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NEW CLASSES OF SEIDEL EQUIENERGETIC GRAPHS

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ABSTRACT. In this paper, we give the complete characterization of the $\mathcal{S}$-eigenvalues of the union of the join graph $G_1 \vee G_2$ and the corona product $G_1 \circ G_3$ when G_1, G_2 and G_3 are regular graphs. As an application, we give some new methods to construct $\mathcal{S}$-equienergetic graphs.

1. Introduction

Let Γ be a graph (simple) with vertex set $V(\Gamma)$ and $|V(\Gamma)| = n$. The Seidel matrix of Γ, denoted by $\mathcal{S}(\Gamma)$, is the matrix $\mathcal{S}(\Gamma) = \mathcal{J}_n - I_n - 2\mathcal{A}(\Gamma)$, where $\mathcal{J}_n = [a_{ij}]_{n\times n}$ with $a_{ij} = 1$ for all $1 \leq i, j \leq n$ and $\mathcal{A}(\Gamma)$ is the well-known adjacency matrix of Γ. The eigenvalues of $\mathcal{S}(\Gamma)$ (resp. $\mathcal{A}(\Gamma)$) are called the Seidel eigenvalues or $\mathcal{S}$-eigenvalues (resp. eigenvalues) of Γ. The (Seidel) spectrum of Γ is the list of all (Seidel) eigenvalues of Γ. For studies on spectral properties of Seidel matrix one may refer to [4, 3, 8] and therein cited references. The Seidel energy of Γ, denoted by $\mathcal{E}_{\mathcal{S}}(\Gamma)$, is the sum $\sum_{i=1}^{n} \eta_i$, where η_i's are the $\mathcal{S}$- eigenvalues of Γ. Two graphs Γ_1 and Γ_2 of same order having distinct Seidel spectrum are called Seidel equienegertic (simply, $\mathcal{S}$-equienergetic) if $\mathcal{E}_{\mathcal{S}}(\Gamma_1) = \mathcal{E}_{\mathcal{S}}(\Gamma_2)$. Some methods to construct $\mathcal{S}$-equienergetic graphs are given in [7, 10]. Recent studies on Seidel energy can be found in [9, 2] and therein cited references.

The join of graphs Γ_1 and Γ_2, denoted by $\Gamma_1 \vee \Gamma_2$, is obtained by taking one copies of Γ_1, Γ_2 and then joining each vertex of Γ_1 with every vertices of Γ_2 [5]. In [7], $\mathcal{S}$-spectrum of $\Gamma_1 \vee \Gamma_2$ is computed when Γ_1 and Γ_2 are regular graphs. The corona product [6] of two graphs Γ_1 and Γ_2, denoted by $\Gamma_1 \circ \Gamma_2$, is obtained by taking $|V(\Gamma_1)|$ copies of Γ_2 and then joining the i-th vertex of Γ_1 with all vertices of the i-th copy of Γ_2. The $\mathcal{S}$- eigenvalues and the pertaining Seidel eigenvectors of corona product are described completely in [1]. With this motivation, here we give the complete characterization of the $\mathcal{S}$-eigenvalues of the graph $(G_1 \vee G_2) \cup (G_1 \circ G_3)$, i.e., the union of the join graph $G_1 \vee G_2$ and the corona product $G_1 \circ G_3$ when G_1, G_2 and G_3 are regular graphs. As an application, we give some new methods to construct $\mathcal{S}$-equienergetic graphs.

2. Main Results

Let $J_{p\times q}$ be the $p \times q$ matrix given by $J_{p\times q} = [a_{ij}]$, where $a_{ij} = 1$. Denote by $\mathbb{1}_p$, the column matrix $\begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}^T$ with p elements. Let $e(p, k)$ be the

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column matrix of size p whose only non-zero entry is at its k-th position and is equal to 1. The zero column matrix of order p is denoted by $\mathbf{0}_p$. The following theorem describes the $\mathcal{S}$-eigenvalues of $(G_1 \vee G_2) \cup (G_1 \circ G_3)$ when G_1, G_2 and G_3 are regular graphs.

Theorem 2.1. *Let G_i be an r_i-regular graph on n_i vertices for $i = 1, 2, 3$. Let $\lambda_{ij}, j = 1, 2, \ldots, n_i$ be the spectrum of G_i Then the Seidel spectrum of $(G_1 \vee G_2) \cup (G_1 \circ G_3)$ consists of:*

(i) $-1 - 2\lambda_{3j},\ j = 2, 3, \ldots, n_3$ *with multiplicity* n_1.
(ii) $-1 - 2r_3 - 2t$, *where* $2t = \lambda_{1j} - r_3 \pm \sqrt{(\lambda_{1j} - r_3)^2 + 4n_3}$ *and* $j = 2, 3, \ldots, n_1$.
(iii) $-1 - 2\lambda_{2j},\ j = 2, 3, \ldots, n_3$.
(iv) *Three roots of the polynomial*

$$\det \begin{pmatrix} -1 - 2r_3 + n_1 n_3 - t & n_1 - 2 & n_2 \\ (n_1 - 2)n_3 & n_1 - 1 - 2r_1 - t & -n_2 \\ n_1 n_3 & -n_1 & n_2 - 1 - 2r_2 - t \end{pmatrix} = 0.$$

Proof. Let $\Gamma = (G_1 \vee G_2) \cup (G_1 \circ G_3)$. The $\mathcal{S}$-matrix of $(G_1 \vee G_2) \cup (G_1 \circ G_3)$ is given by

$$\left[\begin{array}{c|c|c} I_{n_1} \otimes \mathcal{S}(G_3) + (J_{n_1} - I_{n_1}) \otimes J_{n_3} & (J_{n_1} - 2I_{n_1}) \otimes \mathbb{1}_{n_3} & J_{n_1 \times n_2} \otimes \mathbb{1}_{n_3} \\ \hline (J_{n_1} - 2I_{n_1}) \otimes \mathbb{1}_{n_3}^T & \mathcal{S}(G_1) & -J_{n_1 \times n_2} \\ \hline J_{n_2 \times n_1} \otimes \mathbb{1}_{n_3}^T & -J_{n_2 \times n_1} & \mathcal{S}(G_2) \end{array}\right].$$

Let $i = 1, 2, 3$ and $\{Z_{ij} : j = 1, 2, \ldots, n_i\}$ be a set of orthogonal eigenvectors of the adjacency matrix $\mathcal{A}(G_i)$ corresponding to the eigenvalues $\lambda_{ij}, j = 1, 2, \ldots, n_i$. Since G_i for $i = 1, 2, 3$ is regular, we can assume that $Z_{i1} = \mathbb{1}_{n_i}$.
For $j = 2, 3, \ldots, n_3$ and $k = 1, 2, \ldots, n_1$, we have

$$\mathcal{S} \begin{bmatrix} e(n_1, k) \otimes Z_{3j} \\ \mathbf{0}_{n_1} \\ \mathbf{0}_{n_2} \end{bmatrix} = \eta_{kj} \begin{bmatrix} e(n_1, k) \otimes Z_{3j} \\ \mathbf{0}_{n_1} \\ \mathbf{0}_{n_2} \end{bmatrix},$$

where $\eta_{kj} = -1 - 2\lambda_{3j}$. Thus, $\begin{bmatrix} e(n_1, k) \otimes Z_{3j} \\ \mathbf{0}_{n_1} \\ \mathbf{0}_{n_2} \end{bmatrix}$, $k = 1, 2, \ldots, n_1$ and $j = 2, 3, \ldots, n_3$ form a set of $n_1(n_3 - 1)$ orthogonal eigenvectors corresponding to the eigenvalue η_{kj}.
Further, let $j = 2, 3, \ldots, n_1$ and δ_j be some scalar. Then

$$\mathcal{S} \begin{bmatrix} Z_{1j} \otimes \mathbb{1}_{n_3} \\ \delta_j Z_{1j} \\ \mathbf{0}_{n_2} \end{bmatrix} = \begin{bmatrix} (-1 - 2r_3 - 2\delta_j) Z_{1j} \otimes \mathbb{1}_{n_3} \\ (-2n_3 - (1 + 2\lambda_{1j})\delta_j) Z_{1j} \\ \mathbf{0}_{n_2} \end{bmatrix} = (-1 - 2r_3 - 2\delta_j) \begin{bmatrix} Z_{1j} \otimes \mathbb{1}_{n_3} \\ \delta_j Z_{1j} \\ \mathbf{0}_{n_2} \end{bmatrix}$$

for $2\delta_j = \lambda_{1j} - r_3 \pm \sqrt{(\lambda_{1j} - r_3)^2 + 4n_3}$. Thus, $\begin{bmatrix} Z_{1j} \otimes \mathbb{1}_{n_3} \\ \delta_j Z_{1j} \\ \mathbf{0}_{n_2} \end{bmatrix}$ $j = 2, 3, \ldots, n_1$ form a set of $2(n_1 - 1)$ orthogonal eigenvectors of $\mathcal{S}$ corresponding to the eigenvalue $(-1 - 2r_3 - 2\delta_j) \begin{bmatrix} Z_{1j} \otimes \mathbb{1}_{n_3} \\ \delta_j Z_{1j} \\ \mathbf{0}_{n_2} \end{bmatrix}$.

Also, for $j = 2, 3, \dots, n_3$, we have

$$S \begin{bmatrix} \mathbf{0}_{n_1 n_3} \\ \mathbf{0}_{n_1} \\ Z_{2j} \end{bmatrix} = (-1 - 2\lambda_{2j}) \begin{bmatrix} \mathbf{0}_{n_1 n_3} \\ \mathbf{0}_{n_1} \\ Z_{2j} \end{bmatrix}.$$

Thus, $\begin{bmatrix} \mathbf{0}_{n_1 n_3} \\ \mathbf{0}_{n_1} \\ Z_{2j} \end{bmatrix}$ for $j = 2, 3, \dots, n_3$ form a set of $n_2 - 1$ orthogonal eigenvectors corresponding to the eigenvalue $-1 - 2\lambda_{2j}$.

Henceforth, we have listed $n_1 n_3 + n_1 + n_2 - 3$ orthogonal eigenvectors of $\mathcal{S}$. Since the order of the graph Γ is $n_1 n_3 + n_1 + n_2$, we need to determine 3 more $\mathcal{S}$-eigenvalues of Γ. Let these $\mathcal{S}$-eigenvalues be ζ_i for $i = 1, 2, 3$ corresponding to the eigenvectors X_i. Observe that the listed $n_1 n_3 + n_1 + n_2 - 3$ orthogonal eigenvectors of S along with the vectors $\begin{bmatrix} \mathbb{1}_{n_1} \otimes \mathbb{1}_{n_3} \\ \mathbf{0}_{n_1} \\ \mathbf{0}_{n_2} \end{bmatrix}$, $\begin{bmatrix} \mathbf{0}_{n_1 n_3} \\ \mathbb{1}_{n_1} \\ \mathbf{0}_{n_2} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{0}_{n_1 n_3} \\ {}_{n_1} \\ \mathbb{1}_{n_2} \end{bmatrix}$ form an orthogonal set of n vectors. Thus, $X_i = \begin{bmatrix} a_i \mathbb{1}_{n_1} \otimes \mathbb{1}_{n_3} \\ b_i \mathbb{1}_{n_1} \\ c_i \mathbb{1}_{n_2} \end{bmatrix}$ for some scalars a_i, b_i and c_i.

Therefore, from the equation, $\mathcal{S} X_i = \zeta_i X_i$, we get

$$\det \begin{pmatrix} -1 - 2r_3 + n_1 n_3 - \zeta_i & n_1 - 2 & n_2 \\ (n_1 - 2) n_3 & n_1 - 1 - 2r_1 - \zeta_i & -n_2 \\ n_1 n_3 & -n_1 & n_2 - 1 - 2r_2 - \zeta_i \end{pmatrix} = 0.$$

Thus the three more eigenvalues of $\mathcal{S}$ are roots of above polynomial equation in ζ_i. □

Corollary 2.2. *Let Γ_1 and Γ_2 be arbitrary regular graphs. Let Γ_3 and Γ_4 be two $\mathcal{S}$-equienergetic r-regular graphs. Then the graphs*

(i) $(\Gamma_1 \vee \Gamma_3) \cup (\Gamma_1 \circ \Gamma_2)$ and $(\Gamma_1 \vee \Gamma_4) \cup (\Gamma_1 \circ \Gamma_2)$ are $\mathcal{S}$-equienergetic.
(ii) $(\Gamma_1 \vee \Gamma_2) \cup (\Gamma_1 \circ \Gamma_3)$ and $(\Gamma_1 \vee \Gamma_2) \cup (\Gamma_1 \circ \Gamma_4)$ are $\mathcal{S}$-equienergetic.

Lemma 2.3. [7] *The graphs as shown in Figure 1 are $\mathcal{S}$-equienergetic 3-regular graphs on 12 vertices.*

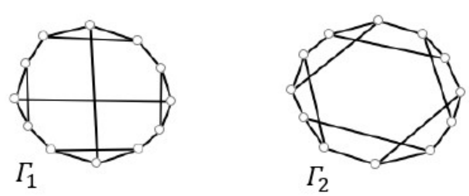

FIGURE 1. $\mathcal{S}$-equienergetic graphs Γ_1 and Γ_2 on 12 vertices.

Corollary 2.4. *There exists $\mathcal{S}$-equienergetic graph on $2n$ vertices for $n > 12$.*

Proof. Let Γ_1 and Γ_2 be graphs as shown in Fig. 1. Then by Lemma 2.3, Γ_1 and Γ_2 are S-equienergetic 3-regular graphs on 12 vertices. Therefore by Corollary 2.4 the graphs $(\overline{K_m} \vee \Gamma_1) \cup (\overline{K_m} \circ K_1)$ and $(\overline{K_m} \vee \Gamma_2) \cup (\overline{K_m} \circ K_1)$ are $\mathcal{S}$-equienergetic graphs on $2m + 12$ vertices for all $m \geq 1$. □

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