Global and Stochastic Analysis
Vol. 12 No. 4 (July, 2025)

Received: 08th February 2025 *Revised: 16th May 2025* *Accepted: 25th May 2025*

DYNAMICALLY DEFINED SET FOR A DETERMINISTIC RANDOM WALK

I. INOQUIO-RENTERIA

Abstract. In this note, we present an example of a fibered dynamical system, also referred to as a deterministic random walk, generated by a piecewise linear expanding map on the first coordinate and a sign function on the second. We focus on the set of points whose orbits under this system remain non-negative in the second coordinate. Using the reflection principle from probability theory, we establish a connection between the topological pressure and a counting problem over constrained random walks. Furthermore, we show that the Hausdorff dimension of this set is invariant across horizontal sections and compute its Hausdorff measure.

1. Introduction

We study a fibered dynamical system, referred to as a homogeneous deterministic random walk, defined by the map $F : [0,1] \times \mathbb{Z} \to [0,1] \times \mathbb{Z}$,

$$F(x, n) = (f(x), \varphi(x) + n),$$

where f is a piecewise expanding map with two linear branches, and φ is the sign function determined by the domain of monotonicity of f. More precisely, for parameters $0 < p < q < 1$ satisfying $p + q = 1$, the map $f : [0,1] \to [0,1]$ is defined by,

$$f(x) = \begin{cases} \frac{x}{p}, & \text{if } x \in [0,p], \\ \frac{x-p}{q}, & \text{if } x \in (p,1]. \end{cases} \tag{1.1}$$

This map is piecewise linear and uniformly expanding map, with minimal expansion rate $\inf_{x\in[0,1]} |f'(x)| = \frac{1}{q} > 1$.
The sign function $\varphi : [0,1] \to \mathbb{Z}$ is then defined as

$$\varphi(x) = \begin{cases} +1, & \text{if } x \in [0,p], \\ -1, & \text{if } x \in (p,1]. \end{cases}$$

2000 *Mathematics Subject Classification.* Primary 37D35, 60G50; Secondary 37C45.

Key words and phrases. Deterministic random walk, topological pressure, reflection principle, Hausdorff dimension.

The author acknowledges the financial support of ANID/ULS, through the Project InES Género INGE220009.

For $m \geq 1$ and any $x \in [0,1]$, we denote by $f^m(x)$ the m-fold composition of f applied to x, and define the partial sum of φ along the orbit of x as

$$S_m\varphi(x) := \sum_{j=0}^{m-1} \varphi(f^j(x)).$$

The m-th iterate of a point $(x,r) \in [0,1] \times \mathbb{Z}$ under F is then given by

$$F^m(x,r) = (f^m(x), S_m\varphi(x) + r).$$

Let

$$\Lambda_+(F) := \bigcap_{j \geq 0} F^{-j}\left([0,1] \times (\mathbb{N} \cup \{0\})\right)$$

be a subset of $[0,1] \times \mathbb{N} \cup \{0\}$ dynamically defined by F.

If π_2 denotes the projection onto the second coordinate, then $(x,n) \in \Lambda_+(F)$ if and only if

$$\pi_2(F^j(x,n)) \geq 0, \text{ for all } j \geq 0.$$

This set is central in linking deterministic dynamical systems with stochastic processes. The function $\varphi(x)$ determines the evolution of the second coordinate, generating a sequence of ± 1 values that, despite being deterministic, resemble a Bernoulli process. Therefore, $\Lambda_+(F)$ acts as a "trapping region" for the dynamics, ensuring that the second coordinate of any orbit within it never becomes negative. This property mirrors survival sets in constrained random walks, where a walker remains in non-negative states (see [4]).

Although the system is deterministic, the iterates of $\varphi(x)$ induce an oscillatory behavior in the second coordinate, exhibiting behavior similar to fluctuations of a stochastic process. Specifically, the sequence $S_n\varphi(x)$ behaves like a biased random walk, where the increments are governed by the piecewise expanding map f.

The map F is sometimes referred to in the literature in various ways, including as a skew-product between f and the translation on the group $\mathbb{Z}$, a group extension of f, or even as a deterministic random walk generated by f. Moreover, the dynamics of F serves as a geometric model for studying complex systems, such as renormalizable unimodal maps (see [6, 7]).

Skew-product systems and their deterministic analogues of random walks have been central to the development of both dynamical systems and probability theory. In the probabilistic setting, they naturally arise in the study of random walks in random environments and in the analysis of stochastic processes influenced by underlying deterministic dynamics. These frameworks provide powerful models for understanding how complex, seemingly random behavior can emerge from purely deterministic rules. Consequently, skew-product systems have become a central object in the intersection of ergodic theory and probability, bridging concepts such as entropy, large deviations, and invariant measures with probabilistic notions like recurrence and survival probabilities.

In ergodic theory, this type of map is particularly interesting because there is a connection to topological pressure and constrained random walks. Moreover, if $\Lambda^+(F)$ has a fractal structure, it may have implications for the Hausdorff dimension of the confined trajectories, linking the system to multifractal analysis and large deviations theory (see [3]). Therefore, in this short note, we aim to highlight the usefulness of probabilistic tools in understanding the topological pressure from ergodic theory. In particular, we focus on the reflection principle, which provides an interesting tool for analyzing constrained random walks and their connection to dynamical systems.

The topological pressure of F is associated with the potential $\varphi = -t \log |Df| \circ \pi_1$ with $t \geq 0$, where π_1 is the projection on the first coordinate, is defined as follows

$$P(F, \varphi) := \lim_{n \to \infty} \frac{1}{n} \log \sup_{\substack{x \in [0,1] \\ k \in [-n,n]}} \sum_{(y,r) \in \Lambda^+(F) \cap F^{-n}(x,k)} |Df^n(\pi_1(y,r))|^{-t}.$$

Thus, this quantity can be analyzed by relating it to a sum over paths of a random walk, using the reflection principle. The principle states that paths crossing zero can be reflected to contribute to the count of paths that remain strictly non-negative. More precisely, let $N_n^{\neq -1}(0, j)$ denote the number of paths from 0 to j in n steps that do not cross below zero. Then, the asymptotic behavior of the topological pressure is given by the following

Theorem 1.1. *Let p, q be such that $0 < p < q < 1$ with $p + q = 1$, and let $F : [0,1] \times \mathbb{Z} \to [0,1] \times \mathbb{Z}$, be the map generated by f and φ, as in (1.1). Then, for all $n \geq 0$ and $t \geq 0$, the pressure $P(F, \varphi)$, for $\varphi = -t \log |Df| \circ \pi_1$ satisfies*

$$\exp(nP(F, \varphi)) = \left(\sum_{j=0}^{n} N_n^{\neq -1}(0, j) (p^t)^{\frac{n+j}{2}} (q^t)^{\frac{n-j}{2}} \right) [1 + o(1)].$$

Moreover, the value of t that solves the equation $P(F, -t \log Df \circ \pi_1) = 0$ is given by $t_0 = \frac{-2 \ln 2}{\ln p + \ln q}$.

Definition 1.2. Given a metric space X, the Hausdorff dimension of a subset $E \subset X$ is defined by

$$\dim_H(E) := \inf\{s > 0 : \mathcal{H}^s(E) = 0\} = \sup\{s > 0 : \mathcal{H}^s(E) = +\infty\},$$

where $\mathcal{H}^s(E)$ denotes the s-dimensional Hausdorff measure of E, defined by

$$\mathcal{H}^s(E) := \lim_{\delta \to 0} \inf \left\{ \sum_i |U_i|^s : E \subset \bigcup_i U_i, \ |U_i| < \delta \right\}.$$

Here, $|U_i|$ denotes the diameter of each set U_i in a countable cover $\{U_i\}$ of E. (See [2], [5].)

For $k \geq 0$, let $\Lambda_+^k(F)$ denote the horizontal section of $\Lambda_+(F)$, given by

$$\Lambda_+^k(F) := \Lambda_+(F) \cap \pi_2^{-1}(\{k\}).$$

Then, the Hausdorff dimension $\dim_H(\Lambda_+(F))$ of set $\Lambda_+(F)$ remains invariant across all horizontal levels k, and its Hausdorff measure is equal to zero.

Theorem 1.3. *Let F be as above, then, for all $k \geq 1$, the Hausdorff dimension remains invariant across all horizontal levels: $\dim_H(\Lambda_+^k(F)) = \dim_H(\Lambda_+(F)) = \frac{-2 \ln 2}{\ln p + \ln q}$. Moreover, if $t_0 = \dim_H(\Lambda_+(F))$, then the t_0-Hausdorff measure of $\Lambda_+(F)$ is zero.*

While the Hausdorff dimension of restricted trajectories can often be computed directly through dynamical arguments, our main emphasis here is on establishing a deeper connection between this problem and the reflection principle in probability theory.

2. Preliminaries

2.1. Reflection Principle. The reflection principle is a classical tool in probability theory used to count trajectories in simple random walks. This principle simplifies complex random walk analyses by reflecting paths and creating new paths from existing ones. More precisely, for any integers a, b and $r \geq 1$, we denote by $N_r(a,b)$ the number of paths from a to b in r steps, given by:

$$N_r(a,b) = \begin{cases} \binom{r}{\frac{r+b-a}{2}}, & \text{if } r+b-a \text{ is even,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Let $N_r^{\neq 0}(a,b)$ denote the number of paths from a to b in r steps that do not visit 0, and $N_r^0(a,b)$ denote the number of paths that do visit 0. The following decomposition holds:

$$N_r(a,b) = N_r^0(a,b) + N_r^{\neq 0}(a,b).$$

If $a > 0$ and $b < 0$, then every path from a to b must visit 0, implying that $N_r^{\neq 0}(a,b) = 0$ and $N_r^0(a,b) = N_r(a,b)$.

There exists a one-to-one correspondence between paths that start at a, end at b in r steps, and cross the x-axis, and paths that start at $-a$ and end at b. This correspondence is the essence of the reflection principle, which states: For $a > 0$ and $b > 0$, the number of paths from a to b in r steps that visit 0 is given by $N_r^0(a,b) = N_r(-a,b)$. Hence, the number of paths that do not visit 0 satisfies

$$N_r^{\neq 0}(a,b) = N_r(a,b) - N_r(-a,b).$$

Denote by $N_r^{\neq -1}(a,b)$ the number of paths from a to b in r steps that do not visit -1. Then, we have the following relationship:

$$N_r^{\neq -1}(a,b) = N_r^{\neq 0}(a+1,b+1).$$

A classical application of the reflection principle is in solving the *ballot problem*, which determines the number of paths that stay on the positive side throughout their trajectory and the probability that they never enter the negative side.

Consider an election where candidate A receives a votes and candidate B receives b votes with $a \geq b$. Assuming all voting sequences are equally likely, we model the vote counts as a random walk of size $a+b$, starting at 0 and ending at $S_{a+b} = a-b$. The total number of such paths is given by:

$$N_{a+b}(0, a-b) = \binom{a+b}{a}.$$

Using the reflection principle, the number of paths that never return to zero (i.e., where candidate A is always ahead) is given by:

$$N_{a+b}^{\neq 0}(0, a-b) = N_{a+b}^{\neq 0}(1, a+1-b) = \binom{a+b}{a}\left(\frac{a-b}{a+b}\right).$$

Thus, the probability that candidate A remains ahead throughout the counting process is: $\frac{a-b}{a+b}$. Since $N^{\neq -1}_{a+b}(0, a-b) = N^{\neq 0}_{a+b}(1, a+1-b)$, the probability that candidate B never leads during the counting process is: $\frac{a+1-b}{a+1}$.

2.2. Some preliminary lemmas. This section outlines some lemmas that will be used in the proof of the results.

Lemma 2.1. *Stirling's Approximation: The asymptotic formula for factorials is given by:*

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \Big[1 + o(1)\Big], \quad \text{as } n \to \infty.$$

For binomial coefficients, with $\alpha \in (0,1)$,

$$\binom{m}{\lfloor \alpha m \rfloor} = \frac{1}{\sqrt{2\pi\alpha(1-\alpha)m}} \cdot \frac{1}{\alpha^{\alpha m}(1-\alpha)^{(1-\alpha)m}}\Big[1 + o(1)\Big], \quad \text{as } n \to \infty. \tag{2.2}$$

In particular, for $\alpha = 1/2$,

$$\binom{m}{\lfloor m/2 \rfloor} = \frac{2^m}{\sqrt{\pi m}}\Big[1 + o(1)\Big], \quad n \to \infty.$$

Lemma 2.2 (Approximation of tail probabilities)**.** *Let $\{X_k\}_{k=1}^{+\infty}$ be i.i.d. Bernoulli random variables, with mean μ and variance σ^2. For every real number $\alpha > \mu$ we have,*

$$\mathbb{P}\left(\frac{1}{n}S_n > \alpha\right) = \mathbb{P}\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} > \frac{\sqrt{n}(\alpha-\mu)}{\sigma}\right) \asymp 1 - \Phi\left(\frac{\sqrt{n}(\alpha-\mu)}{\sigma}\right) \to 0,$$

as $n \to +\infty$, where, $\Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt$, for $x \in \mathbb{R}$, is the normal distribution function with zero-mean and unit variance Gaussian.

Lemma 2.3 (Refinement Cramer's theorem)**.** *Let $\{X_i\}$ be i.i.d. Bernoulli random variables with moment generating function $M_{X_1}(\theta) = \mathbb{E}(e^{\theta X_1})$. For $a > \mathbb{E}(X_1)$, Cramér's theorem states that*

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{P}(S_n \geq na) = -I(a),$$

where S_n is the sum of n i.i.d. Bernoulli variables, and

$$I(a) := \sup_{\theta \in \mathbb{R}}\{\theta a - \lambda(\theta)\}, \quad \lambda(\theta) = \ln M_{X_1}(\theta).$$

The function $I(a)$ is the Fenchel-Legendre transform *of $\lambda(\theta)$.*

A refinement of Cramér's theorem, known as the Bahadur-Rao theorem, provides a sharper estimate for large deviation probabilities:

$$\mathbb{P}(S_n \geq na) = \frac{e^{-nI(a)}}{\tilde{\theta}\sqrt{2\pi n \lambda''(\tilde{\theta})}}\Big[1 + o(1)\Big], \quad n \to \infty, \tag{2.3}$$

where $\tilde{\theta}$ maximizes $\theta a - \lambda(\theta)$, and $\lambda''(\tilde{\theta})$ is the second derivative of $\lambda(\theta)$. These parameters depend on a (See [1]).

Remark 2.4. Let $\{X_k\}_{k\geq 1}$ be i.i.d. Bernoulli random variables with $P(X_i = 1) = \tau, \quad P(X_i = 0) = 1 - \tau, \quad 0 < \tau \leq \frac{1}{2}$. The logarithmic cumulant function is: $\lambda(\theta) = \ln\left(\tau e^{\theta} + (1-\tau)\right)$.

For any $x \in [\tau, 1)$, the optimal $\tilde{\theta}$ solving $\sup_{\theta\in\mathbb{R}} \{\theta x - \lambda(\theta)\}$ is: $\tilde{\theta} = \ln\left(\frac{x(1-\tau)}{\tau(1-x)}\right)$. Moreover, the rate function is given by $I(x) = x\ln\left(\frac{x}{\tau}\right) + (1-x)\ln\left(\frac{1-x}{1-\tau}\right)$, $\lambda''(\tilde{\theta}(1/2)) = 1/4$, and $\tilde{\theta}\left(\frac{1}{2}\right) = \ln\left(\frac{1-\tau}{\tau}\right)$. From Lemma 2.3.

$$\sum_{k\geq \frac{m}{2}} \binom{m}{k} \tau^k (1-\tau)^{m-k} = \frac{2e^{-mI(1/2)}}{\sqrt{2\pi m \ln\left(\frac{1-\tau}{\tau}\right)}}\Big[1 + o(1)\Big].$$

3. Proofs of Results

Proof of Theorem 1.1. Let $t \geq 0$ and consider the potential $\varphi = -t\log|Df| \circ \pi_1$ on $[0,1]\times\mathbb{Z}$. For $n \in \mathbb{N}$, fixing $(x,k) \in [0,1]\times\mathbb{Z}$, we observe that

$$\Lambda^+(F) \cap F^{-n}(x,k) = \left\{(y,r) : y \in f^{-n}(x), r = k - S_n\varphi(y), \forall j \geq 0,\ S_j\varphi(y) + k - S_n\varphi(y) \geq 0\right\}.$$

Since $\Lambda^+(F)$ consists of points whose orbits under F remain non-negative, a necessary condition for the set to be non-empty is that $k \geq -n$.

To prevent arbitrary growth in k, we refine the definition of topological pressure by restricting k to a symmetric interval around zero:

$$P(F,\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sup_{\substack{x\in[0,1]\\ k\in[-n,n]}} \sum_{(y,r)\in\Lambda^+(F)\cap F^{-n}(x,k)} |Df^n(\pi_1(y,r))|^{-t}.$$

For every $(y,r) \in \Lambda^+(F) \cap F^{-n}(x,k)$ and $j \geq 0$, the sequence $S_j(y,r)$ defined by $S_j(y,r) := S_j\varphi(y) + r \geq 0$ describes a biased random walk on $\mathbb{Z}$. The terms of the sequence are given as

$$S_0(y,r) = k - S_n\varphi(y),$$
$$S_1(y,r) = S_0(y,r) + \varphi(y),$$
$$S_j(y,r) = S_0(y,r) + \sum_{i=0}^{j-1} \varphi(f^i(y)).$$

Since $S_n\varphi(y) \in \{0,1,\dots,n\}$, we count elements of $\Lambda^+(F)\cap F^{-n}(x,k)$ by counting the number of paths from $k - S_n\varphi(y)$ to $a \in \{0,1,\dots,n\}$ in n steps that do not visit -1, that is,

$$\#\Lambda^+(F) \cap F^{-n}(x,k) = \sum_{a=0}^{n} N_n^{\neq -1}(k - S_n\varphi(y), a).$$

Without loss of generality, we can restrict our attention to the main terms, which occur when $k - S_n\varphi(y) = 0$, leading us to consider only the contribution of $\sum_{a=0}^{n} N_n^{\neq -1}(0,a)(p^t)^{\frac{n+a}{2}}(q^t)^{\frac{n-a}{2}}$. Thus,

$$\exp(nP(F,\varphi)) \asymp \sum_{a=0}^{n} N_n^{\neq -1}(0,a)(p^t)^{\frac{n+a}{2}}(q^t)^{\frac{n-a}{2}}. \tag{3.1}$$

Counting Paths in the Biased Walk; for every $n \geq 1$ and $a \in \{0, 1, \ldots, n\}$, we use the following relation $N_n^{\neq -1}(0, a) = N_n(0, a) - N_n(-1, a)$, where

$$N_n(0, a) = \binom{n}{\frac{n+a}{2}} \text{ and } N_n(-1, a) = \binom{n}{\frac{n+a}{2} + 1}.$$

Thus, $N_n^{\neq -1}(0, a) = \binom{n}{\frac{n+a}{2}} - \binom{n}{\frac{n+a}{2}+1}$. Substituting in (3.1), we have

$$\exp(nP(F, \varphi)) \asymp \sum_{a=0}^{n} \left(\binom{n}{\frac{n+a}{2}} - \binom{n}{\frac{n+a}{2} + 1} \right) p^{t(n+a)/2} q^{t(n-a)/2}.$$

Rewriting the sum in terms of ℓ, where $\ell = (n + a)/2$, we obtain:

$$\exp(nP(F, \varphi)) \asymp \sum_{\ell=\lceil n/2 \rceil}^{n} \left(\binom{n}{\ell} - \binom{n}{\ell + 1} \right) p^{t\ell} q^{t(n-\ell)}.$$

The last term can be rewritten as,

$$= \left(1 - \frac{q^t}{p^t}\right) \sum_{k=\lfloor n/2 \rfloor}^{n} \binom{n}{k} (p^t)^k (q^t)^{(n-k)} + \left(\frac{q^t}{p^t}\right) \binom{n}{\lfloor n/2 \rfloor} (p^t)^{\lfloor n/2 \rfloor} (q^t)^{n - \lfloor n/2 \rfloor}.$$

For notational convenience, we omit the integer part symbols $\lfloor \cdot \rfloor$ and proceed as if the expressions are integers, noting that this simplification does not affect the asymptotic estimates. By Applying (2.2) Lemma 2.3, and Remark 2.4, we have

$$\begin{aligned} \exp(nP(F, \varphi)) = & \left(1 - \frac{q^t}{p^t}\right) (p^t + q^t)^n \frac{e^{-nI_t(1/2)}}{\sqrt{2\pi n}\sqrt{\tilde{\theta}^2 \lambda''(\tilde{\theta})}} \Big[1 + o(1)\Big] \\ & + \left(\frac{q^t}{p^t}\right) (p^t)^{n/2} (q^t)^{n/2} \frac{2^n}{\sqrt{\pi n}} \Big[1 + o(1)\Big] \end{aligned} \quad (3.2)$$

where $I_t(1/2) = -\frac{1}{2} \ln \left(\frac{p^t}{p^t+q^t}\right) - \frac{1}{2} \ln \left(\frac{q^t}{p^t+q^t}\right) - \ln 2$, $\tilde{\theta} = \ln \left(q^t/p^t\right)$ and $\lambda''(\theta) = 1/4$.

Note that

$$(p^t)^{n/2} (q^t)^{n/2} \frac{2^n}{\sqrt{\pi n}} \Big[1 + o(1)\Big] = (p^t + q^t)^n \frac{e^{-nI_t(1/2)}}{\sqrt{2\pi n}} \Big[1 + o(1)\Big]$$

Then from (3.2) we get:

$$\exp(nP(F, \varphi)) = (p^t + q^t)^n \frac{e^{-n(I_t(1/2))}}{\sqrt{2\pi n}} \left\{ \frac{q^t}{p^t} + \frac{1}{\ln\left(\frac{q^t}{p^t}\right)} \left(1 - \frac{q^t}{p^t}\right) \right\} \Big[1 + o(1)\Big].$$

Taking logarithms and passing to the limit as $n \to \infty$, we obtain $P(F, \varphi) = \ln(p^t + q^t) - I_t(1/2)$. To determine the value t_0 for which the topological pressure vanishes, we solve $P(F, \varphi) = 0$, yielding $t_0 = \frac{-2 \ln 2}{\ln p + \ln q}$.

Proof of Theorem 1.3. By definition, for $(z,k) \in \Lambda_+^k(F)$, we have that, for every $j \geq 1$, $S_j\varphi(z) + k \geq 0$. Let $y \in [p,1)$ such that $f(y) = z$, then $\varphi(y) = -1$, and for every $j \geq 0$,

$$S_{j+1}\varphi(y) + k + 1 = S_j\varphi(x) + k \geq 0.$$

Then, for all $k \geq 1$, $\Lambda_+^k(F) \subset F(\Lambda_+^{k+1}(F) \cap [p,1) \times \{k+1\})$.

Conversely, if $(y, k+1) \in \Lambda_+^{k+1}(F)$ and let $x \in [0,p]$ such that $f(x) = y$. Since $\varphi(x) = 1$, then, for all $j \geq 0$, $S_j\varphi(x) + k = S_j\varphi(y) + k + 1 \geq 0$,, then

$$\Lambda_+^{k+1}(F) \subset F(\Lambda_+^k(F) \cap [0,p] \times \{k\}).$$

Since F is a piecewise linear expanding map, it induces a bi-Lipschitz correspondence between $\Lambda_+^k(F)$ and $\Lambda_+^{k+1}(F)$, preserving the Hausdorff dimension. Hence,

$$\dim_H \Lambda_+^k(F) = \dim_H \Lambda_+^{k+1}(F) = \dim_H \Lambda_+(F).$$

On the other hand, the map $f : [0,1] \to [0,1]$ induces a symbolic encoding in the space of binary sequences $\Sigma = \{-1,1\}^{\mathbb{N}}$. Each point $x \in [0,1]$ has a symbolic representation given by the sequence of signs

$$\omega(x) = (\varphi(f^j(x)))_{j \geq 0} \in \{-1,1\}^{\mathbb{N}},$$

The set of finite words of length n in this space is $\Sigma_n = \{-1,1\}^{\{0,1,\ldots,n-1\}}$. For each finite word $\omega_n \in \Sigma_n$, we define the partial sum $\mathcal{S}(\omega_n) = \sum_{i=1}^n \omega_i$. According to Lemma 2.2, for every integer m, there exists an integer $N(m)$ such that

$$\mathbb{P}\left(\mathcal{S}(w_{N(m)}) < -m\right) \asymp \frac{2}{3}.$$

This implies that among the $2^{N(m)}$ words of length $N(m)$, fewer than $\frac{2}{3} \cdot 2^{N(m)}$ satisfy $\mathcal{S}(\omega_{N(m)}) < -m$, ensuring that at least $(1 - 2/3)2^{N(m)} = \frac{1}{3}2^{N(m)}$ words satisfy $\mathcal{S}(\omega_{N(m)}) \geq -m$.

To construct the covering family of intervals, we fix an initial value m_0 and define the sequence $(m_k)_{k \geq 1}$ recursively by $m_k := N(m_0 + \cdots + m_{k-1}), \forall k \geq 1$.

The accumulated length of sequences up to step k is given by, $n_k := m_1 + \cdots + m_k$. We define the word τ_{n_k} as the concatenation

$$\tau_{n_k} = \alpha_{m_1}\beta_{m_2}\ldots\omega_{m_k},$$

where each block satisfies a lower bound on its partial sum:

α_{m_1} is a word for $m = m_0$ satisfying $\mathcal{S}(w_{m_1}) \geq -m_0$.

β_{m_2} is a word for $m = m_0 + m_1$ satisfying $\mathcal{S}(w_{m_2}) \geq -(m_0 + m_1)$.

ω_{m_k} is a word for $m = m_0 + n_{k-1}$ satisfying $\mathcal{S}(w_{m_k}) \geq -(m_0 + n_{k-1})$.

The family of intervals $I[\tau_{n_k}]$ forms a covering of $\pi_1(\Lambda_{m_0}^+(F))$. This is because the sequences τ_{n_k} are constructed to ensure that the second coordinate never becomes negative.

There are at most $(2/3)^k 2^{n_k}$ intervals in this covering. At each step, we select only those words satisfying $\mathcal{S}(w_{m_k}) \geq -m$, so the number of selected words decreases as $(2/3)^k 2^{n_k}$.

Since, $p < q$, the length of each such interval satisfies $|I[\tau_{n_k}]|^s \leq 2^{-n_k}$. Therefore, the s-dimensional Hausdorff measure of the projection satisfies,

$$H^s(\pi_1(\Lambda^+_{m_0}(F))) \leq \lim_{k\to\infty} (2/3)^k 2^{n_k} \sup_{\tau_{n_k}} |I[\tau_{n_k}]|^s \leq \lim_{k\to\infty} (2/3)^k 2^{n_k} 2^{-n_k} = 0.$$

Since this estimate holds for any choice of m_0, it follows the same bound holds for the full set $\Lambda_+(F)$.

4. Conclusions

In this work, we established a connection between deterministic dynamical systems and stochastic processes by studying the set of orbits that stay non-negative under a fibered dynamical map. Using tools from asymptotic probability, we computed the topological pressure and Hausdorff dimension of such dynamically defined sets, showing their connection to random walks and the reflection principle.

The system studied here is based on a map with two linear branches and a simple sign function. It is natural to consider generalizations involving maps with more than two branches, where the step sizes in the second coordinate may vary. Such extensions could reveal aspects of the structure of more intricate subsets where the dynamics remain non-negative, and may give rise to richer fractal geometry.

Another direction involves replacing the linear components with nonlinear expanding maps. In such cases, the distortion introduced by the map would affect the behavior of the orbits and would require more advanced tools to analyze the pressure and Hausdorff dimension.

Acknowledgments. The author acknowledges financial support of ANID/ULS, through the Project InES Género INGE220009.

The author is deeply grateful to the anonymous referee of an earlier version of this manuscript for their thorough and constructive review, which greatly helped improve this work.

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I. INOQUIO-RENTERIA

Departamento de Matemáticas, Facultad de Ciencias, Universidad de La Serena, Avda. Juan Cisternas 1200, La Serena, 1700000, Chile.

Email address: irene.inoquio@userena.cl