Global and Stochastic Analysis
Vol. 10 No. 1 (January-June, 2023)

Received: 24th March 2023 ***Revised: 29th April 2023*** ***Accepted: 16th May 2023***

TWO PARAMETER CALCULATION OF SOME IDENTITIES OF THETA-FUNCTIONS

SHRUTHI, R. NARENDRA, AND B. R. SRIVATSA KUMAR*

ABSTRACT. In the present paper, we calculate two parameters $g_{k,n}$ and $g'_{k,n}$ of some P-Q type theta-function $\psi(q)$ for some positive real numbers k and n. Also, we evaluate Ramanujan-Göllnitz-Gordon continued fraction during this process.

1. Introduction

Always, it is accurately considered in the sequel that $|q| < 1$. For $q := e^{2\pi iz}$, $Im(z) > 0$, define

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = 2^{-1}q^{-1/8}\Theta_2(0,z)$$

and

$$f(-q) := (q;q)_\infty = q^{-1/24}\eta(z),$$

where Θ_2 is the classical theta-function [16] and $\eta(z)$ represents the Dedekind eta-function and

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k).$$

Recently, J. Yi [13–15] evaluated several new values of $\varphi(q)$, $\psi(q)$ and $f(q)$ applying modular identities, transformation formulae for theta-functions and the parameters $c_{k,n}$ and $c'_{k,n}$ is defined as follows:

Definition 1.1. For all $k, n \in \mathbb{Z}$, we have

$$c_{k,n} := \frac{\psi(-q)}{k^{1/4}q^{(k-1)/8}\psi(-q^k)} \qquad q = e^{-\pi\sqrt{n/k}}, \tag{1.1}$$

$$c'_{k,n} := \frac{\psi(q)}{k^{1/4}q^{(k-1)/8}\psi(q^k)} \qquad q = e^{-\pi\sqrt{n/k}}. \tag{1.2}$$

Also, the following results holds true.

i. $c_{k,1} = 1$,
ii. $c_{k,1/n} = c_{k,n}^{-1}$,
iii. $c_{k,n} = c_{n,k}$.

2000 *Mathematics Subject Classification.* 33D10, 11F27.

Key words and phrases. Theta function, Modular equation, Ramanujan-Göllnitz-Gordon continued fraction.

The Ramanujan-Göllnitz-Gordan continued fraction $H(q)$ be defined by

$$H(q) := \frac{q^{1/2}}{1+q_{+}}\frac{q^2}{1+q^3{}_{+}}\frac{q^4}{1+q^5{}_{+\dots}}.$$

The atop identity was first found by S. Ramanujan in his second notebook [8, p. 229]. H. Göllnitz [6] and B. Gordon [7] reclaimed $H(q)$ without knowing work of Ramanujan. Ramanujan also documented the following two identities for $H(q)$ in his second notebook [8, p. 229].

$$\frac{1}{H(q)} - H(q) = \frac{\varphi(q^2)}{q^{1/2}\psi(q^4)}$$

and

$$\frac{1}{H(q)} + H(q) = \frac{\varphi(q)}{q^{1/2}\psi(q^4)}.$$

Validation of the atop two identities can be seen in [2, p. 221]. H. H. Chan and S. S. Huang [5], entrenched several relations for $H(q)$, which are similar to the results of distinguished Roger-Ramanujan continued fraction and Ramanujan's cubic continued fraction. Chan and Huang [5] likewise obtained many accurate formulas for evaluating $H(e^{-\pi\sqrt{n}/2})$ in terms of Ramanujan-Weber class invariants. Recently C. Adiga et. al. [1] obtained many modular identities for the Rogers-Ramanujan type functions of order eleven which are analogues to Ramanujan's forty identities and also they found some partition theoretic interpretations. Inspired by the atop mentioned work, in the present paper, we find some general formulas for the explicit calculation of $c'_{2,n}, c_{3,n}$ and $c'_{3,n}$. In [3], S. Bhargava et al. established the following:

Lemma 1.1. *If*

$$A_n := \frac{1}{\sqrt[4]{3}}\frac{\psi(-q)}{\psi(-q^3)} \qquad\qquad q = e^{-\pi\sqrt{n/3}},$$

then

i. $A_n A_{1/n} = 1,$

ii. $A_1 = 1,$

iii. $H(q) = \dfrac{1}{\sqrt[3]{3A_n^4+1}}.$

2. Preliminaries

We state some P-Q type theta-function identities in this section, which we need in sequel.

Theorem 2.1. *[3] If*

$$P := \frac{\psi(q)}{q^{1/4}\psi(q^3)} \qquad \textit{and} \qquad Q := \frac{\psi(q^3)}{q^{3/4}\psi(q^9)}$$

then

$$(PQ)^2 + \frac{9}{(PQ)^2} = 3 + 6\frac{Q^2}{P^2} + \frac{Q^4}{P^4}.$$

Theorem 2.2. *[12] If*

$$P := \frac{\psi(q)}{q^{1/4}\psi(q^3)} \qquad \text{and} \qquad Q := \frac{\psi(q^2)}{q^{1/2}\psi(q^6)}$$

then

$$P^2 + \frac{3}{P^2} = \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2.$$

Theorem 2.3. *[12] If*

$$P := \frac{\psi(q)}{q^{1/8}\psi(q^2)} \qquad \text{and} \qquad Q := \frac{\psi(q^3)}{q^{3/8}\psi(q^6)}$$

then

$$(PQ)^2 + \frac{16}{(PQ)^2} = 4\left[\left(\frac{P}{Q}\right)^4 - \left(\frac{Q}{P}\right)^4\right] + 9.$$

Theorem 2.4. *[9] If*

$$P := q^{3/8}\frac{\psi(-q)\psi(-q^6)}{\psi(-q^2)\psi(-q^3)} \qquad \text{and} \qquad Q := q^{3/4}\frac{\psi(-q^2)\psi(-q^{12})}{\psi(-q^4)\psi(-q^6)}$$

then

$$\left(\sqrt{\frac{P}{Q}} + \sqrt{\frac{Q}{P}}\right)\left(\sqrt{PQ} + \frac{1}{\sqrt{PQ}}\right) - 8 = 0.$$

Theorem 2.5. *[9] If*

$$P := q^{1/4}\frac{\psi(-q)\psi(-q^{15})}{\psi(-q^3)\psi(-q^5)} \qquad \text{and} \qquad Q := q^{1/2}\frac{\psi(-q^2)\psi(-q^{30})}{\psi(-q^6)\psi(-q^{10})}$$

then

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + \frac{P}{Q} + \frac{Q}{P} - \left(\frac{1}{\sqrt{PQ}} - \sqrt{PQ}\right)$$
$$\times\left(\sqrt{\frac{P}{Q}} + \sqrt{\frac{Q}{P}} + \sqrt[3]{\frac{P}{Q}} + \sqrt[3]{\frac{Q}{P}}\right) = PQ + \frac{1}{PQ}.$$

Theorem 2.6. *[9] If*

$$P := q^{1/2}\frac{\psi(-q)\psi(-q^9)}{\psi^2(-q^3)} \qquad \text{and} \qquad Q := q\frac{\psi(-q^2)\psi(-q^{18})}{\psi^2(-q^6)}$$

then

$$\left(\frac{P}{Q}\right)^4 + \left(\frac{Q}{P}\right)^4 + \left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 - \left(3PQ - \frac{1}{PQ}\right)\left(\left(\frac{P}{Q}\right)^3 + \left(\frac{Q}{P}\right)^3\right)$$
$$-3\left(3PQ - \frac{1}{PQ}\right)\left(\frac{P}{Q} + \frac{Q}{P}\right) - \frac{1}{(PQ)^2} - 9\,(PQ)^2 - 6 = 0.$$

3. Evaluations of $g_{k,n}$ and $g^{'}_{k,n}$

Theorem 3.1. *We have*

$$\text{i.} \qquad c_{3,18} = \sqrt{5+3\sqrt{2}+\frac{A}{3}+\sqrt[3]{2}B},$$

$$\text{ii.} \qquad c'_{3,1/9} = \frac{1}{\sqrt{6}}\sqrt{\sqrt[6]{32}(\sqrt{3}+1)+\sqrt{6}+2\sqrt[6]{2}-\sqrt{2}},$$

where

$$A = \left(9612+6804\sqrt{2}-108\sqrt{99+70\sqrt{2}}\right)^{1/3},$$

$$B = \left(89+63\sqrt{2}+\sqrt{99+70\sqrt{2}}\right)^{1/3}.$$

Proof. On applying the interpretation of $c'_{k,n}$ in Theorem 2.1, we obtain

$$3\left(c'_{3,n}c'_{3,9n}\right)^2 + \frac{3}{\left(c'_{3,n}c'_{3,9n}\right)^2} = 3+6\left(\frac{c'_{3,9n}}{c'_{3,n}}\right)^2 + \left(\frac{c'_{3,9n}}{c'_{3,n}}\right)^4. \qquad (3.1)$$

Setting $n=2$ in the above, we obtain

$$3\left(c'_{3,2}c'_{3,18}\right)^2 + \frac{3}{\left(c'_{3,2}c'_{3,18}\right)^2} = 3+6\left(\frac{c'_{3,18}}{c'_{3,2}}\right)^2 + \left(\frac{c'_{3,18}}{c'_{3,2}}\right)^4. \qquad (3.2)$$

From [4], we have

$$c'_{3,2} = \left(1+\sqrt{2}\right)^{1/2}. \qquad (3.3)$$

Using (3.3) in (3.2) and after simplification, we obtain $c'_{3,18}$. Further on setting $n=1/9$ in (3.1), we obtain

$$3(c'_{3,1/9}c'_{3,1})^2 + \frac{3}{(c'_{3,1/9}c'_{3,1})^2} = 3+6\left(\frac{c'_{3,1}}{c'_{3,1/9}}\right)^2 + \left(\frac{c'_{3,1}}{c'_{3,1/9}}\right)^4 \qquad (3.4)$$

From [4], we have

$$c'_{3,1} = \sqrt{\frac{\sqrt{3}+1}{\sqrt{2}}}. \qquad (3.5)$$

Using (3.5) in (3.4) and after simplification we obtain $c'_{3,1/9}$. □

Corollary 3.1. *We have*

$$H(e^{-\sqrt{6}\pi}) = \frac{1}{\sqrt[4]{3\left(5+3\sqrt{2}+\frac{A}{3}+\sqrt[3]{2}B\right)^2+1}}.$$

Proof. On applying Theorem 3.1(i) to Lemma 1.1(iii), we obtain the above result. □

Theorem 3.2. *We have*

$$c'_{3,1/2} = \sqrt[4]{\frac{\sqrt{3}a - a^2}{\sqrt{3}a - 1}}$$

where

$$a = 1 + \sqrt{2}.$$

Proof. Applying the interpretation of $c'_{k,n}$ in Theorem 2.2, we have

$$\sqrt{3}(c'_{3,n})^2 + \frac{\sqrt{3}}{(c'_{3,n})^2} = \left(\frac{c'_{3,4n}}{c'_{3,n}}\right)^2 + \left(\frac{c'_{3,n}}{c'_{3,4n}}\right)^2. \tag{3.6}$$

On setting $n = 1/2$ in (3.6), we have

$$\sqrt{3}(c'_{3,1/2})^2 + \frac{\sqrt{3}}{(c'_{3,1/2})^2} = \left(\frac{c'_{3,2}}{c'_{3,1/2}}\right)^2 + \left(\frac{c'_{3,1/2}}{c'_{3,2}}\right)^2. \tag{3.7}$$

On using (3.3) in (3.7) and after simplification, we obtain $c'_{3,1/2}$. □

Theorem 3.3. *We have*

i. $$c'_{2,36} = \sqrt{\frac{9a + \sqrt{418(1+\sqrt{2})}}{8(2+\sqrt{2})}},$$

ii. $$c'_{2,72} = \sqrt{\frac{9(2+\sqrt{2}) + \sqrt{2790 + 2116\sqrt{2}}}{8(5+2\sqrt{2})}},$$

where

$$a = \sqrt{\sqrt{2}-1} + \sqrt{\sqrt{2}+1}.$$

Proof. Applying the interpretation of $c'_{k,n}$ in Theorem 2.3, we obtain

$$2\left(c'_{2,n}c'_{2,9n}\right)^2 + \frac{8}{\left(c'_{2,n}c'_{2,9n}\right)^2} = 4\left[\left(\frac{c'_{2,n}}{c'_{2,9n}}\right)^2 - \left(\frac{c'_{2,9n}}{c'_{2,n}}\right)^2\right] + 9. \tag{3.8}$$

Setting $n = 4$ in the above, we have

$$2\left(c'_{2,4}c'_{2,36}\right)^2 + \frac{8}{\left(c'_{2,4}c'_{2,36}\right)^2} = 4\left[\left(\frac{c'_{2,4}}{c'_{2,36}}\right)^2 - \left(\frac{c'_{2,36}}{c'_{2,4}}\right)^2\right] + 9. \tag{3.9}$$

From [4], we have

$$c'_{2,4} = \sqrt{\sqrt{\sqrt{2}-1} + \sqrt{\sqrt{2}+1}}. \tag{3.10}$$

On using (3.10) in (3.9) and after simplification, we obtain $c'_{2,36}$. Further on setting $n = 8$ in (3.8), we have

$$2\left(c'_{2,8}c'_{2,72}\right)^2 + \frac{8}{\left(c'_{2,8}c'_{2,72}\right)^2} = 4\left[\left(\frac{c'_{2,8}}{c'_{2,72}}\right)^2 + \left(\frac{c'_{2,72}}{c'_{2,8}}\right)^2\right] + 9. \tag{3.11}$$

From [4], we have

$$c'_{2,8} = \sqrt{2+\sqrt{2}}. \tag{3.12}$$

On using (3.12) in (3.11) and after simplification we obtain $c'_{3,72}$. □

Theorem 3.4. *We have*

i. $c_{3,4} = 2+\sqrt{3}$,
ii. $c_{3,16} = 7+4\sqrt{3}$.

Proof. Applying the interpretation of $c_{k,n}$ in Theorem 2.4, we obtain

$$\left(\frac{\sqrt{c_{k,n}c_{k,16n}}}{c_{k,4n}} + \frac{c_{k,4n}}{\sqrt{c_{k,n}c_{k,16n}}}\right)\left(\sqrt{\frac{c_{k,n}}{c_{k,16n}}} + \sqrt{\frac{c_{k,16n}}{c_{k,n}}}\right) - 8 = 0. \tag{3.13}$$

On letting $k=3$ and $n=1/4$ in the above and the definition of $c_{k,n}$, we obtain

$$c_{3,4} + \frac{1}{c_{3,4}} - 4 = 0.$$

On solving, we obtain Theorem 3.4(i). Further, on setting $k=3$ and $n=1$ in (3.13), we obtain

$$\left(\frac{c_{3,4}}{\sqrt{c_{3,16}}} + \frac{\sqrt{c_{3,16}}}{c_{3,4}}\right)\left(\frac{1}{\sqrt{c_{3,16}}} + \sqrt{c_{3,16}}\right) - 8 = 0.$$

Using $c_{3,4} = 2+\sqrt{3}$ and by the definition of $c_{k,n} > 1$, we obtain Theorem 3.4(ii). □

Corollary 3.2. *We have*

i. $H(e^{-2\pi/\sqrt{3}}) = \dfrac{1}{\sqrt[4]{292+168\sqrt{3}}}$,
ii. $H(e^{-4\pi/\sqrt{3}}) = \dfrac{1}{\sqrt[4]{56452+32592\sqrt{3}}}$.

Proof. On applying Theorem 3.4 to Lemma 1.1(iii), we obtain the result. □

Theorem 3.5. *We have*

i. $c_{3,10} = \left(\sqrt{3}+\dfrac{\sqrt{6}}{2}-\dfrac{1}{\sqrt{2}}-1\right)^{1/2}\left(\sqrt{6}+\sqrt{5}\right)^{1/4}$,
ii. $c_{3,2/5} = \dfrac{\left(\sqrt{6}+\sqrt{5}\right)^{1/4}}{\left(\sqrt{3}+\frac{\sqrt{6}}{2}-\frac{1}{\sqrt{2}}-1\right)^{1/2}}$.

Proof. On using the definition of $c_{k,n}$ in Theorem 2.5, we have

$$\left(\frac{c_{k,n}c_{k,100n}}{c_{k,4n}c_{k,25n}}\right)^2+\left(\frac{c_{k,4n}c_{k,25n}}{c_{k,n}c_{k,100n}}\right)^2+\frac{c_{k,n}c_{k,100n}}{c_{k,4n}c_{k,25n}}+\frac{c_{k,4n}c_{k,25n}}{c_{k,n}c_{k,100n}}$$
$$-\left(\left(\frac{c_{k,n}c_{k,100n}}{c_{k,4n}c_{k,25n}}\right)^{1/2}+\left(\frac{c_{k,4n}c_{k,25n}}{c_{k,n}c_{k,100n}}\right)^{1/2}+\left(\frac{c_{k,n}c_{k,100n}}{c_{k,4n}c_{k,25n}}\right)^{3/2}+\left(\frac{c_{k,4n}c_{k,25n}}{c_{k,n}c_{k,100n}}\right)^{3/2}\right)$$
$$\times\left(\left(\frac{c_{k,25n}c_{k,100n}}{c_{k,n}c_{k,4n}}\right)^{1/2}-\left(\frac{c_{k,n}g_{k,4n}}{c_{k,25n}c_{k,100n}}\right)^{1/2}\right)=\frac{c_{k,25n}c_{k,100n}}{c_{k,n}c_{k,4n}}+\frac{c_{k,n}c_{k,4n}}{c_{k,25n}c_{k,100n}}. \tag{3.14}$$

On letting $k=3$, $n=1/10$ in (3.14), by the definition $h_{k,n}$

$$\frac{c_{3,10}^2}{c_{3,2/5}^2}+\frac{c_{3,2/5}^2}{c_{3,10}^2}+4\left(\frac{c_{3,10}}{c_{3,2/5}}-\frac{c_{3,2/5}}{c_{3,10}}\right)-4=0.$$

Since $c_{k,n}$ is increasing in n, we have $\dfrac{c_{3,10}}{c_{3,2/5}}>1$. Hence

$$\frac{c_{3,10}}{c_{3,2/5}}=\sqrt{3}+\frac{\sqrt{6}}{2}-\frac{1}{\sqrt{2}}-1. \tag{3.15}$$

From [3], if

$$P:=\frac{\psi(-q)}{q^{1/4}\psi(-q^3)}\qquad\text{and}\qquad Q:=\frac{\psi(-q^5)}{q^{5/4}\psi(-q^{15})}$$

then

$$(PQ)^2+\frac{9}{(PQ)^2}=\left(\frac{Q}{P}\right)^3-\left(\frac{P}{Q}\right)^3+5\left(\left(\frac{Q}{P}\right)^2+\left(\frac{P}{Q}\right)^2\right)+5\left(\frac{Q}{P}-\frac{P}{Q}\right).$$

Applying the interpretation of $c_{k,n}$ in the above, we obtain

$$\frac{3}{\left(c_{3,10}c_{3,2/5}\right)^2}+3\left(c_{3,10}c_{3,2/5}\right)^2=\left(\frac{c_{3,10}}{c_{3,2/5}}\right)^3-\left(\frac{c_{3,2/5}}{c_{3,10}}\right)^3$$
$$+5\left(\left(\frac{c_{3,10}}{c_{3,2/5}}\right)^2+\left(\frac{c_{3,2/5}}{c_{3,10}}\right)^2\right)+5\left(\frac{c_{3,10}}{c_{3,2/5}}-\frac{c_{3,2/5}}{c_{3,10}}\right). \tag{3.16}$$

Using (3.15) in (3.16), we obtain

$$c_{3,10}c_{3,2/5}=\sqrt{\sqrt{6}+\sqrt{5}}. \tag{3.17}$$

Finally, from (3.15) and (3.17), we obtain the required result. □

Corollary 3.3. *We have*

i. $$H(e^{-\pi\sqrt{10/3}})=\frac{1}{\sqrt{5}+\sqrt{6}}\sqrt[4]{\frac{4}{16\sqrt{2}-12\sqrt{3}-8\sqrt{6}+28}},$$

ii. $$H(e^{-\pi\sqrt{2/15}})=\sqrt[4]{\frac{6+4\sqrt{2}-3\sqrt{3}-2\sqrt{6}}{6+4\sqrt{2}-3\sqrt{3}+3\sqrt{5}+\sqrt{6}}}.$$

Proof. Utilizing Theorem 3.5 to Lemma 1.1(*iii*), we obtain the result. □

Theorem 3.6. *We have*

i. $$c_{3,6}=\left((3\sqrt{2}-3)(2+\sqrt{3})\right)^{1/4},$$

ii. $$c_{3,2/3}=\left(\frac{2+\sqrt{3}}{3\sqrt{2}-3}\right)^{1/4}.$$

Proof. Applying the interpretation on $c_{k,n}$ in Theorem 2.6, we have

$$\begin{aligned}&\left(\frac{c_{k,n}c_{k,36n}}{c_{k,4n}c_{k,9n}}\right)^4+\left(\frac{c_{k,4n}c_{k,9n}}{c_{k,n}c_{k,36n}}\right)^4+\left(\frac{c_{k,n}c_{k,36n}}{c_{k,4n}c_{k,9n}}\right)^2+\left(\frac{c_{k,4n}c_{k,9n}}{c_{k,n}c_{k,36n}}\right)^2\\&-\left(3\frac{c_{k,n}c_{k,4n}}{c_{k,9n}c_{k,36n}}-\frac{c_{k,9n}c_{k,36n}}{c_{k,n}c_{k,4n}}\right)\left[\left(\frac{c_{k,n}c_{k,36n}}{c_{k,4n}c_{k,9n}}\right)^3+\left(\frac{c_{k,4n}c_{k,9n}}{c_{k,n}c_{k,36n}}\right)^3\right]\\&-3\left(3\frac{c_{k,n}c_{k,4n}}{c_{k,9n}c_{k,36n}}-\frac{c_{k,9n}c_{k,36n}}{c_{k,n}c_{k,4n}}\right)\left(\frac{c_{k,n}c_{k,36n}}{c_{k,4n}c_{k,9n}}+\frac{c_{k,4n}c_{k,9n}}{c_{k,n}c_{k,36n}}\right)\\&-\left(9\left(\frac{c_{k,n}c_{k,4n}}{c_{k,9n}c_{k,36n}}\right)^2+\left(\frac{c_{k,9n}c_{k,36n}}{c_{k,n}c_{k,4n}}\right)^2\right)-6=0.\end{aligned}\tag{3.18}$$

On letting $k=3$ and $n=1/6$ in (3.18), by the definition of $c_{k,n}$, we obtain

$$\left(9\frac{c_{3,2/3}^4}{c_{3,6}^4}+\frac{c_{3,6}^4}{c_{3,2/3}^4}\right)+8\left(3\left(\frac{c_{3,2/3}}{c_{3,6}}\right)^2-\left(\frac{c_{3,6}}{c_{3,2/3}}\right)^2\right)+2=0.$$

Since $c_{k,n}$ is increasing in n, we have $\frac{c_{3,6}}{c_{3,2/3}}>1$. Hence

$$\frac{c_{3,6}}{c_{3,2/3}}=\sqrt{3\sqrt{2}-3}.\tag{3.19}$$

Replacing q to $-q$ in Theorem 2.1 and on using the definition of $c_{k,n}$, we obtain

$$3(c_{3,6}c_{3,2/3})^2+\frac{3}{(c_{3,6}c_{3,2/3})^2}=3+6\left(\frac{c_{3,6}}{c_{3,2/3}}\right)^2+\left(\frac{c_{3,6}}{c_{3,2/3}}\right)^4.\tag{3.20}$$

Using (3.19) in (3.20), we obtain

$$c_{3,6}c_{3,2/3}=\sqrt{2+\sqrt{3}}.\tag{3.21}$$

Finally, from (3.19) and (3.21), we obtain the required result. □

Corollary 3.4. *We have*

i. $$H(e^{-\sqrt{2}\pi})=\frac{1}{\sqrt[4]{18\sqrt{2}-9\sqrt{3}+9\sqrt{6}-17}},$$

ii. $$H(e^{-\sqrt{2}\pi/3})=\sqrt[4]{\frac{\sqrt{2}-1}{\sqrt{2}+\sqrt{3}-1}}.$$

Proof. Applying Theorem 3.6 to Lemma 1.1(*iii*), we obtain the result. □

References

1. Adiga, C., Bulkhali, N.A.S., Ranganatha, D. and Srivastava, H.M.: Some new modular relations for the Rogers-Ramanujan type of order eleven with applications to partitions, *J. of Number Theory* **158**, (2016) 281–297 .
2. Berndt, B.C.: *Ramanujan's Notebooks*, Part III, Springer, New York, 1991.
3. Bhargava, S., Vasuki, K.R. and Sreeramamurthy, T.G.: Some evaluations of Ramanujan's cubic continued fraction, *Indian J. Pure Appl. Math* **35(8)** (2004) 1003–1025.
4. Baruah, N.D. and Siakia, N.: Two parameters for Ramanujan's theta functions and their explicit values, *Rocky Mountain J. of Mathematics* **37(6)** (2007) 1747–1790.
5. Chan, H.H. and Huang, S.S.: On the Ramanujan-Göllnitz-Gordan continued fraction, *The Ramanujan J.* **1** (1997) 75–90.
6. Göllnitz, H.: Partition mit Differnzenbendingungen, *J. Reine Angew. Math.* **25** (1967) 154–190.
7. Gordon, B.: Some continued fractions of the Roger's Ramanujan type, *Duke Math. J.* **32** (1965) 741–748.
8. Ramanujan, S.: *Notebooks (2 Volumes)*, Tata Institute of Fundamental Research, Bombay, 1957.
9. Srivatsa Kumar, B.R. and Rajappa, G.N.: Modular equations for Ramanujan's cubic continued fraction and its evaluation, *Int. J. of Mathematical Combinatorics* **3** (2013) 1–15.
10. Srivatsa Kumar, B.R., Sreeramamurthy, T.G. and Rajappa, G.N.: Evaluations of Ramanujan's cubic continued fraction G(q) by modular equations, *J. of Ramanujan Society of Math. and Math. Sc.* **1(1)** (2012) 49–68.
11. Vasuki, K.R. and Srivatsa Kumar, B.R.: Evaluations of the Ramanujan-Gollnitz-Gordan continued fraction H(q) by modular equations, *Indian J. of Mathematics* **48(3)** (2006) 275–300.
12. Vasuki, K.R. and Kahtan, A.A.A.: On Certain Theta Function Identities Analogous to Ramanujan's P-Q Eta Function Identities, *Applied Mathematics* **2** (2011) 874–882.
13. Yi, J.: *The construction and Applications of Modular equations*, Ph. D. Thesis, University of Illinois at Urbana-Champaign, 2001.
14. Yi, J.: Theta-function identities and the explicit formulas for theta-function and their applications, *J. Math. Anal. Appl.* **292** (2004) 381–400.
15. Yi, J., Lee, Y. and Paek, D.H.: The explicit formulas and evaluations of Ramanujan's theta-function ψ, *J. Math. Anal. Appl.* **321(1)** (2006) 157–181.
16. Whittaker, E.T. and Watson, G.N.: *A course of modern analysis*, Cambridge University Press, Cambridge, 1966.

Shruthi: Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal-576 104, Karnataka India

Email address: shruthikarranth@gmail.com

R. Narendra: Department of Mathematics, Acharya Institute of Technology, Soladevanahalli, Bengaluru 560 107, Karnataka, India

Email address: narendrarnaidu@yahoo.com

B. R. Srivatsa Kumar*: Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal-576 104, Karnataka India

Email address: srivatsa.kumar@manipal.edu *Corresponding author