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THE E-TWIN CONNECTIVITY INDEX OF TENSOR PRODUCT OF SOME GRAPHS

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ABSTRACT. Let $G = (V, E)$ be a simple undirected graph, where $V = V(G)$ denotes the vertex set and $E = E(G)$ the edge set. For any unordered pair of distinct vertices $u, v \in V$, we define the *e-twin set*, denoted by Ec_{uv}, as follows:

$$\mathrm{Ec}_{uv} = \{w \in N(u) \cap N(v) : \deg(w) \equiv 0 \pmod 2\},$$

where $N(u)$ and $N(v)$ represent the open neighborhoods of the vertices u and v, respectively, and $\deg(w)$ is the degree of the vertex w.

The *e-twin number*, denoted ec_{uv}, is the cardinality of the set Ec_{uv}, i.e., $ec_{uv} = |\,\mathrm{Ec}_{uv}\,|$.

Using these local contributions, we define a global graph invariant called the *e-twin connectivity index*, or $\mathscr{EC}$-index for short. It is given by summing ec_{uv} over all unordered pairs of distinct vertices in the graph:

$$\mathscr{EC}(G) = \sum_{\{u,v\} \subset V(G)} ec_{uv}.$$

In this work, we explore and analyse the behavior of the $\mathscr{EC}$-index for specific graph operations, particularly focusing on the tensor product and the join of certain families of graphs. The structural properties of these operations are leveraged to derive explicit formulas for their corresponding $\mathscr{EC}$-indices.

1. Introduction

Graph theory serves as a fundamental mathematical tool for representing and analysing molecular structures, especially with regard to their symmetry, stability, and structural redundancy [5, 6]. Within the domain of chemical graph theory, a molecule is typically modelled as a graph: vertices represent atoms, while edges denote covalent bonds. For example, saturated hydrocarbons such as alkanes can be depicted as acyclic (tree-like) graphs, whereas aromatic compounds like benzenoid hydrocarbons correspond to planar graphs composed of hexagonal ring systems.

Among the many graph-theoretic descriptors employed in molecular studies, degree-based invariants play a crucial role in characterizing connectivity patterns and evaluating the relative stability of molecular frameworks. In this setting, the *e-twin connectivity index* [14] emerges as a novel and insightful parameter. It is defined based on the count of common neighbors with even degrees between pairs

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of atoms, providing a measure of local structural symmetry and redundancy within the bonding network.

This index proves especially informative when applied to molecules where symmetry and delocalization are central features, such as in conjugated systems like benzenoids. Additionally, it can offer valuable insights in the analysis of alkanes, where the branching structure significantly influences chemical reactivity and stability. Consequently, the e-twin connectivity index enriches the tool kit available to theoretical chemists exploring structure–property relationships across diverse classes of molecular graphs.

The search for robust numerical descriptors of molecular structure began more than half a century ago with the Wiener index [10], introduced in 1947 as the sum of distances over all vertex pairs in a molecular graph, and quickly followed by the first and second Zagreb indices in the 1970s, which measure degree-based connectivity at each end of an edge [2, 11, 12]. In the 1990s, the Randić index and its harmonic variant added further nuance by weighting edges inversely by the product (or sum) of their endpoint degrees; the past decade has witnessed an explosion of new descriptors such as the Sombor index [13], defined by Euclidean norms of degree-pairs, and its various augmented and geometric extensions. Parallel to these developments, spectral indices—most notably graph energy, the Estrada index [4], and the recently formulated Zagreb energy—have provided a global perspective by summing functions of eigenvalues of the adjacency or Laplacian matrices [3]. The article [9] summarises recent progress about link prediction algorithms, emphasizing on the contributions from physical perspectives and approaches, such as the random-walk based methods and the maximum likelihood methods.

Within this broad framework, the e-twin connectivity index takes a complementary approach: rather than aggregating local degree contributions or global spectral information, it focuses on pairwise interactions through the lens of shared neighborhoods, counting only those common neighbors whose degree is even. This restriction to even-degree intersection points gives rise to novel structural and spectral phenomena, especially when graphs are constructed via standard product operations [7]. In particular, the tensor (or categorical) product which model the simultaneous occurrence of two adjacency relations, play a pivotal role in both theoretical graph theory and applications to complex molecular frameworks. Tensor products preserve and intertwine parity properties of degrees in a way that makes the e-twin connectivity index especially tractable.

Let $G = (V, E)$ be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. If v_i and v_j are adjacent in G, we denote it as $v_i \sim v_j$ or $v_i \sim_G v_j$. The common neighborhood graph of G is the graph with the same vertex set as that of G where two vertices are adjacent if they have at least one common neighbor in G. The common neighborhood graph of G is also known as congraph of G, and is denoted by $con(G)$.

Let the common neighborhood of two distinct vertices v_i and v_j, be denoted by $c(v_i, v_j)$ and is the set of vertices adjacent to both v_i and v_j other than v_i and v_j. Then the common neighborhood matrix of G denoted by $CN(G)$ is an $n \times n$ matrix whose (i,j)th entry is 0 or $|c(v_i, v_j)|$ according as $i = j$ or $i \neq j$, respectively. The even degree common neighborhood graph of G, denoted by $econ(G)$, is the

graph with vertex set $V(G)$ and edge set
$E_{econ} = \{\{v_i, v_j\} \subseteq V \mid \exists\, u \in N(v_i) \cap N(v_j) \text{ and } d_G(u) \equiv 0 \pmod 2\}$.
Hence, $(econ)(G) = (V, E_{econ})$.

The *ECN-matrix* of (G), is an $n \times n$ matrix whose (i,j)th entry is the number of even degree common neighborhood vertices between v_i and v_j in G for $i \neq j$ and zero otherwise.
The number of common neighborhood vertices of even degree between u and v is called e-twin number of the vertices u and v, and is denoted by ec_{uv}. Then the e-twin connectivity index of the graph G is defined as

$$\mathscr{EC}(G) = \sum_{\{u,v\} \subset V(G)} ec_{uv},$$

where ec_{uv} is the e-twin number for $u, v \in V(G)$.

In this paper, we have computed the e-twin connectivity index of tensor product and join of some standard graphs such as the complete graph K_n, path graph P_n, cycle graph C_n, wheel graph W_n and star graph S_n.

2. Preliminaries

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $v \in V(G)$ is denoted by $deg(v)$ or $deg_G(v)$.

Definition 2.1. [14] Let G be a graph with vertex set $V(G)$, and let $u, v \in V(G)$. Define the *e-twin set* of u and v, denoted Ec_{uv}, as

$$Ec_{uv} = \{w \in N(u) \cap N(v) : \deg(w) \equiv 0 \pmod 2\}.$$

The *e-twin number* of the pair u, v, denoted ec_{uv}, is defined as the cardinality of the set Ec_{uv}, that is, $ec_{uv} = |Ec_{uv}|$.

Definition 2.2. [14] Let G be a graph. Define a topological index, *e-twin connectivity index* by

$$\mathscr{EC}(G) = \sum_{\{u,v\} \subset V(G)} ec_{uv}$$

where ec_{uv} is the e-twin number for $u, v \in V(G)$.

Lemma 2.3. [14] *Let K_n be the complete graph of order n. Then*

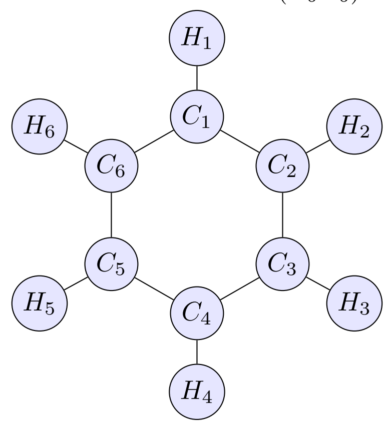
$$\mathscr{EC}(K_n) = \begin{cases} \frac{1}{2}n(n-1)(n-2) & if\, n\, is\, odd, \\ 0 & if\, n\, is\, even. \end{cases}$$

Let us consider the examples of Benzene and Butane. We find the e-twin connectivity index of these molecular structures and analyse their structural property.

Benzene (C_6H_6). Benzene consists of a 6-carbon ring with alternating single and double bonds. Each carbon atom is bonded to two other carbon atoms and one hydrogen atom, yielding all carbon atoms of degree 3 (odd) and all hydrogen atoms of degree 1 (odd), see Figure 1.

As no vertex has even degree, there can be no common even-degree neighbors. Thus, $\mathscr{EC}(C_6H_6) = 0$.

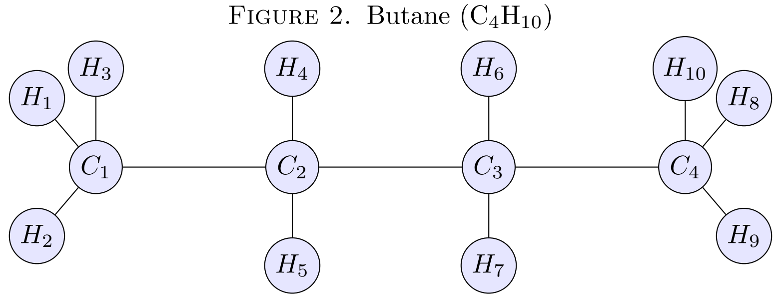
FIGURE 1. Benzene (C_6H_6)



Significance: The zero e-twin connectivity index reflects a uniformly odd-degree structure. While benzene exhibits high symmetry and aromatic stability, structurally it lacks even-degree bonding hubs. This reinforces the uniqueness of benzene's bonding: no two atoms share an even-degree intermediate.

Butane (C_4H_{10}). Butane is a straight-chain alkane with four carbon atoms. The degrees are as follows: terminal carbon atoms (C_1 and C_4)have degree 4 (even), internal carbon atoms (C_2 and C_3) have degree 4 (even), and all hydrogen have degree 1 (odd), see Figure 2.

FIGURE 2. Butane (C_4H_{10})



Hydrogen atoms bonded to the same carbon (terminal or internal) share that carbon as an even-degree common neighbor. Since degree of each carbon is 4, there are $\binom{4}{2}$ pairs. Hence,

$$\mathscr{E}\mathscr{C}(C_4H_{10}) = 4 \times \binom{4}{2} = 24.$$

Significance: The e-twin connectivity index captures the presence of shared bonding centers in the molecule. Here, it reflects structural regularity and modularity in the methyl groups, and reveals even-degree carbon atoms as bonding hubs.

3. The e-twin connectivity index of tensor product of some graphs

The tensor (or categorical) product of graphs produces a structure where adjacency is determined by simultaneous adjacency in both factor graphs. This operation often leads to highly regular and symmetric graphs, making it an interesting settings for studying degree-based invariants. In particular, the e-twin connectivity index reflects how local neighborhood overlaps with even-degree vertices emerge in such composite structures.

In this section, we compute the e-twin connectivity index $\mathscr{E}\mathscr{C}(G)$ for the tensor product of common graph families: complete graphs (K_n), paths (P_n), cycles (C_n), wheels (W_n), and stars (S_n). These results help illustrate how the interaction between the structural properties of each factor graph influences the overall index in the product graph.

Definition 3.1. [1] The tensor product $G \times H$ of two graphs G and H is a graph with vertex set $V(G \times H) = V(G) \times V(H)$, where two vertices (g, h) and (g', h') are adjacent in $G \times H$ if and only if $g \sim g'$ in G and $h \sim h'$ in H .

Proposition 3.2. [7] *For* $(u, v) \in V(G \times H)$, $deg_{G\times H}(u, v) = deg_G(u)\, deg_H(v)$.

In the tensor product $G \times H$, a vertex (u, v) is adjacent exactly to those (u', v') for which $u' \sim_G u \quad \text{and} \quad v' \sim_H v$.
Therefore, $N_{G\times H}(u, v) = \big\{(u', v') : u' \in N_G(u),\ v' \in N_H(v)\big\} = N_G(u) \times N_H(v)$.
Thus, $deg_{G\times H}(u, v) = \big|N_{G\times H}(u, v)\big| = \big|N_G(u) \times N_H(v)\big| = \big|N_G(u)\big|\ \big|N_H(v)\big| = deg_G(u)\, deg_H(v)$.

Proposition 3.3. *Let* $G = K_m \times K_n$. *Then,*

$$\mathscr{E}\mathscr{C}(G) = \begin{cases} mn\dbinom{(m-1)(n-1)}{2} & if\ (m-1)(n-1)\ even, \\ 0 & if\ (m-1)(n-1)\ odd. \end{cases}$$

Proof. In G, each vertex (u, v) has degree

$$deg_G(u, v) = deg_{K_m}(u)\, deg_{K_n}(v) = (m-1)(n-1).$$

Now, each vertex of even degree d contributes $\binom{d}{2}$ to $\mathscr{E}\mathscr{C}(G)$. There are mn vertices, so $\mathscr{E}\mathscr{C}(G) = mn\binom{(m-1)(n-1)}{2}$ whenever $(m-1)(n-1)$ is even, and is zero otherwise. □

Proposition 3.4. *Let* P_m *and* P_n *be path graphs on* $m, n \geq 2$ *vertices, and let* $G = P_m \times P_n$. *Then,* $\mathscr{E}\mathscr{C}(G) = 2(n-2) + 2(m-2) + 6\,(m-2)(n-2)$.

Proof. In P_m, its two endpoints have degree 1 and its $m-2$ internal vertices have degree 2, and likewise for P_n. Now $deg_G(u, v) = deg_{P_m}(u)\, deg_{P_n}(v) \in \{1, 2, 4\}$. If $\deg_G(u, v) = 2$ (which happens when one factor is an endpoint and the other is internal), then there are $2(n-2)$ vertices of the form (u, v), u an endpoint of P_m

and v an internal vertex of P_n, and $2(m-2)$ vertices of the form (u, v), u an internal vertex of P_m and v an endpoint of P_n, contributing $2(n-2) \cdot 1 + 2(m-2) \cdot 1$.

If $deg_G(u, v) = 4$ (both the factors are internal), then there are $(m-2)(n-2)$ such vertices, contributing $(m-2)(n-2)\binom{4}{2} = (m-2)(n-2) \times 6$.

Summing these gives, $\mathscr{E}\mathscr{C}(P_m \times P_n) = 2(n-2) + 2(m-2) + 6\,(m-2)(n-2)$. □

Theorem 3.5. *Let C_m and C_n be cycle graphs of orders $m, n \geq 3$. Then the e-twin connectivity index of their tensor product is*

$$\mathscr{E}\mathscr{C}(C_m \times C_n) = 6mn.$$

Proof. Let $G = C_m \times C_n$, then $V(G) = V(C_m) \times V(C_n)$, and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in G if and only if $u_1 \sim u_2$ in C_m and $v_1 \sim v_2$ in C_n.

Since each vertex in C_m and C_n has degree 2, it follows that each vertex (u, v) in G has degree, $\deg_G(u, v) = \deg_{C_m}(u) \cdot \deg_{C_n}(v) = 2 \cdot 2 = 4$. Thus, all vertices in G have even degree 4.

By definition, a vertex of even degree d contributes $\binom{d}{2}$ to the e-twin connectivity index. Here, each vertex contributes $\binom{4}{2} = 6$. Since there are mn vertices in total, the total e-twin connectivity index is $\mathscr{E}\mathscr{C}(C_m \times C_n) = mn \cdot 6 = 6mn$. □

The following theorem presents an exact formula for computing the e-twin connectivity index $\mathscr{E}\mathscr{C}(G)$ when G is the tensor product of two wheel graphs, W_m and W_n, with $m, n \geq 4$. The expression for $\mathscr{E}\mathscr{C}(W_m \times W_n)$ crucially depends on the parity (even or odd nature) of the orders m and n.

Notably, when both m and n are odd, the index attains its maximum value among all four cases, reflecting the highest possible redundancy due to the presence of even-degree common neighbors across both wheel structures.

Theorem 3.6. *Let W_m and W_n be wheel graphs of orders $m, n \geq 4$, respectively, and let $G = W_m \times W_n$. Then,*

$$\mathscr{E}\mathscr{C}(W_m \times W_n) = \begin{cases} 0 & \text{if } m \text{ and } n \text{ even,} \\ \binom{(m-1)(n-1)}{2} + (n-1)\binom{3(m-1)}{2} & \text{if } m \text{ odd and } n \text{ even,} \\ \binom{(m-1)(n-1)}{2} + (m-1)\binom{3(n-1)}{2} & \text{if } m \text{ even and } n \text{ odd,} \\ \binom{(m-1)(n-1)}{2} + (n-1)\binom{3(m-1)}{2} + (m-1)\binom{3(n-1)}{2} & \text{if } m \text{ and } n \text{ odd.} \end{cases}$$

Proof. In W_m, the center vertex has degree $m-1$ and each of the $m-1$ rim vertices has degree 3, and similarly in W_n.

Now, $deg_G(x, y) = deg_{W_m}(x)\, deg_{W_n}(y) \in \{(m-1)(n-1),\, 3(m-1),\, 3(n-1),\, 9\}$. There are one center-center vertex of degree $(m-1)(n-1)$, $m-1$ rim-center vertices of degree $3(n-1)$, $n-1$ center-rim vertices of degree $3(m-1)$, and $(m-1)(n-1)$ rim-rim vertices of degree 9. If $(m-1)(n-1)$ is even then, at least one of m, n is odd; if so, the center-center vertex contributes $\binom{(m-1)(n-1)}{2}$. If $3(m-1)$ is even when m is odd; if so, the $n-1$ vertices of type center-rim contribute $(n-1)\binom{3(m-1)}{2}$. If $3(n-1)$ is even when n is odd; if so, the $m-1$ vertices of type rim-center contribute $(m-1)\binom{3(n-1)}{2}$. Summing these contributions we get the result. □

Theorem 3.7. *Let S_m and S_n be star graphs on $m, n \geq 2$ vertices (each with one center and $m-1$ or $n-1$ leaves). Let $G = S_m \times S_n$. Then*

$$\mathscr{E}\mathscr{C}(S_m \times S_n) = \begin{cases} 0 & \text{if } m \text{ and } n \text{ even,} \\ \binom{(m-1)(n-1)}{2} + (n-1)\binom{m-1}{2} & \text{if } m \text{ odd and } n \text{ even,} \\ \binom{(m-1)(n-1)}{2} + (m-1)\binom{n-1}{2} & \text{if } m \text{ even and } n \text{ odd,} \\ \binom{(m-1)(n-1)}{2} + (n-1)\binom{m-1}{2} + (m-1)\binom{n-1}{2} & \text{if } m \text{ and } n \text{ odd.} \end{cases}$$

Proof. In S_m, the center has degree $m-1$ and each of the $m-1$ leaves has degree 1, and similarly for S_n. In the tensor product $G = S_m \times S_n$, for each vertex (x, y)

$$deg_G(x, y) = deg_{S_m}(x)\, deg_{S_n}(y) \in \{(m-1)(n-1),\ m-1,\ n-1,\ 1\}.$$

Each even-degree vertex w contributes $\binom{deg_G(w)}{2}$ to $\mathscr{E}\mathscr{C}(G)$. There are one vertex of degree $(m-1)(n-1)$ (center-center), $n-1$ vertices of degree $m-1$ (center of S_m times leaves of S_n), $m-1$ vertices of degree $n-1$ (leaves of S_m times center of S_n) and $(m-1)(n-1)$ vertices of degree 1 (leaf-leaf), which do not contribute. $(m-1)(n-1)$ even if at least one of m, n is odd. $m-1$ even if m is odd. $n-1$ even if n is odd. Hence in each of the four parity cases, summing the corresponding binomial contributions we get the formula. □

The above theorem gives an explicit formula for the e-twin connectivity index of the tensor product of two star graphs. The value of $\mathscr{E}\mathscr{C}(S_m \times S_n)$ depends entirely on the parity of m and n. When both are even, no even-degree common neighbors exist, leading to a zero index. In all other cases, the index increases based on interactions among the leaf vertices, with additional contributions arising when one or both stars have an odd number of vertices. This highlights how the structural imbalance introduced by odd degrees enhances local redundancy in the product graph.

Theorem 3.8. *Let K_m be the complete graph on $m \geq 2$ vertices and P_n the path graph on $n \geq 2$ vertices. Let $G = K_m \times P_n$, then*

$$\mathscr{E}\mathscr{C}\big(K_m \times P_n\big) = m\,(n-2)\,(m-1)\,(2m-3) + \begin{cases} m\,(m-1)\,(m-2) & \text{if } m \text{ odd,} \\ 0 & \text{if } m \text{ even.} \end{cases}$$

Proof. In K_m, each vertex has degree $m-1$. In P_n, the two endpoints have degree 1 and the $n-2$ internal vertices have degree 2.

Hence, $deg_G(u, v) = deg_{K_m}(u)\, deg_{P_n}(v) = (m-1) \times \begin{cases} 1 & if\ v \text{ an end vertex,} \\ 2 & if\ v \text{ an internal vertex.} \end{cases}$

When v is internal in P_n, $deg_G(u, v) = 2(m-1)$, which is always even. There are $m(n-2)$ such vertices, and each contributes $\binom{2(m-1)}{2} = \frac{2(m-1)\big(2(m-1)-1\big)}{2} = (m-1)(2m-3)$. Thus their total contribution to $\mathscr{E}\mathscr{C}(G)$ is $m(n-2)(m-1)(2m-3)$. When v is an endpoint of P_n, $deg_G(u, v) = m-1$, which is even exactly when m is odd. If m is odd, there are $2m$ such vertices (two endpoints times m choices for u), each contributing $\binom{m-1}{2} = \frac{(m-1)(m-2)}{2}$, and so their total contribution is $2m \cdot \frac{(m-1)(m-2)}{2} = m(m-1)(m-2)$. When m is even, there is no contribution. Summing these two parts will give the result. □

Theorem 3.9. *Let K_m and C_n be the complete and cycle graphs of orders $m \geq 2$ and $n \geq 3$, respectively. Then, the e-twin connectivity index of their tensor product is $\mathscr{EC}(K_m \times C_n) = mn(m-1)(2m-3)$.*

Proof. Let $G = K_m \times C_n$. In K_m, each vertex has degree $m-1$, and in C_n, each vertex has degree 2. Therefore, in G, the degree of each vertex is $\deg_G(u,v) = \deg_{K_m}(u) \cdot \deg_{C_n}(v) = (m-1) \cdot 2 = 2(m-1)$. Thus, all vertices in G have even degree $2(m-1)$. A vertex of even degree d contributes $\binom{d}{2}$ to the e-twin connectivity index. Here, $\binom{2(m-1)}{2} = (m-1)(2m-3)$. Since there are mn vertices in total, the total contribution to the e-twin connectivity index is $\mathscr{EC}(K_m \times C_n) = mn \cdot (m-1)(2m-3)$. □

Theorem 3.10. *Let K_m be the complete graph on $m \geq 2$ vertices and W_n the wheel graph on $n \geq 4$ vertices. Let $G = K_m \times W_n$, then*

$$\mathscr{EC}(K_m \times W_n) = \begin{cases} 0 & \textit{if both } m \textit{ and } n \textit{ even}, \\ m\binom{(m-1)(n-1)}{2} & \textit{if } m \textit{ even and } n \textit{ odd}, \\ m\binom{(m-1)(n-1)}{2} + m(n-1)\binom{3(m-1)}{2} & \textit{if } m \textit{ odd and } n \textit{ even or odd.} \end{cases}$$

Proof. In K_m, every vertex has degree $m-1$. In W_n, the center vertex has degree $n-1$ and each of the $n-1$ rim vertices has degree 3. Therefore, in the tensor product $G = K_m \times W_n$, every vertex (u,v) satisfies

$$deg_G(u,v) = deg_{K_m}(u)\ deg_{W_n}(v) = (m-1) \times \begin{cases} (n-1) & \text{if } v \text{ the center vertex}, \\ 3 & \text{if } v \text{ a rim vertex.} \end{cases}$$

There are m (center–type) vertices of degree $(m-1)(n-1)$ and $m(n-1)$ (rim–type) vertices of degree $3(m-1)$.
Now we check the conditions for the even degree vertices as they only contribute to $\mathscr{EC}(G)$. The integer $(m-1)(n-1)$ is even if at least one of m,n is odd and $3(m-1)$ is even if $m-1$ is even, that is when m is odd. Accordingly, if m and n are both even, neither degree is even, so $\mathscr{EC}(G) = 0$. If m is even and n is odd, then only the center-type vertices (all m of them) contribute, each by $\binom{(m-1)(n-1)}{2}$. Hence, $\mathscr{EC}(G) = m\binom{(m-1)(n-1)}{2}$. If m is odd (regardless of n), then both center-type and rim-type vertices contribute. There are m center-type giving $m\binom{(m-1)(n-1)}{2}$, and $m(n-1)$ rim-type giving $m(n-1)\binom{3(m-1)}{2}$. Summing these we get the result. □

Theorem 3.11. *Let $G = K_m \times S_n$, where K_m be the complete graph on $m \geq 2$ vertices and S_n the star graph on $n \geq 2$ vertices. Then,*

$$\mathscr{EC}(K_m \times S_n) = \begin{cases} 0 & \textit{if } m \textit{ and } n \textit{ even}, \\ m\binom{(m-1)(n-1)}{2} & \textit{if } m \textit{ even and } n \textit{ odd}, \\ m\binom{(m-1)(n-1)}{2} + m(n-1)\binom{m-1}{2} & \textit{if } m \textit{ odd and } n \textit{ even or odd.} \end{cases}$$

Proof. In K_m, each vertex has degree $m-1$. In S_n, the center has degree $n-1$ and each of the $n-1$ leaves has degree 1. In the tensor product $G = K_m \times S_n$,

$$\deg_G(u,v) = \deg_{K_m}(u)\ \deg_{S_n}(v) = (m-1) \times \begin{cases} (n-1) & if\ v\ the\ center\ vertex, \\ 1 & if\ v\ a\ leaf. \end{cases}$$

There are m vertices of degree $(m-1)(n-1)$ (center-type), each contributing $\binom{(m-1)(n-1)}{2}$ whenever $(m-1)(n-1)$ is even and $m(n-1)$ vertices of degree $m-1$ (every u with a leaf), each contributing $\binom{m-1}{2}$ whenever $m-1$ is even. Now, $(m-1)(n-1)$ is even if at least one of m, n is odd and $m-1$ is even if m is odd. Summing these contributions case by case gives the result. □

4. The e-twin connectivity index of join of some graphs

The join operation is a fundamental graph construction that significantly increases connectivity by adding edges between every vertex of two disjoint graphs. This operation combines structural features from both components and creates new interactions that influence graph invariants. In the context of the e-twin connectivity index, the join operation can introduce even-degree common neighbors between vertex pairs that were previously disconnected or structurally distant.

In this section, we compute the e-twin connectivity index $\mathscr{EC}(G)$ for the join of various well-known graph families, including the complete graph K_n, path P_n, cycle C_n, wheel W_n, and star S_n. These graphs offer diverse degree patterns and topological characteristics, allowing us to observe how the index behaves under different structural scenarios. The results provide insight into the role of vertex degrees and neighborhood intersections in determining local redundancy in the joined graphs.

Definition 4.1. [8] The join of two simple graphs G and H, denoted by $G+H$ is a graph formed by taking the disjoint union of G and H and adding an edge between every vertex of G and every vertex of H.

Theorem 4.2. *Let G and H be simple, undirected graphs with $|V(G)| = p$ and $|V(H)| = q$. For each vertex $x \in V(G)$, define*

$$t_G(x) = \binom{\deg_G(x)}{2},\ A_G = \{x \in V(G) : \deg_G(x) + q \text{ is even}\},\ D_G = \sum_{x \in A_G} \deg_G(x).$$

Similarly, for each $y \in V(H)$, define

$$t_H(y) = \binom{\deg_H(y)}{2},\ A_H = \{y \in V(H) : \deg_H(y) + p \text{ is even}\},\ D_H = \sum_{y \in A_H} \deg_H(y).$$

Then the e-twin connectivity index of the join graph $G+H$ is given by

$$\mathscr{EC}(G+H) = \sum_{x \in A_G} t_G(x) + \sum_{y \in A_H} t_H(y) + qD_G + pD_H + \binom{p}{2} \cdot |A_H| + \binom{q}{2} \cdot |A_G|.$$

Proof. In the join graph $G+H$, each vertex in G is adjacent to every vertex in H, and vice versa. For $x \in V(G)$, $\deg_{G+H}(x) = \deg_G(x) + q$, and similarly, $\deg_{G+H}(y) = \deg_H(y) + p$ for $y \in V(H)$.

The e-twin connectivity index counts unordered vertex pairs $\{u, v\} \subset V(G+H)$, which has the number of common neighbors having even degree in $G + H$. We classify these pairs into three types:

(1) Pairs within G: For $u, v \in V(G)$, the even-degree common neighbors are:

- vertices $x \in A_G$ that are common neighbors of u and v in G, contributing $\sum_{x\in A_G} \binom{\deg_G(x)}{2}$,
- all vertices $y \in A_H$, since every vertex in H is adjacent to all vertices in G, so shared by all such pairs. For each $\{u, v\} \in \binom{V(G)}{2}$, there are $|A_H|$ such common neighbors. Thus, the total contribution is $\binom{p}{2} \cdot |A_H|$.

(2) Pairs within H: Similarly, for $u, v \in V(H)$, the contribution is

$$\sum_{y\in A_H} \binom{\deg_H(y)}{2} + \binom{q}{2} \cdot |A_G|.$$

(3) Pairs across G and H: For each pair $u \in V(G), v \in V(H)$, the common neighbors are:

- all vertices in G — shared since $v \in H$ is adjacent to all of G,
- all vertices in H — shared since $u \in G$ is adjacent to all of H.

But only those common neighbors of even degree are being counted here.
Hence, $\sum_{x\in A_G} \deg_G(x)$ contributes to q such pairs is qD_G, and $\sum_{y\in A_H} \deg_H(y)$ contributes to p such pairs is pD_H.

Summing all these contributions, we get

$$\mathscr{E}\mathscr{C}(G + H) = \sum_{x\in A_G} \binom{\deg_G(x)}{2} + \sum_{y\in A_H} \binom{\deg_H(y)}{2} + qD_G + pD_H + \binom{p}{2} \cdot |A_H| + \binom{q}{2} \cdot |A_G|.$$

□

Theorem 4.3.

$$\mathscr{E}\mathscr{C}\big(K_m + K_n\big) = \mathscr{E}\mathscr{C}\big(K_{m+n}\big) = \begin{cases} \dfrac{(m+n)(m+n-1)(m+n-2)}{2} & if\ m+n\ odd, \\ 0 & if\ m+n\ even. \end{cases}$$

Proof. We have

$$K_m + K_n \cong K_{m+n}.$$

Then, by Theorem 2.3, we get the result. □

Theorem 4.4. *Let P_m and P_n be path graphs on $m, n \geq 2$ vertices, and let $G = P_m + P_n$ be their join. Then,*

$$\mathscr{E}\mathscr{C}(G) = \begin{cases} n(n+1) \ + \ m(m+1) & if\ m\ and\ n\ odd, \\ n(n+1) \ + \ \dfrac{(n-2)(m+2)(m+1)}{2} & if\ m\ even\ and\ n\ odd, \\ \dfrac{(m-2)(n+2)(n+1)}{2} \ + \ m(m+1) & if\ m\ odd\ and\ n\ even, \\ \dfrac{(m-2)(n+2)(n+1)}{2} \ + \ \dfrac{(n-2)(m+2)(m+1)}{2} & if\ m\ and\ n\ even. \end{cases}$$

Proof. In $G = P_m + P_n$, each vertex of P_m gains n new neighbors and each of P_n gains m. Thus,

$$deg_G(w) = \begin{cases} 1+n & \text{if } w \in \{\text{endpoints of } P_m\}, \\ 2+n & \text{if } w \in \{\text{internals of } P_m\}, \\ 1+m & \text{if } w \in \{\text{endpoints of } P_n\}, \\ 2+m & \text{if } w \in \{\text{internals of } P_n\}. \end{cases}$$

Now, there are four types of vertices in G and we calculate its contribution to $\mathscr{EC}(G)$. Also, each even-degree vertex w contributes $\binom{deg_G(w)}{2}$ to $\mathscr{EC}(G)$.
Type 1: $w \in G$ are endpoints of P_m
Here, $deg_G(w) = 1 + n$, is even if and only if n is odd. Then, each such vertex contributes $\binom{n+1}{2} = \frac{n(n+1)}{2}$. Since, there are 2 such vertices, total contribution of them is $n(n+1)$.
Type 2: $w \in G$ are internal vertices of P_m
Here, $deg_G(w) = 2 + n$, is even if and only if n is even. Then, each of the $m-2$ such vertices contributes

$$(m-2)\binom{n+2}{2} = \left(\frac{(m-2)(n+2)(n+1)}{2}\right).$$

Type 3: $w \in G$ are endpoints of P_n
Here, $deg_G(w) = 1 + m$, is even if and only if m is odd. Then their combined contribution is

$$2\binom{m+1}{2} = m(m+1).$$

Type 4: $w \in G$ are internal vertices of P_n
Here, $deg_G(w) = 2 + m$ is even if and only if m is even. Then the $n-2$ such vertices contribute

$$(n-2)\binom{m+2}{2} = \frac{(n-2)(m+2)(m+1)}{2}.$$

Now, by combining exactly those terms whose parity-conditions hold gives the formula for $\mathscr{EC}(P_m + P_n)$. □

The join of two cycle graphs C_m and C_n forms a densely connected structure where each vertex in one cycle is adjacent to every vertex in the other. This operation significantly alters the degree distribution and introduces new common neighbors between vertex pairs. The e-twin connectivity index of such a join reflects how even-degree vertices contribute to local redundancy across the combined graph. The following theorem provides explicit formulas for $\mathscr{EC}(C_m + C_n)$, highlighting the strong dependence of the index on the parity of m and n.

Theorem 4.5. *Let C_m and C_n be cycle graphs of orders $m, n \geq 3$, and let $G = C_m + C_n$ be their join. Then, the e-twin connectivity index of G is given by*

$$\mathscr{E}\mathscr{C}(C_m + C_n) = \begin{cases} \frac{m(n+2)(n+1)}{2} + \frac{n(m+2)(m+1)}{2} & \textit{if } m \textit{ and } n \textit{ even },\\ \frac{m(n+2)(n+1)}{2} & \textit{if } m \textit{ odd and } n \textit{ even},\\ \frac{n(m+2)(m+1)}{2} & \textit{if } m \textit{ even and } n \textit{ odd},\\ 0 & \textit{if } m \textit{ and } n \textit{ odd}. \end{cases}$$

Proof. Each vertex in a cycle graph has degree 2. In the join graph $G = C_m + C_n$, each vertex of C_m gains n new neighbors, and each vertex of C_n gains m new neighbors. Thus, the degree of any vertex becomes:

$$deg_G(w) = \begin{cases} 2+n & \text{if } w \in C_m,\\ 2+m & \text{if } w \in C_n. \end{cases}$$

We now count the contribution of vertices of even degree to $\mathscr{E}\mathscr{C}(G)$. Recall that a vertex w of even degree contributes $\binom{deg_G(w)}{2}$ to the e-twin connectivity index. We analyze the cases based on the parity of m and n.

Case 1: n is even. Then all vertices of C_m have even degree $2+n$. Since there are m such vertices, their total contribution is:

$$m \cdot \binom{n+2}{2} = \frac{m(n+2)(n+1)}{2}.$$

Case 2: m is even. Then all vertices of C_n have even degree $2+m$. Since there are n such vertices, their total contribution is:

$$n \cdot \binom{m+2}{2} = \frac{n(m+2)(m+1)}{2}.$$

If both m and n are even, we add both contributions. If both are odd, then all degrees in G are odd and $\mathscr{E}\mathscr{C}(G) = 0$.

Hence, the result follows. □

Theorem 4.6. *Let W_m and W_n be the wheel graphs of orders $m, n \geq 4$, and let $G = W_m + W_n$. Then,*

$$\mathscr{E}\mathscr{C}(G) = \begin{cases} (m-1)\binom{n+3}{2} + (n-1)\binom{m+3}{2} & \textit{for } m \textit{ and } n \textit{ odd},\\ 2\binom{m+n-1}{2} + (n-1)\binom{m+3}{2} & \textit{for } m \textit{ odd and } n \textit{ even},\\ 2\binom{m+n-1}{2} + (m-1)\binom{n+3}{2} & \textit{for } m \textit{ even and } n \textit{ odd},\\ 0 & \textit{for } m \textit{ and } n \textit{ even}. \end{cases}$$

Proof. In the join $G = W_m + W_n$, each vertex of W_m gains all n vertices of W_n as new neighbors, and vice versa. Thus,

$$deg_G(w) = \begin{cases} (m-1)+n & if\ w\ is\ the\ center\ of\ W_m, \\ 3+n & if\ w\ is\ a\ rim\ vertex\ of\ W_m, \\ (n-1)+m & if\ w\ is\ the\ center\ of\ W_n, \\ 3+m & if\ w\ is\ a\ rim\ vertex\ of\ W_n. \end{cases}$$

Each even-degree vertex w contributes $\binom{d_G(w)}{2}$ to $\mathscr{EC}(G)$. Even degree vertices occurs when $(m-1)+n$ or $3+n$ or $3+m$ is even. That is, when $(m+n)$ or n or m is odd. Hence, the two center vertices (one from W_m, one from W_n) each contribute $\binom{m+n-1}{2}$ precisely when $m+n-1$ is even, i.e., when $m+n$ is odd. Together they give $2\binom{m+n-1}{2}$, which simplifies to $\binom{m+n-1}{2}$ in the piecewise cases below.
Each of the $m-1$ rim-vertices of W_m has degree $n+3$, so if n is odd these contribute $(m-1)\binom{n+3}{2}$.
Each of the $n-1$ rim-vertices of W_n has degree $m+3$, so if m is odd these contribute $(n-1)\binom{m+3}{2}$. Considering the four parity cases for m, n; we get the result. □

Theorem 4.7. *Let S_m and S_n be star graphs on $m \geq 2$ and $n \geq 2$ vertices (each with one center and $m-1$, respectively. $n-1$, leaves), and $G = S_m + S_n$. Then,*

$$\mathscr{EC}(G) = \begin{cases} 2\binom{m+n-1}{2} + (m-1)\binom{n+1}{2} & for\ m\ even\ and\ n\ odd, \\ 2\binom{m+n-1}{2} + (n-1)\binom{m+1}{2} & for\ m\ odd\ and\ n\ even, \\ (m-1)\binom{n+1}{2} + (n-1)\binom{m+1}{2} & for\ m\ and\ n\ odd, \\ 0 & for\ m\ and\ n\ even. \end{cases}$$

Proof. In the join $G = S_m + S_n$, each center gains all n (respectively m) vertices of the other star, so its degree becomes $(m-1)+n = m+n-1$. Each leaf of S_m (resp. S_n) gains n (resp. m) new neighbors, so its degree is $1+n$ (resp. $1+m$). Since, each even-degree vertex w contributes $\binom{\deg_G(w)}{2}$ to $\mathscr{EC}(G)$,

- The two centers contribute $2\binom{m+n-1}{2}$ exactly when $m+n-1$ is even, that is when $m+n$ odd.
- The $m-1$ leaves of S_m contribute $(m-1)\binom{n+1}{2}$ exactly when $n+1$ is even, that is when n odd.
- The $n-1$ leaves of S_n contribute $(n-1)\binom{m+1}{2}$ exactly when $m+1$ is even, that is when m odd.

Summing these contributions under the four parity - conditions of m, n, we get the result. □

The join of a complete graph and a path graph, denoted $K_m + P_n$, yields a highly connected structure combining dense and linear components. Studying the e-twin connectivity index of such graphs offers insight into how the interaction between fully connected and sequential substructures affects local redundancy.

The following theorem provides closed-form expressions for $\mathscr{EC}(K_m + P_n)$, with values determined by the parity of m and n.

Theorem 4.8. *Let $G = K_m + P_n$ be the join of the complete graph on m vertices and the path graph on n vertices, with $m, n \geq 2$. Then its e-twin connectivity index is*

$$\mathscr{EC}(K_m + P_n) = \begin{cases} m\binom{m+n-1}{2} \ + \ (n-2)\binom{m+2}{2} & for\ m\ even\ and\ n\ odd, \\ (n-2)\binom{m+2}{2} & for\ m\ and\ n\ even, \\ m\binom{m+n-1}{2} \ + \ 2\binom{m+1}{2} & for\ m\ odd\ and\ n\ even, \\ 2\binom{m+1}{2} & for\ m\ and\ n\ odd. \end{cases}$$

Proof. Let $A = V(K_m), \quad |A| = m, \quad B = V(P_n), \quad |B| = n.$

$$deg_G(w) = \begin{cases} (m-1)+n & if\ w \in A, \\ m+2 & if\ w \in B \text{ is an internal vertex}, \\ m+1 & if\ w \in B \text{ is an endvertex}. \end{cases}$$

Thus, each of the m vertices in A has degree $m+n-1$. In the path P_n there are $n-2$ internal vertices (of degree $m+2$ in G) and 2 endpoints (of degree $m+1$). For $w \in A$, $deg(w)$ is even if $m+n-1$ is even, that is, if n is odd when m is even, or if n is even when m is odd. For, $w \in B$, $deg(w) = m+2$ (internals) is even if m is even, $deg(w) = m+1$ (endpoints) is even if m is odd. Thus,

$$\mathscr{EC}(G) = \sum_{\substack{w \in A \\ \deg(w) \text{ even}}} \binom{m+n-1}{2} + \sum_{\substack{w \in B \\ \deg(w) \text{ even}}} \binom{\deg_G(w)}{2}.$$

Then there are four cases:

- *Case 1: m even, n odd.* $deg_A(w) = m+n-1$ is even, internals are even, endpoints are odd.
$$\mathscr{EC}(G) = m\binom{m+n-1}{2} \ + \ (n-2)\binom{m+2}{2}.$$
- *Case 2: m even, n even.* $deg_A(w) = m+n-1$ is odd, internals are even, endpoints are odd.
$$\mathscr{EC}(G) = (n-2)\binom{m+2}{2}.$$
- *Case 3: m odd, n even.* $deg_A(w) = m+n-1$ is even, internals are odd, endpoints are even.
$$\mathscr{EC}(G) = m\binom{m+n-1}{2} \ + \ 2\binom{m+1}{2}.$$
- *Case 4: m odd, n odd.* $deg_A(w) = m+n-1$ is odd, internals are odd, endpoints are even.
$$\mathscr{EC}(G) = 2\binom{m+1}{2}.$$

Thus, the result. □

Theorem 4.9. *Let K_m be the complete graph on $m \geq 2$ vertices and C_n be the cycle graph on $n \geq 3$ vertices. Let $G = K_m + C_n$ be their join. Then the e-twin connectivity index of G is given by :*

$$\mathscr{E}\mathscr{C}(K_m+C_n) = \begin{cases} \dfrac{n(m+2)(m+1)}{2} & \text{if } m \text{ even and } m+n \text{ even,} \\ \dfrac{n(m+2)(m+1)}{2} + \dfrac{m(m+n-1)(m+n-2)}{2} & \text{if } m \text{ even and } m+n \text{ odd,} \\ \dfrac{m(m+n-1)(m+n-2)}{2} & \text{if } m \text{ odd and } m+n \text{ odd,} \\ 0 & \text{if } m \text{ odd and } m+n \text{ even.} \end{cases}$$

Proof. The e-twin connectivity index of a graph G is given by:

$$\mathscr{E}\mathscr{C}(G) = \sum_{\substack{v \in V(G) \\ \deg(v) \text{ even}}} \binom{\deg(v)}{2}.$$

In the join $G = K_m + C_n$, the degrees of vertices change as follows:

- Each vertex $v \in K_m$ has degree $\deg(v) = m - 1 + n = m + n - 1$.
- Each vertex $u \in C_n$ has degree $\deg(u) = 2 + m = m + 2$.

Now consider the contribution of each set of vertices to the e-twin connectivity index. The vertices in K_m contribute if $\deg(v) = m + n - 1$ is even, i.e., if $m + n$ is odd. The vertices in C_n contribute if $\deg(u) = m + 2$ is even, i.e., if m is even. Hence, the total e-twin connectivity index becomes:

- If m is even and $m + n$ is even: only C_n contributes,

$$\mathscr{E}\mathscr{C}(G) = n \cdot \binom{m+2}{2} = \frac{n(m+2)(m+1)}{2}.$$

- If m is even and $m + n$ is odd: both K_m and C_n contribute,

$$\begin{aligned} \mathscr{E}\mathscr{C}(G) &= n \cdot \binom{m+2}{2} + m \cdot \binom{m+n-1}{2} \\ &= \frac{n(m+2)(m+1)}{2} + \frac{m(m+n-1)(m+n-2)}{2}. \end{aligned}$$

- If m is odd and $m + n$ is odd: only K_m contributes,

$$\mathscr{E}\mathscr{C}(G) = m \cdot \binom{m+n-1}{2} = \frac{m(m+n-1)(m+n-2)}{2}.$$

- If m is odd and $m + n$ is even: no contribution (all degrees are odd), so $\mathscr{E}\mathscr{C}(G) = 0$.

Thus, the result. □

Theorem 4.10. *Let $G = K_m + W_n$ be the join of the complete graph on m vertices and the wheel graph of order $n \geq 4$. Then*

$$\mathscr{EC}(K_m + W_n) = \begin{cases} (n-1)\binom{m+3}{2} & if\ m\ and\ n\ odd, \\ m\binom{m+n-1}{2} + \binom{m+n-1}{2} + (n-1)\binom{m+3}{2} & if\ m\ odd\ and\ n\ even, \\ m\binom{m+n-1}{2} + \binom{m+n-1}{2} & if\ m\ even\ and\ n\ odd, \\ 0 & if\ m\ and\ n\ even. \end{cases}$$

Proof. Let $A = V(K_m)$, $|A| = m$, $B = V(W_n)$, $|B| = n$. In the join $G = K_m + W_n$ each vertex in A acquires all n vertices of W_n as new neighbours, and likewise each vertex in B acquires the m vertices of K_m. Hence the degrees in G are

$$deg_G(w) = \begin{cases} (m-1)+n & if\ w \in A, \\ (n-1)+m & if\ w \text{ is the center of } W_n, \\ 3+m & if\ w \text{ is a rim-vertex of } W_n. \end{cases}$$

Since,

$$\mathscr{EC}(G) = \sum_{\substack{w \in V(G) \\ \deg_G(w) \text{ even}}} \binom{\deg_G(w)}{2},$$

we will check the even degree vertices. A vertex in A has even degree if $m-1+n$ is even, then its contribution to e-twin connectivity index is $m\ \binom{m-1+n}{2}$.
The center of W_n has even degree if $m+n-1$ is even, then its contribution to e-twin connectivity index is $\binom{m+n-1}{2}$.
A rim-vertex of W_n has even degree if $m+3$ is even. That is if m is odd. Then, they contribute $(n-1)\ \binom{m+3}{2}$. Summing these three terms we get the formula. □

Theorem 4.11. *Let $G = K_m + S_n$ be the join of the complete graph on m vertices and the star graph on n vertices ($n \geq 2$). Then*

$$\mathscr{EC}(K_m + S_n) = \begin{cases} 0 & if\ m\ and\ n\ even, \\ (m+1)\binom{m+n-1}{2} & if\ m\ even\ and\ n\ odd, \\ (m+1)\binom{m+n-1}{2} + (n-1)\binom{m+1}{2} & if\ m\ odd\ and\ n\ even, \\ (n-1)\binom{m+1}{2} & if\ m\ and\ n\ odd. \end{cases}$$

Proof. Let $A = V(K_m)$, $B = V(S_n)$, with $|A| = m$ and S_n has one center and $n-1$ leaves. In the join $G = K_m + S_n$,

$$deg_G(w) = \begin{cases} (m-1)+n & if\ w \in A, \\ m+(n-1) & if\ w \text{ is the center of } S_n, \\ m+1 & if\ w \text{ is a leaf of } S_n. \end{cases}$$

Since,

$$\mathscr{E}\mathscr{C}(G) = \sum_{\substack{w \in V(G) \\ \deg_G(w) \text{ even}}} \binom{\deg_G(w)}{2},$$

we will check the even degree vertices. A vertex in A has even degree if $m+n$ is odd. Each contributes $\binom{m+n-1}{2}$. Hence, their total contribution is $m\ \binom{m+n-1}{2}$. The center of S_n contributes $\binom{m+n-1}{2})$ when $m+n$ is odd. The leaves of S_n contributes $\binom{m+1}{2}$ when m is odd. Their total contribution is $(n-1)\ \binom{m+1}{2}$. Summing these three contributions we get the result. □

5. Conclusion

The e-twin connectivity index is a new graph invariant based on the concept of even-degree common neighbors. It is a topological invariant that counts, for each unordered pair of vertices, the number of their shared neighbors of even degree. In this paper we calculated the e-twin connectivity index of tensor product and join of some standard graphs like complete graphs, paths, cycles, stars and wheels. By focusing on two fundamental graph operations — the tensor product and the join — we derived closed - form expressions for $\mathscr{E}\mathscr{C}(G \times H)$ and $\mathscr{E}\mathscr{C}(G+H)$ in terms of the degree sequences of G and H. Our results reveal how parity constraints in the individual factors propagate through these constructions, yielding simple combinatorial formulas when one or both factors are among the classical families K_n, P_n, C_n, W_n and S_n.

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