_T^{(i)} = S_0^{(i)}e^{X_T^{(i)}} \quad (7.4)$$

where

$$S_0^{(i)} = \alpha^{(i)} \quad (7.5)$$

with

$$\alpha^{(i)} = e^{D^{(i)}+\tilde{H}^{(i)}} \quad (7.6)$$

such that for $i = 1, 2, ..., n-1$

$$D^{(i)} = \left(b-\frac{\sigma^2}{2a^2}\right)\left(A^{(i)}-i\right)-\frac{\sigma^2}{4a}\left(A^{(i)}\right)^2 \quad (7.7)$$

with

$$A^{(i)} = \frac{1-e^{-ai}}{a} \quad (7.8)$$

and

$$\tilde{H}^{(i)} = \left(\frac{\rho\sigma\xi}{ac}-\frac{\xi^2}{2c^2}\right)\left(\tilde{G}^{(i)}-i\right)+\frac{\rho\sigma\xi}{ac}\left(A^{(i)}-\phi^{(i)}\right)+\frac{\xi^2}{4c}\left(\tilde{G}^{(i)}\right)^2 \quad (7.9)$$

with

$$\tilde{G}^{(i)} = \frac{e^{ci} - 1}{c} \tag{7.10}$$

and

$$\phi^{(i)} = \frac{1 - e^{(a-c)i}}{a - c}. \tag{7.11}$$

Further $\left\{X_T^{(i)}\right\}_{i=1,2,\ldots,n-1}$ is defined as:

$$X_T^{(i)} = -\left(A^{(i)} r_T + \tilde{G}^{(i)} \mu_T\right), \tag{7.12}$$

where $A^{(i)}$ and $\tilde{G}^{(i)}$ are defined respectively in equations (7.8) and (7.10). Here we have [c.f. 62] under the probability measure $\tilde{Q}$,

$$(r_T,\, \mu_T) \sim \text{BVN}\left(\mu_{r_T}, \mu_{\mu_T}, \sigma^2_{r_T}, \sigma^2_{\mu_T}, \rho\left(r_T, \mu_T\right)\right) \tag{7.13}$$

where BVN stands for bivariate normal distribution and

$$\begin{aligned}
\mu_{r_T} &= \tilde{E}\left[r_T\right] \\
&= e^{-aT} r_0 + b\left(1 - e^{-aT}\right) - \frac{\sigma^2}{2a^2}\left(1 - e^{-aT}\right)^2 \\
&- \frac{\rho\sigma\xi}{c}\left[\frac{e^{cT}\left(e^{-cT} - e^{-aT}\right)}{a - c} - \frac{1 - e^{-aT}}{a}\right],
\end{aligned} \tag{7.14}$$

$$\sigma^2_{r_T} = \frac{\sigma^2}{2a}\left(1 - e^{-2aT}\right), \tag{7.15}$$

$$\mu_{\mu_T} = \tilde{E}\left[\mu_T\right] = e^{cT}\mu_0 - \frac{\xi^2}{2c^2}\left(1 - e^{cT}\right)^2 - \frac{\rho\sigma\xi}{a}\left[\frac{e^{-aT}\left(e^{aT} - e^{cT}\right)}{a - c} - \frac{e^{cT} - 1}{c}\right], \tag{7.16}$$

$$\sigma^2_{\mu_T} = \frac{\xi^2}{2c}\left(e^{2cT} - 1\right), \tag{7.17}$$

and

$$Cov\left[r_T, \mu_T\right] = \frac{\rho\sigma\xi}{a - c}\left(1 - e^{-(a-c)t}\right) \tag{7.18}$$

with Cov standing for covariance. In light of equation (7.12) and (7.13), it is clear that

$$X_T^{(i)} \sim N\left(\mu^{(i)}, \left(\sigma^{(i)}\right)^2\right) \tag{7.19}$$

where $\mu^{(i)}$ and $\left(\sigma^{(i)}\right)^2$ are defined respectively in equations (5.12) and (5.13) are given as follows in the context of the Vasicek model.

$$\mu^{(i)} = -\left(A^{(i)}\mu_{r_T} + \tilde{G}^{(i)}\mu_{\mu_T}\right) \tag{7.20}$$

$$\left(\sigma^{(i)}\right)^2 = \left(A^{(i)}\right)^2 \sigma^2_{r_T} + \left(\tilde{G}^{(i)}\right)^2 \sigma^2_{\mu_T} + 2A^{(i)}\tilde{G}^{(i)} Cov\left[r_T, \mu_T\right]. \tag{7.21}$$

In fact, one may write

$$X_T^{(i)} = -W_T^{(i)}, \tag{7.22}$$

where $W_T^{(i)} \sim N\left(-\mu^{(i)}, \left(\sigma^{(i)}\right)^2\right)$. Finally, for $i \neq j$ we note that

$$\rho^{(ij)} = Corr\left(X_T^{(i)}, X_T^{(j)}\right) = Corr\left(W_T^{(i)}, W_T^{(j)}\right) \tag{7.23}$$

where $Corr$ stands for correlation and for $i \neq j = 1, 2, ..., n-1$

$$\rho^{(ij)} = \frac{A^{(i)}A^{(j)}\sigma_{r_T}^2 + \left(A_i\tilde{G}^{(j)} + A^{(j)}\tilde{G}^{(i)}\right) Cov\left[r_T, \mu_T\right] + \tilde{G}^{(i)}\tilde{G}^{(j)}\sigma_{\mu_T}^2}{\sigma^{(i)}\sigma^{(j)}}. \tag{7.24}$$

The computation of the price bounds for GAO hinges upon the availability of the price of SZCBs $SZ(0,T)$. We refer to [62] for the price of these instruments under the Vasicek model and note that

$$SZ(0,T) = \alpha^{(0)}e^{V^{(0)}} \tag{7.25}$$

with

$$\alpha^{(0)} = e^{D^{(0)}+\tilde{H}^{(0)}} \tag{7.26}$$

where

$$D^{(0)} = \left(b - \frac{\sigma^2}{2a^2}\right)\left(A^{(0)} - T\right) - \frac{\sigma^2}{4a}\left(A^{(0)}\right)^2 \tag{7.27}$$

with

$$A^{(0)} = \frac{1 - e^{-aT}}{a} \tag{7.28}$$

and

$$\tilde{H}^{(0)} = \left(\frac{\rho\sigma\xi}{ac} - \frac{\xi^2}{2c^2}\right)\left(\tilde{G}^{(0)} - T\right) + \frac{\rho\sigma\xi}{ac}\left(A^{(0)} - \phi^{(0)}\right) + \frac{\xi^2}{4c}\left(\tilde{G}^{(0)}\right)^2 \tag{7.29}$$

with

$$\tilde{G}^{(0)} = \frac{e^{cT} - 1}{c} \tag{7.30}$$

and

$$\phi^{(0)} = \frac{1 - e^{(a-c)T}}{a - c}. \tag{7.31}$$

and finally

$$V^{(0)} = -\left(A^{(0)}r_0 + \tilde{G}^{(0)}\mu_0\right), \tag{7.32}$$

where $A^{(0)}$ and $\tilde{G}^{(0)}$ are defined respectively in equations (7.28) and (7.30) and r_0 and μ_0 are the initial (time 0) values of the interest rate and mortality rate. We now derive lower and upper bounds for the Vasicek model on the lines of $\mathrm{GAOLB}_j^{(2)}$ and $\mathrm{GAOUB}_j^{(2)}$ respectively.

7.0.1. *The Lower Bound $GAOLB_j^{(VS)}$.* We know that if a 2-dimensional random vector $(X, Y) \sim \text{BVN}\left(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho\right)$, the conditional distribution of the log-normal random variable e^X, given the event $e^Y = y$ is given as

$$F_{e^X|e^Y=y}(x) = \Phi\left(\frac{\log_e x - \left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(\log_e y - \mu_Y)\right)}{\sigma_X\sqrt{1-\rho^2}}\right). \tag{7.33}$$

where Φ denotes the c.d.f. of standard normal distribution. Clearly for two assets, say the ith and jth asset in the basket considered above, it is evident from (7.19) and (7.23) that $(X_i, X_j) \sim \text{BVN}\left(\mu^{(i)}, \mu^{(j)}, \left(\sigma^{(i)}\right)^2, \left(\sigma^{(j)}\right)^2, \rho^{(ij)}\right)$. Further from equation (7.4) as $S_T^{(i)} = S_0^{(i)} e^{X_T^{(i)}}$, we have from equation (7.33) that the distribution function of $S_T^{(i)}$ conditional on the event $S_T^{(j)} = s$ is given as

$$F_{S_T^{(i)}|S_T^{(j)}=s}(x) = \Phi(a(x))$$

where $a(x)$ is given by

$$a(x) = \frac{\log_e x - \left(\log\left(S_0^{(i)}\right) + \mu^{(i)} + \rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}\left(\log\left(\frac{s}{S_0^{(j)}}\right) - \mu^{(j)}\right)\right)}{\sigma^{(i)}\sqrt{\left(1-\left(\rho^{(ij)}\right)^2\right)}}. \tag{7.34}$$

As the differentiation of c.d.f. yields the p.d.f., therefore the conditional density function of $S_T^{(i)}$ given $S_T^{(j)} = s$ satisfies the following equation:

$$f_{S_T^{(i)}|S_T^{(j)}=s}(x) = \frac{1}{x\sigma^{(i)}\sqrt{\left(1-\left(\rho^{(ij)}\right)^2\right)}}\phi(a(x)), \tag{7.35}$$

where ϕ denotes the p.d.f. of standard normal distribution. As a result, the conditional expectation of $S_T^{(i)}$ given $S_T^{(j)}$ is given by the expression

$$\tilde{E}\left(S_T^{(i)}|S_T^{(j)}\right) = S_0^{(i)}\left(\frac{S_T^{(j)}}{S_0^{(j)}}\right)^{\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2\left(1-\left(\rho^{(ij)}\right)^2\right)}{2} - \rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}\mu^{(j)}} \tag{7.36}$$

Invoking equation (5.15) and denoting the individual components of the sum on the r.h.s. of equation (7.36) as Y_{ij}, we see that under the assumption $\rho^{(ij)} \geq 0\ \forall i, j$, the vector $\left(Y_{1j}, Y_{2j}, ..., Y_{(n-1)j}\right)$ is comonotonic, and so define

$$S_j^{l_3} = \sum_{i=1}^{n-1} Y_{ij} \tag{7.37}$$

and from (7.36), (5.15) and (7.37), it is evident that

$$S \geq_{cx} S_j^{l_3}. \tag{7.38}$$

Further, the stop-loss transform of $S_j^{l_3}$ can be written as the sum of stop-loss transform of its components [see for example in 30], i.e.,

$$\tilde{E}\left[\left(S_j^{l_3}-(K-1)\right)^+\right]=\sum_{i=1}^{n-1}\tilde{E}\left[\left(Y_{ij}-F_{Y_{ij}}^{-1}\left(F_{S_j^{l_3}}(K-1)\right)\right)^+\right] \tag{7.39}$$

where $F_{S_j^{l_3}}(K-1)$ is the distribution function of S^{l_3} evaluated at $K-1$ so that we have:

$$\begin{aligned} F_{S_j^{l_3}}(K-1) &= \mathbf{P}\left[S_j^{l_3}\leq(K-1)\right] \\ &= \mathbf{P}\left[\sum_{i=1}^{n-1}S_0^{(i)}\left(\frac{S_T^{(j)}}{S_0^{(j)}}\right)^{\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}} e^{\mu^{(i)}+\frac{\left(1-\left(\rho^{(ij)}\right)^2\right)}{2}-\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}\mu^{(j)}}\right. \\ &\leq \left.(K-1)\right] \end{aligned} \tag{7.40}$$

In fact $S_j^{l_3}\leq(K-1)$ if and only if $S_T^{(j)}\leq xS_0^{(j)}$ provided that $\rho^{(ij)}\geq 0\ \forall\ i,j$, where we substitute x for $S_j/S_0^{(j)}$ in the above expression and obtain its value by solving the following equation

$$\sum_{i=1}^{n-1}S_0^{(i)}(x)^{\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}} e^{\mu^{(i)}+\frac{\left(\sigma^{(i)}\right)^2\left(1-\left(\rho^{(ij)}\right)^2\right)}{2}-\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}\mu^{(j)}}-(K-1)=0. \tag{7.41}$$

As a result, we have:

$$\begin{aligned} F_{S_j^{l_3}}(K-1) &= F_{S_j}\left(xS_0^{(j)}\right) \\ &= F_{Y_T^{(ij)}}\left(S_0^{(i)}(x)^{\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}} e^{\mu^{(i)}+\frac{\left(\sigma^{(i)}\right)^2\left(1-\left(\rho^{(ij)}\right)^2\right)}{2}-\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}\mu^{(j)}}\right). \end{aligned} \tag{7.42}$$

Using this result in (7.39) along with the convex order relationship between S and $S_j^{l_3}$ as given by equation (7.38), we obtain

$$\begin{aligned} C(0,x,T) &\geq gSZ(0,T)\left(\sum_{i=1}^{n-1}S_0^{(i)}\left(S_0^{(j)}\right)^{-\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}}\right. \\ &\quad\times\left. e^{\mu^{(i)}+\frac{\left(\sigma^{(i)}\right)^2\left(1-\rho_{ij}^2\right)}{2}-\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}\mu^{(j)}}P\left(xS_0^{(j)},T,\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}},j\right)\right) \end{aligned} \tag{7.43}$$

where $\mu^{(i)}$, $\left(\sigma^{(i)}\right)^2$ and $\rho^{(ij)}$ for the Vasicek model are given respectively in (7.20), (7.21) and (7.23) and P is defined in (5.25) so that we have

$$P\left(xS_0^{(j)},T,\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}},j\right) = \left(S_0^{(j)}\right)^{\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}}\left(e^{\rho^{(ij)}\frac{\sigma^{(i)}}{2\sigma^{(i)}}\left(\rho^{(ij)}\sigma^{(i)}\sigma^{(j)}-2\mu^{(j)}\right)}\right.$$

$$\times \Phi\left(d^{(1j)}\right) - x^{\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}}\Phi\left(d^{(2j)}\right)\Bigg), \tag{7.44}$$

where d_{2j} and d_{1j} are given respectively as

$$d^{(2j)} = \frac{\log_e\left(\frac{1}{x}\right) - \mu^{(j)}}{\sigma^{(j)}} \tag{7.45}$$

$$d^{(1j)} = d^{(2j)} + \rho^{(ij)}\sigma^{(i)} \tag{7.46}$$

Inserting (7.44) in (7.43), we achieve the lower bound $\text{GAOLB}_j^{(VS)}$ as follows

$$\begin{aligned} C(0,x,T) \quad &\geq \quad gSZ\left(0,T\right)\left(\sum_{i=1}^{n-1} S_0^{(i)} e^{\mu^{(i)} + \frac{\left(\sigma^{(i)}\right)^2\left(1-\left(\rho^{(ij)}\right)^2\right)}{2} - \rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}\mu^{(j)}} \times\right. \\ &\left.\left(e^{\rho^{(ij)}\frac{\sigma^{(i)}}{2\sigma^{(j)}}\left(\rho^{(ij)}\sigma^{(i)}\sigma^{(j)} - 2\mu^{(j)}\right)}\Phi\left(d^{(1j)}\right) - x^{\rho^{(ij)}\frac{\sigma^{(i)}}{\sigma^{(j)}}}\Phi\left(d^{(2j)}\right)\right)\right) \\ &=: \quad \text{GAOLB}_j^{(VS)}. \end{aligned} \tag{7.47}$$

Since the above lower bound is a lower bound for every j, we can maximise this for $j \in \{1, 2, ..., n-1\}$ to obtain an optimal lower bound for GAO in the Vasicek Case.

7.0.2. *The Improved Upper Bound $GAOUB_j^{(VS)}$.* In section 5.6.2, we have shown that the upper bound SWUB_1 can be improved by assuming that there exists a random variable Λ such that $\text{Cov}\left(S_T^{(i)}, \Lambda\right) \neq 0 \ \forall i$. Suppose this assumption is true here and we choose

$$\Lambda = \sum_{k=1}^{n-1} Y_T^{(k)} \tag{7.48}$$

where

$$Y_T^{(k)} = \frac{X_T^{(k)} - \mu^{(k)}}{\sigma^{(k)}} \tag{7.49}$$

where in the context of the Vasicek Model, $X_T^{(k)}$, $\mu^{(k)}$ and $\sigma^{(k)}$ are defined respectively in equations (7.12), (7.20) and (7.21) and it is evident from (7.19) that

$$Y_T^{(k)} \sim N\left(0,1\right); \quad k = 1, 2, ..., n-1 \tag{7.50}$$

and as a result by the definition of Λ in equation (7.48)

$$\Lambda \sim N\left(0, \sigma_\Lambda^2\right) \tag{7.51}$$

where

$$\sigma_\Lambda^2 = (n-1) + \sum_{\substack{k=1 \\ k\neq l}}^{n-1}\sum_{l=1}^{n-1} \rho^{(kl)} \tag{7.52}$$

where $\rho^{(kl)}$ is defined in equation (7.23). Also simple calculations show that the correlation coefficient between $X_T^{(k)}$ and Λ is given by

$$\rho_{k\Lambda} = \frac{\sum_{l=1}^{n-1} \rho^{(kl)}}{\sqrt{(n-1) + \sum_{\substack{k=1\\k\neq l}}^{n-1} \sum_{l=1}^{n-1} \rho^{(kl)}}}; \quad k = 1, 2, ..., n-1. \tag{7.53}$$

As a result

$$\left(X_T^{(k)},\, \Lambda\right) \sim \text{BVN}\left(\mu^{(k)},\ 0,\ \left(\sigma^{(k)}\right)^2,\ \sigma_\Lambda^2,\ \rho_{k\Lambda}\right). \tag{7.54}$$

Now, from equation (6.17) noting that the marginal cdfs $F_{S_T^{(i)}|\Lambda=\lambda}$ are strictly increasing so that $K_4 = 0$, we see that an upper bound for GAO is given as

$$C(0,x,T) \leq gSZ\left(0,T\right) \sum_{i=1}^{n-1} \int_{-\infty}^{\infty} \mathbf{E}\left[\left(S_T^{(i)} - F^{-1}_{S_T^{(i)}|\Lambda=\lambda}\left(x\right)\right)^+ \middle| \Lambda = \lambda\right] d\Phi\left(\frac{\lambda}{\sigma_\Lambda}\right), \tag{7.55}$$

where using equation (6.16), we see that x is obtained by solving the following equation

$$\sum_{i=1}^{n-1} F^{-1}_{S_T^{(i)}|\Lambda=\lambda}\left(x\right) = K - 1. \tag{7.56}$$

An explicit formula for the conditional inverse distribution function of $S_T^{(i)}$ given the event $\Lambda = \lambda$, is provided by the following result.

Proposition 7.1. *Under the assumptions of the Vasicek model, conditional on the event $\Lambda = \lambda$, the conditional inverse distribution function of $S_T^{(i)}$ for $i = 1, 2, ..., n-1$ is given by*

$$F^{-1}_{S_T^{(i)}|\Lambda=\lambda} = S_0^{(i)} e^{\mu^{(i)} + \rho_{i\Lambda}\frac{\sigma^{(i)}}{\sigma_\Lambda}\lambda + \sigma^{(i)}\sqrt{1-\rho_{i\Lambda}^2}\,\Phi^{-1}(x)}. \tag{7.57}$$

Proof. The proof follows directly from equations (5.29) and (5.30) of Section 5.5.3. □

From equation (7.56), we then wish to solve the following for x

$$\sum_{i=1}^{n-1} S_0^{(i)} e^{\mu^{(i)} + \rho_{i\Lambda}\frac{\sigma^{(i)}}{\sigma_\Lambda}\lambda + \sigma^{(i)}\sqrt{1-\rho_{i\Lambda}^2}\,\Phi^{-1}(x)} = K - 1. \tag{7.58}$$

As a result, using equation (7.55), the improved upper bound for Guaranteed Annuity Option is given by the following set of equations

$$\begin{aligned} C(0,x,T) \quad &\leq \quad gSZ\left(0,T\right) \int_{-\infty}^{\infty} \left(\sum_{i=1}^{n-1} S_0^{(i)} e^{\mu^{(i)} + \rho_{i\Lambda}\frac{\sigma^{(i)}}{\sigma_\Lambda}\lambda + \frac{1}{2}\left(\sigma^{(i)}\right)^2\left(1-\rho_{i\Lambda}^2\right)} \Phi\left(c_1^{(i)}\right) \right. \\ &\qquad \left. - \left(K-1\right)\left(1-x\right) \right) d\Phi\left(\frac{\lambda}{\sigma_\Lambda}\right) \\ &=: \quad \text{GAOUB}_j^{(VS)} \end{aligned} \tag{7.59}$$

and

$$c_1^{(i)} = \sigma^{(i)} \sqrt{(1-\rho_{i\Lambda}^2)} - \Phi^{-1}(x) \quad i = 1, 2, ..., n-1 \tag{7.60}$$

where $x \in (0,1)$ solves equation (7.58).

We have also worked out bounds for the multidimensional Cox-Ingersoll-Ross (CIR) (c.f. [21]) model and the well-known Wishart Model. However, keeping in view the length of the paper, we are not presenting them here. Interested readers can refer to [5].

8. Numerical Results

Now we investigate the applications of the theory derived in the previous sections. We have successfully obtained a number of lower bounds and an upper bound for Guaranteed Annuity Options in Sections 5 and 6. We now test these vis-a-vis the well-known Monte Carlo estimate for the GAO. We carry out this working for the Vasicek model. The nomenclature for the bounds has already been specified in Sections 5, 6 and 7. In all the examples, we have the following 'Contract Specification':

$$g = 11.1\%, \ T = 15, \ n = 35;$$

8.1. Vasicek Model. In table 1, we assume that the interest rate (r_t) and the force of mortality (μ_t) for an insured aged x at time 0 obey the Vasicek model, with dynamics given by the specifications in equations (7.1)-(7.3). We highlight below the parameter choices in accordance with [62]. The value of the correlation coefficient between the interest rate and the force of mortality is varied in table 1. Parameter choices for table 1 are

Interest Rate Model:

$$a = 0.15\%, \ b = 0.045, \ \sigma = 0.03, \ r_0 = b;$$

Mortality Model:

$$c = 0.1\%, \ b = 0.045, \ \xi = 0.0003, \ \mu_0 = 0.006.$$

Using equations (7.15) and (7.17)-(7.18), we see that

$$Corr\left[r_T, \mu_T\right] = \frac{2\rho\sqrt{ac}}{(a-c)} \frac{\left(1-e^{-(a-c)t}\right)}{\sqrt{\left(1-e^{-2aT}\right)\left(e^{2cT}-1\right)}} \tag{8.1}$$

As a result, we infer that the correlation between mortality and interest rate is directly proportional to the ρ which depicts the correlation between the underlying Brownian motions governing these two risks. In table 1, we vary ρ from -0.9 to 0.9 and investigate the effect of changing correlation between the two aforementioned risks on the lower and upper bounds and Monte Carlo estimate for the GAO price under the Vasicek model. To obtain the general lower bound given in section 5.5.3 we adhere to the same choice of Λ as that for the improved upper bound for the Vasicek case given in section 5.7.1. It is evident that when the correlation between these underlying rates grows, the prices of the GAO begin to swell. This finding is in line with the results of [62]. However, while the bounds obtained by these authors are vague, we succeed in deriving tight lower and upper bounds for the GAO price. The numerical findings of Table 1 are portrayed in figures 1-3. While figures 1 and 2 depict comparisons between the bounds, figure 3 portrays the price bounds for the GAO price under the Vasicek Model. We do not work out the third upper bound GAOUB for the Vasicek case since inversion of the distribution function is possible here yielding extremely tight upper bounds especially using the conditioning approach. Table 1 reflects that the relative difference $(= \frac{|bound-MC|}{MC})$ between any bound and the benchmark Monte Carlo estimate decreases with an increase in the correlation between mortality and interest rate. This observation is echoed by figure 1. On the other hand, figure 2 depicts the difference between the Monte Carlo estimate of the GAO price and the derived bounds. The bound $\text{GAOLB}_j^{(VS)}$ fares much better than GAOLB_3, although the former is restricted to the assumption of positive correlation between the two competing risks viz. mortality and interest rate which is by all means a very sensible assumption. The absolute difference between the estimated price and the bounds diminishes as the value of the correlation is increased. The competing worms in figures 1 and 2 show the efficacy of additional information as the ones exploiting extra knowledge completely outperform the thread of trivial lower bound by a huge margin. Finally figure 3 is a testimony to the fact that the bounds are extremely tight. There is indeed a clustering of the bounds around the line depicting Monte Carlo estimator.

9. Conclusions

Our research investigates the designing of price bounds for Guaranteed Annuity Options assuming that mortality and interest rate risk are correlated. The highlight of this paper is that the methodologies devised here allow to get rid with the issue of dealing with sums of a large number of correlated variables. Moreover they are also successful in dealing with cumbersome stochastic processes. In fact, the bounds are extremely tight particularly when the underpinning risks are governed by Vasicek models. This paper is frontrunner in obtaining both the lower and the upper bound in the affine case that depend on the properties of the distribution of the random variables connected to the transformed stochastic processes underlying mortality and interest rate. While the lower bound depicts itself in form of Laplace transform of the underlying random variable, the upper bound is presented in the form of the associated characteristic function. Both of these tools are easy computable vital statistics for any distribution. The most interesting point is that we need to work in one dimension, in contrast to what would have been at least a 34-dimensional set up, assuming that a person lives at least 100 years making n=35. As a result in cases where inversion of the distribution function is unavailable, an upper bound can still be found provided the characteristic function of the log prices of the underlying assets viz. pure endowments in our case is known. This paper serves as the perfect launching pad to deal with experiments to incorporate jumps in the models for mortality to price GAOs. Moreover, the methodology employed for furnishing bounds for GAOs can be extended to obtain bounds for Guaranteed Minimum Income Benefits (GMIBs). A worthy observation is that the stimulant for the present work is the theory of comonotonicity. One can therefore easily extend this approach for computing tight bounds for other longevity linked securities.

TABLE 1. Lower Bounds and Upper Bounds for Guaranteed Annuity Option under the Vasicek Model with parameter choice in accordance with [62]. MC Simulations: 5000000 iterations (Antithetic Method)

ρ	GAOLB	GAOLB_3	$\text{GAOLB}_j^{(VS)}$	MC	$S.E.(MC)$	$\text{GAOUB}_j^{(VS)}$	GAOUB_1
-0.9	0.08204574	0.08807759	—	0.08979374	0.00001268	0.08991707	0.09017503
-0.8	0.08419116	0.09039377	—	0.09203019	0.00001306	0.09237052	0.09237346
-0.7	0.08636037	0.09272434	—	0.09425514	0.00001341	0.09456473	0.09458951
-0.6	0.08855377	0.09507005	—	0.09647159	0.00001375	0.09664983	0.09682426
-0.5	0.09077176	0.09743177	—	0.09876248	0.00001415	0.09907454	0.09907869
-0.4	0.09301474	0.09981041	—	0.10101607	0.00001452	0.10125653	0.10135368
-0.3	0.09528314	0.10220688	—	0.10331535	0.00001492	0.10353471	0.10365005
-0.2	0.09757738	0.10462207	—	0.10565516	0.00001532	0.10574405	0.10596859
-0.1	0.09989789	0.10705689	—	0.10800921	0.00001573	0.10807736	0.10831003
0.0	0.10224512	0.10951218	0.11039418	0.11040279	0.00001616	0.11066465	0.11067510
0.1	0.10461954	0.11198880	0.11279801	0.11281387	0.00001660	0.11301493	0.11306448
0.2	0.10702161	0.11448758	0.11522744	0.11524241	0.00001703	0.11524655	0.11547887
0.3	0.10945182	0.11700932	0.11768307	0.11769731	0.00001748	0.11790709	0.11791892
0.4	0.11191066	0.11955482	0.12016545	0.12019181	0.00001795	0.12033328	0.12038533
0.5	0.11439863	0.12212486	0.12267518	0.12267866	0.00001840	0.12267877	0.12287873
0.6	0.11691627	0.12472023	0.12521286	0.12521445	0.00001889	0.12526952	0.12539980
0.7	0.11946410	0.12734169	0.12777909	0.12779723	0.00001939	0.12781586	0.12794918
0.8	0.12204267	0.12999001	0.13037449	0.13039169	0.00001991	0.13047918	0.13052756
0.9	0.12465253	0.13266595	0.13299970	0.13303942	0.00002043	0.13311967	0.13313559

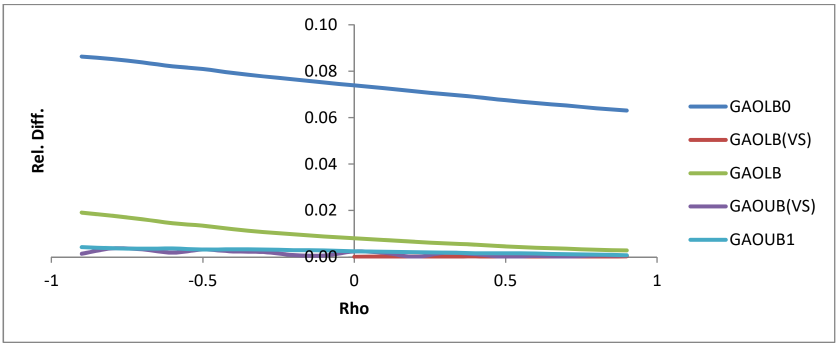

FIGURE 1. Relative Difference of Lower and Upper Bounds w.r.t. MC estimate under Vasicek model with GAOLB0 denoting GAOLB and GAOLB denoting GAOLB_3

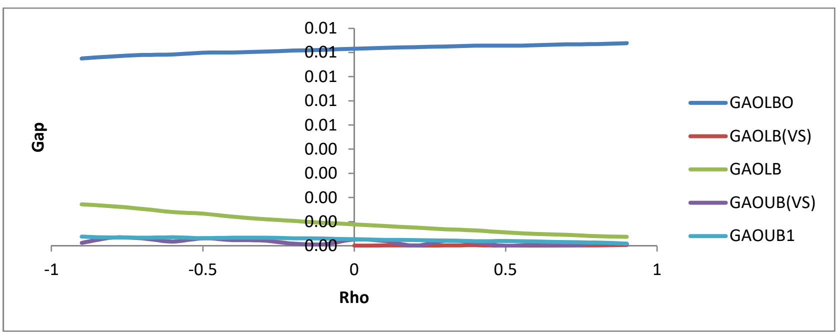

FIGURE 2. Comparison of different bounds under Vasicek Model in terms of difference from MC estimate with GAOLBO denoting GAOLB and GAOLB denoting GAOLB_3

Appendix A. Appendix A

Definition A.1. ***Affine Process*** A time-homogeneous Markov process X relative to some filtration $(\mathcal{F}_s)$ and with state space $(D, \mathcal{D})$ (augmented by Δ) is called affine if

(i) it is stochastically continuous, that is, $\lim_{s \to t} p_s(x, \cdot) = p_t(x, \cdot)$ for all $t \geq 0$ and $x \in D$, and

(ii) its Fourier-Laplace transform has exponential affine dependence on the initial state. This means that there exist functions $\phi : \mathbb{R}_+ \times S_d^+ \to \mathbb{R}_+$ and $\psi : \mathbb{R}_+ \times S_d^+ \to$

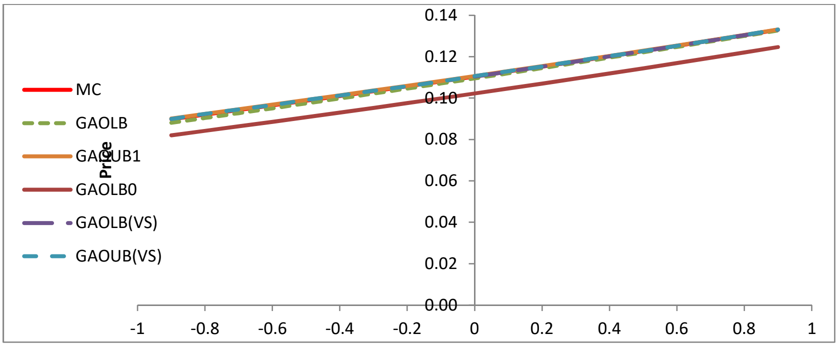

FIGURE 3. GAO Price Bounds under Vasicek model for the parameter choice of Liu(2013) with GAOLB0 denoting GAOLB and GAOLB denoting GAOLB$_3$

S_d^+ such that

$$\mathbb{E}_x\left[e^{\langle u,X_t\rangle}\right] = P_t e^{\langle u,x\rangle} = \int_D e^{\langle u,\xi\rangle} p_t\left(x,d\xi\right) = e^{-\phi(t,u)-\langle\psi(t,u),x\rangle}, \tag{A.1}$$

for all $x \in D$ and $(t,u) \in \mathbb{R}_+ \times \mathbb{R}_d$

Definition A.2. ***Truncation Function*** Let $\chi : S_d \to S_d$ be some bounded continuous truncation function with $\chi(\xi) = \xi$ in the neighborhood of 0. An admissible parameter set given by $\left(\alpha, b, \beta^{ij}, c, \gamma, m, \mu\right)$ associated with χ consists of:

- a linear diffusion coefficient
$$\alpha \in S_d^+, \tag{A.2}$$
- a constant drift term
$$b \succeq (d-1)\,\alpha, \tag{A.3}$$
- a constant killing rate term
$$c \in \mathbb{R}^+, \tag{A.4}$$
- a linear killing rate coefficient
$$\gamma \in S_d^+, \tag{A.5}$$
- a constant jump term: a Borel measure m on $S_d^+ \setminus \{0\}$ satisfying
$$\int_{S_d^+\setminus\{0\}} \left(\parallel \xi \parallel \wedge 1\right) m\left(d\xi\right) < \infty, \tag{A.6}$$
- a linear jump coefficient: a $d\times d$ matrix $\mu = (\mu_{ij})$ of finite signed measures on $S_d^+ \setminus \{0\}$ such that $\mu\left(E\right) \in S_d^+$ for all $E \in \mathcal{B}\left(S_d^+ \setminus \{0\}\right)$ and the kernel
$$M\left(x,d\xi\right) := \frac{\langle x, \mu\left(d\xi\right)\rangle}{\parallel \xi \parallel^2 \wedge 1} \tag{A.7}$$

satisfies

$$\int_{S_d^+ \backslash \{0\}} \langle \chi(\xi), u \rangle M(x, d\xi) < \infty \text{ for all } x, u \in S_d^+ \text{with } \langle x, u \rangle = 0, \tag{A.8}$$

- a linear drift coefficient: a family $\beta^{ij} = \beta^{ji} \in S_d$ such that the linear map $B : S_d \to S_d$ of the form

$$B(x) = \sum_{i,j} \beta^{ij} x_{ij} \tag{A.9}$$

satisfies

$$\langle B(x), u \rangle - \int_{S_d^+ \backslash \{0\}} \langle \chi(\xi), u \rangle M(x, d\xi) \geq 0 \text{ for all } x, u \in S_d^+ \text{with } \langle x, u \rangle = 0. \tag{A.10}$$

Definition A.3. ***Generator*** For an affine process X taking values in $S_d^+ \subset S_d$ the infinitesimal generator is defined as

$$\begin{aligned} \mathcal{A}f(x) &= \lim_{t \to 0^+} \frac{\mathbb{E}[f(X_t^x)] - f(x)}{t} \text{ for } x \in S_d^+, \ f \in \mathcal{C}^2(S_d, \mathbb{R}_d) \\ & \quad \text{with bounded derivatives.} \end{aligned} \tag{A.11}$$

Acknowledgments. The authors gratefully acknowledge the financial support of the Institute and Faculty of Actuaries.

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