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ON THE FRACTIONAL MIXED FRACTIONAL BROWNIAN MOTION TIME CHANGED BY INVERSE α-STABLE SUBORDINATOR

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ABSTRACT. A time-changed fractional mixed fractional Brownian motion by inverse α-stable subordinator with index $\alpha \in (0,1)$ is an iterated process $L^{H_1 H_2}_{T^\alpha}(a,b)$ constructed as the superposition of fractional mixed fractional Brownian motion $N^{H_1 H_2}(a,b)$ and an independent inverse α-stable subordinator T^α. In this paper we prove that the process $L^{H_1 H_2}_{T^\alpha}(a,b)$ exhibit long range dependence property under some condition on the Hirst indices H_1 and H_2 of tow independents fractional Brownian motions.

1. Introduction

A mixed fractional Brownian motion (mfBm for short) of parameters a, b and H is the process $M^H(a,b) = \{M^H_t(a,b),\ t \geq 0\}$, defined on the probability space $(\Omega, \mathcal{F}, P)$ by

$$M^H_t(a,b) = aB_t + bB^H_t, \quad t \geq 0,$$

where $B = \{B_t, t \geq 0\}$ is a Brownian motion, $B^H = \left\{B^H_t, t \geq 0\right\}$ is an independent fractional Brownian motion of Hurst index $H \in (0,1)$ and a, b two real constants. The mfBm was introduced by Cheridito [6], with stationary increments exhibit a long-range dependence for $H > \frac{1}{2}$. The mfBm has been discussed in [6] to present a stochastic model of the discounted stock price in some arbitrage-free and complete financial markets. This model is the process

$$X_t = X_0 \exp\{\mu t + \sigma(aB_t + bB^H_t)\},$$

where μ is the rate of the return and σ is the volatility. We refer also to [8, 17, 31] for further information and applications on the mfBm.

The time-changed mixed fractional Brownian motion by inverse α-stable subordinator with index $\alpha \in (0,1)$ is defined as below

$$L^H_{T^\alpha}(a,b) = \{M^H_{T^\alpha_t}(a,b),\ t \geq 0\},$$

where the parent process $M^H(a,b)$ is a mfBm with parameters $a, b, H \in (0,\ 1)$ and $T^\alpha = \{T^\alpha_t,\ t \geq 0\}$ is an inverse α-stable subordinator assumed to be independent of both Brownian and fractional Brownian motion. If $H = \frac{1}{2}$, the process $L^{\frac{1}{2}}_{T^\alpha}(0,1)$ is called subordinated Brownian motion, it was investigated in [9, 19, 22, 26].

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When $a = 0$, $b = 1$ then $L^H_{T^\alpha}(0,1)$ it is the process considered in [15, 16] called subordinated fractional Brownian motion.

Time-changed process is constructed by taking superposition of tow independent stochastic systems. The evolution of time in external process is replaced by a non-decreasing stochastic process, called subordinator. The resulting time-changed process very often retain important properties of the external process, however certain characteristics might change. This idea of subordination was introduced by Bochner [5] and was explored in many papers (see [11, 12, 15, 21]).

The time-changed mixed fractional Brownian motion has been discussed in [10] to present a stochastic Black-Scholes model, whose price of the underlying stock is the process

$$S_t = S_0 \exp\{\mu T^\alpha_t + \sigma(aB_{T^\alpha_t} + bB^H_{T^\alpha_t})\},$$

where μ is the rate of the return, σ is the volatility and T^α is the α-inverse stable subordinator. Also the time-changed processes have found many interesting applications, for example in finance [10, 14, 27, 29, 32].

C. Elnouty [8] propose a generalisation of the mfBm called fractional mixed fractional Brownian motion (fmfBm) of parameters a, b and $H_1, H_2 \in (0,1)$. A fmfBm is a process $N^{H_1H_2}(a,b) = \{N^{H_1H_2}_t(a,b),\ t \geq 0\}$, defined on the probability space $(\Omega, \mathcal{F}, P)$ by

$$N^{H_1H_2}_t(a,b) = aB^{H_1}_t + bB^{H_2}_t, \quad t \geq 0,$$

where $B^{H_1} = \{B^{H_1}_t, t \geq 0\}$ and $B^{H_2} = \{B^{H_2}_t, t \geq 0$ are independents fractional Brownian motions and a, b real constants not both equal to zero. Also the fmfBm was study by Miao, Y et al. [25].

The time-changed fractional mixed fractional Brownian motion is defined as $\{N^{H_1H_2}_{\beta_t}(a,b),\ t \geq 0\}$, where the parent process $N^{H_1H_2}(a,b)$ is a fmfBm with parameters a, b, and $H_1, H_2 \in (0,\, 1)$ and the subordinator $\beta = \{\beta_t,\, t \geq 0\}$ is assumed to be independent of both fractional Brownian motions B^{H_1} and B^{H_2}.

Our goal in this parer is to study the main properties of the time-changed fractional mixed fractional Brownian motion by inverse α-stable subordinator paying attention to the long range dependence property.

2. Main results and proofs

We begin by defining the inverse α-stable subordinator.

Definition 2.1. The inverse α-stable subordinator $T^\alpha = \{T^\alpha_t,\ t \geq 0\}$ is defined in the following way

$$T^\alpha_t = inf\{r > 0,\ \eta^\alpha_r \geq t\}, \tag{2.1}$$

where $\eta^\alpha = \{\eta^\alpha_r,\ r \geq 0\}$ is the α-stable subordinator [28, 30] with Laplace transform

$$E(e^{-u\eta^\alpha_r}) = e^{-ru^\alpha}, \quad \alpha \in (0,1).$$

The inverse α-stable subordinator is a non-decreasing Lévy process, starting from zero, has a stationary and independent increments with α-self similar. Specially, when $\alpha \uparrow 1$, T^α_t reduces to the physical time t.

Let T^α be an inverse α-stable subordinator with index $\alpha \in (0,1)$. From [18, 20], we know that

$$E(T_t^\alpha) = \frac{t^\alpha}{\Gamma(\alpha+1)} \quad and \quad E((T_t^\alpha)^n) = \frac{t^{n\alpha} n!}{\Gamma(n\alpha+1)}.$$

Lemma 2.2. *Let T^α be an inverse α-stable subordinator with index $\alpha \in (0,1)$ and B^H be a fBm. Then, by α-self-similar and non-decreasing sample path of T_t^α, we have*

$$E(B_{T_t^\alpha})^2 = \frac{t^\alpha}{\Gamma(\alpha+1)} \quad and \quad E(B^H_{T_t^\alpha})^2 = \left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^{2H}.$$

Proof. See [14, 20]. □

Definition 2.3. Let $N^{H_1H_2}(a,b) = \{N_t^{H_1H_2}(a,b),\ t \geq 0\}$ be a fmfBm of parameters a, b and $H_1, H_2 \in (0,1)$ and let T^α be an inverse α-stable subordinator with index $\alpha \in (0,1)$. The subordinated of $N^{H_1H_2}(a,b)$ by means of T^α is the process $L_{T^\alpha}^{H_1H_2}(a,b) = \{L_{T_t^\alpha}^{H_1H_2},\ t \geq 0\}$ defined by:

$$L_{T_t^\alpha}^{H_1H_2} = N_{T_t^\alpha}^{H_1H_2}(a,b) = aB_{T_t^\alpha}^{H_1} + bB_{T_t^\alpha}^{H_2}, \tag{2.2}$$

where the subordinator T_t^α is assumed to be independent of both fractional Braownian motions B^{H_1}, B^{H_2} and a, b real constants not both equal to zero.

Remark 2.4. When $\alpha \uparrow 1$, the processes $B_{T_t^\alpha}$ and $B^H_{T_t^\alpha}$ degenerate to B_t and B_t^H.

Notation 2.5. Let U and V be two centered random variables defined on the same probability space. Let

$$Corr(U,V) = \frac{Cov(U,V)}{\sqrt{E(U^2)E(V^2)}}, \tag{2.3}$$

denote the correlation coefficient between U and V.

Now we discuss the long range dependent behavior of $L_{T^\alpha}^{H_1H_2}(a,b)$.

Definition 2.6. A finite variance stationary process $\{X_t,\ t \geq 0\}$ is said to have long range dependence property [7], if $\sum_{k=0}^\infty \gamma_k = \infty$, where

$$\gamma_k = Cov(X_k, X_{k+1}).$$

In the following definition we give the equivalent definition for a non-stationary process $\{X_t,\ t \geq 0\}$.

Definition 2.7. Let $s > 0$ be fixed and $t > s$. Then process $\{X_t,\ t \geq 0\}$ is said to have long range dependence property property if

$$Corr(X_t, X_s) \sim c(s)t^{-d}, \quad as \quad t \to \infty,$$

where $c(s)$ is a constant depending on s and $d \in (0,1)$.

The main result can be stated as follows.

Theorem 2.8. *Let $N^{H_1H_2}(a,b) = \{N_t^{H_1H_2}(a,b),\ t \geq 0\}$ be the fractional mixed fractional Brownian motion of parameters a, b, H_1 and H_2 with $H_1 < H_2$. Let $T^\alpha = \{T_t^\alpha,\ t \geq 0\}$ be an inverse α-stable subordinator with index $\alpha \in (0,1)$ assumed to be independent of both fractional Brownian motions B^{H_1} and B^{H_2}. Then the time-changed fractional mixed fractional Brownian motion by means of T^α has long range dependence property if $0 < 2\alpha H_1 - \alpha H_2 < 1$.*

Proof. Let $T^\alpha = \{T_t^\alpha,\ t \geq 0\}$ be an inverse α-stable subordinator with index $\alpha \in (0,1)$ assumed to be independent of B^{H_i}, $i = 1,2$. Let $L_{T^\alpha}^{H_1H_2}(a,b)$ be the time-changed fractional mixed fractional Brownian motion by means of the inverse α-stable subordinator T^α with index $\alpha \in (0,1)$. The process $L_{T^\alpha}^{H_1H_2}(a,b)$ is not stationary hence Definition 2.7 will be used to establish the long range dependence property.

Step 1: Let $s > 0$ be fixed and let $s \leq t$. Since B^{H_1} and B^{H_2} has stationary increments, then we have

$$\begin{aligned}
Cov(L_{T_t^\alpha}^{H_1H_2}, L_{T_s^\alpha}^{H_1H_2}) &= E(L_{T_t^\alpha}^{H_1H_2} L_{T_s^\alpha}^{H_1H_2}) \\
&= \frac{1}{2}E\left[(L_{T_t^\alpha}^{H_1H_2})^2 + (L_{T_s^\alpha}^{H_1H_2})^2 - (L_{T_t^\alpha}^{H_1H_2} - L_{T_s^\alpha}^{H_1H_2})^2\right] \\
&= \frac{1}{2}E\left[(N_{T_t^\alpha}^{H_1H_2}(a,b))^2 + (N_{T_s^\alpha}^{H_1H_2}(a,b))^2\right] \\
&\quad -\frac{1}{2}E\left[(N_{T_t^\alpha}^{H_1H_2}(a,b) - N_{T_s^\alpha}^{H_1H_2}(a,b))^2\right] \\
&= \frac{1}{2}E\left[(aB_{T_t^\alpha}^{H_1} + bB_{T_t^\alpha}^{H_2})^2 + (aB_{T_s^\alpha}^{H_1} + bB_{T_s^\alpha}^{H_2})^2\right] \\
&\quad -\frac{1}{2}E\left[\Big(a(B_{T_t^\alpha}^{H_1} - B_{T_s^\alpha}^{H_1}) + b(B_{T_t^\alpha}^{H_2} - B_{T_s^\alpha}^{H_2})\Big)^2\right] \\
&= \frac{1}{2}E\left[(aB_{T_t^\alpha}^{H_1} + bB_{T_t^\alpha}^{H_2})^2 + (aB_{T_s^\alpha}^{H_1} + bB_{T_s^\alpha}^{H_2})^2\right] \\
&\quad -\frac{1}{2}E\left[(aB_{T_{t-s}^\alpha}^{H_1} + bB_{T_{t-s}^\alpha}^{H_2})^2\right] \\
&= \frac{1}{2}E\left[(aB_{T_t^\alpha}^{H_1} + (bB_{T_t^\alpha}^{H_2})^2 + 2(aB_{T_t^\alpha}^{H_1} bB_{T_t^\alpha}^{H_2})\right] \\
&\quad +\frac{1}{2}E\left[(aB_{T_s^\alpha}^{H_1})^2 + (bB_{T_s^\alpha}^{H_2})^2 + 2(aB_{T_s^\alpha}^{H_1} bB_{T_s^\alpha}^{H_2})\right] \\
&\quad -\frac{1}{2}E\left[(aB_{T_{t-s}^\alpha}^{H_1})^2 + (bB_{T_{t-s}^\alpha}^{H_2})^2 + 2(aB_{T_{t-s}^\alpha}^{H_1} bB_{T_{t-s}^\alpha}^{H_2})\right].
\end{aligned}$$

Since $B_t^{H_1}$ and $B_t^{H_2}$ are independent and using Lemma 2.2 we get

$$\begin{aligned}
E(L_{T_t^\alpha}^{H_1H_2} L_{T_s^\alpha}^{H_1H_2}) &= \frac{a^2}{2}\left[E(B_{T_t^\alpha}^{H_1})^2 + E(B_{T_s^\alpha}^{H_1})^2 - E(B_{T_{t-s}^\alpha}^{H_1})^2\right] \\
&\quad +\frac{b^2}{2}\left[E(B_{T_t^\alpha}^{H_2})^2 + E(B_{T_s^\alpha}^{H_2})^2 - E(B_{T_{t-s}^\alpha}^{H_2})^2\right].
\end{aligned}$$

Hence,

$$\begin{aligned} E(L^{H_1H_2}_{T^\alpha_t} L^{H_1H_2}_{T^\alpha_s}) &= \frac{a^2}{2}\left[\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^{2H_1} + \left(\frac{s^\alpha}{\Gamma(\alpha+1)}\right)^{2H_1} - \left(\frac{(t-s)^\alpha}{\Gamma(\alpha+1)}\right)^{2H_1}\right] \\ &\quad + \frac{b^2}{2}\left[\left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^{2H_2} + \left(\frac{s^\alpha}{\Gamma(\alpha+1)}\right)^{2H_2} - \left(\frac{(t-s)^\alpha}{\Gamma(\alpha+1)}\right)^{2H_2}\right] \\ &= \frac{a^2\left[t^{2\alpha H_1} + s^{2\alpha H_1} - (t-s)^{2\alpha H_1}\right]}{2[\Gamma(\alpha+1)]^{2H_1}} + \frac{b^2\left[t^{2\alpha H_2} + s^{2\alpha H_2} - (t-s)^{2\alpha H_2}\right]}{2[\Gamma(\alpha+1)]^{2H_2}}. \end{aligned}$$

Hence for all $s \leq t$ and $H_1 < H_2$ we have

$$E(L^{H_1H_2}_{T^\alpha_t} L^{H_1H_2}_{T^\alpha_s}) = \frac{a^2\left[t^{2\alpha H_1} + s^{2\alpha H_1} - (t-s)^{2\alpha H_1}\right]}{2[\Gamma(\alpha+1)]^{2H_1}} + \frac{b^2\left[t^{2\alpha H_2} + s^{2\alpha H_2} - (t-s)^{2\alpha H_2}\right]}{2[\Gamma(\alpha+1)]^{2H_2}}.$$

Step 2: Let s be fixed. Then by Taylor's expansion we have for large t

$$\begin{aligned} E(L^{H_1H_2}_{T^\alpha_t} L^{H_1H_2}_{T^\alpha_s}) &\sim \frac{a^2}{2[\Gamma(\alpha+1)]^{2H_1}} t^{2\alpha H_1}\left[2\alpha H_1 \frac{s}{t} + s^{2\alpha H_1} t^{-2\alpha H_1} + O(t^{-2})\right] \\ &\quad + \frac{b^2}{2[\Gamma(\alpha+1)]^{2H_2}} t^{2\alpha H_2}\left[2\alpha H_2 \frac{s}{t} + s^{2\alpha H_2} t^{-2\alpha H_2} + O(t^{-2})\right] \\ &\sim \frac{a^2 t^{2\alpha H_1}}{2[\Gamma(\alpha+1)]^{2H_1}}\left[2\alpha H_1 \frac{s}{t} + (\frac{s}{t})^{2\alpha H_1} + O(t^{-2})\right] \\ &\quad + \frac{b^2 t^{2\alpha H_2}}{2[\Gamma(\alpha+1)]^{2H_2}}\left[2\alpha H_2 \frac{s}{t} + (\frac{s}{t})^{2\alpha H_2} + O(t^{-2})\right] \\ &\sim \frac{a^2 \alpha s}{(\Gamma(\alpha+1))^{2H_1}} t^{2\alpha H_1 - 1} + \frac{b^2 \alpha s}{(\Gamma(\alpha+1))^{2H_2}} t^{2\alpha H_2 - 1}. \end{aligned}$$

Then for fixed s and large t, $L^{H_1H_2}_{T^\alpha_t}$ satisfies

$$E(L^{H_1H_2}_{T^\alpha_t} L^{H_1H_2}_{T^\alpha_s}) \sim \frac{a^2 \alpha s}{(\Gamma(\alpha+1))^{2H_1}} t^{2\alpha H_1 - 1} + \frac{b^2 \alpha s}{(\Gamma(\alpha+1))^{2H_2}} t^{2\alpha H_2 - 1}.$$

Step 3: Let $H_1 < H_2$. Using Eqs. (2.3), (2.4) and by Taylor's expansion we get, as $t \to \infty$

$$\begin{aligned} Corr(L^{H_1H_2}_{T^\alpha_t}, L^{H_1H_2}_{T^\alpha_s}) &\sim \frac{\frac{a^2 \alpha s}{(\Gamma(\alpha+1))^{2H_1}} t^{2\alpha H_1 - 1} + \frac{b^2 \alpha s}{(\Gamma(\alpha+1))^{2H_2}} t^{2\alpha H_2 - 1}}{\left[\frac{a^2\alpha}{(\Gamma(\alpha+1))^{2H_1}} t^{2\alpha H_1} + \frac{b^2\alpha}{(\Gamma(\alpha+1))^{2H_2}} t^{2\alpha H_2}\right]^{\frac{1}{2}} [E(L^{T^\alpha}_s)^2]^{\frac{1}{2}}} \\ &= \frac{\frac{a^2 \alpha s}{(\Gamma(\alpha+1))^{2H_1}} t^{2\alpha H_1 - 1} + \frac{b^2 \alpha s}{(\Gamma(\alpha+1))^{2H_2}} t^{2\alpha H_2 - 1}}{\frac{|b|\alpha^{\frac{1}{2}} t^{\alpha H_2}}{(\Gamma(\alpha+1))^{H_2}}\left[\frac{a^2}{2b^2(\Gamma(\alpha+1))^{1-2H_2}} t^{2\alpha H_1 - 2\alpha H_2} + 1\right]^{\frac{1}{2}} [E(L^{T^\alpha}_s)^2]^{\frac{1}{2}}} \\ &\sim \frac{a^2 \alpha^{\frac{1}{2}} s t^{2\alpha H_1 - \alpha H_2 - 1}}{|b|(\Gamma(\alpha+1))^{2H_1 - H_2} [E(L^{T^\alpha}_s)^2]^{\frac{1}{2}}} + \frac{|b|\alpha^{\frac{1}{2}} s t^{\alpha H_2 - 1}}{(\Gamma(\alpha+1))^{H_2} [E(L^{T^\alpha}_s)^2]^{\frac{1}{2}}}. \end{aligned}$$

Hence, for every $H_1 < H_2$ we have

$$Corr(L^{H_1H_2}_{T^\alpha_t}, L^{H_1H_2}_{T^\alpha_s}) \sim \frac{a^2 \alpha^{\frac{1}{2}} s\, t^{2\alpha H_1 - \alpha H_2 - 1}}{|b|(\Gamma(\alpha+1))^{2H_1 - H_2} [E(L^{T^\alpha}_s)^2]^{\frac{1}{2}}} + \frac{|b|\alpha^{\frac{1}{2}} s\, t^{\alpha H_2 - 1}}{(\Gamma(\alpha+1))^{H_2} [E(L^{T^\alpha}_s)^2]^{\frac{1}{2}}}. \quad (2.4)$$

Then the correlation function of the stochastic process $L^{H_1H_2}_{T^\alpha_t}$ decays like a mixture of power law $t^{-(2\alpha H_1-\alpha H_2-1)}+t^{-(1-\alpha H_2)}$. Since $0<2\alpha H_1-\alpha H_2<1$ then the first term tends to zero as $t\to\infty$. Then the time-changed process $L^{H_1H_2}_{T^\alpha}(a,b)$ exhibits long range dependence property for all $H_1<H_2$ and $0<2\alpha H_1-\alpha H_2<1$. □

Remark 2.9. When $a=0$ and $b=1$ in Eqs. (2.4) and (2.4) we get

$$E(L^{H_1H_2}_{T^\alpha_t}L^{H_1H_2}_{T^\alpha_s}) = E(B^{H_2}_{T^\alpha_t}B^{H_2}_{T^\alpha_s}) \sim \frac{\alpha s t^{2\alpha H_2-1}}{(\Gamma(\alpha+1))^{2H_2}}, \quad as \quad t\to\infty,$$

$$Corr(L^{H_1H_2}_{T^\alpha_t}, L^{H_1H_2}_{T^\alpha_s}) \sim \frac{\alpha^{\frac{1}{2}} s t^{\alpha H_2-1}}{(\Gamma(\alpha+1))^{H_2}\sqrt{E(B^{H_2}_{T^\alpha_s})^2}}, \quad as\ t\to\infty.$$

Hence we obtain the following result.

Corollary 2.10. *The fractional Brownian motion time changed by inverse α-stable subordinator with index $\alpha\in(0,1)$ is of long range dependence for the Hurst index $H\in(0,1)$.*

Similar result as Corollary 2.10 was obtained in [15] ([16]) in the case of fractional Brownian motion time changed by tempered stable subordinator (gamma subordinator).

As application to the original process we obtain the following. .

Corollary 2.11. *Let $H_2=H>H_1=\frac{1}{2}$. When $\alpha\uparrow 1$, in Eqs.* (2.4) *and* (2.4) *we have, as $t\to\infty$*

$$\lim_{\alpha\to 1} E(L^{H_1H_2}_{T^\alpha_t}L^{H_1H_2}_{T^\alpha_s}) = \frac{a^2 s}{2} + b^2 s t^{2H-1},$$

$$\lim_{\alpha\to 1} Corr(L^{H}_{T^\alpha_t}, L^{H_1H_2}_{T^\alpha_s}) = \frac{a^2 s\, t^{-H}}{2|b|\sqrt{E(N^{H_1H_2}_s(a,b))^2}} + \frac{|b| s\, t^{H-1}}{\sqrt{E(N^{H_1H_2}_s(a,b))^2}}.$$

Hence using Remark 2.4 and Corollary 2.11 we can see that the mixed fractional Brownian motion of parameters a,b and H has long range dependence property for all $H>\frac{1}{2}$ in sense of Definition 2.7.

Remark 2.12. (1) Let $H\in(0,1)$. Then

$$Corr(B^H_t, B^H_s) \sim \frac{s t^{H-1}}{\sqrt{E(B^H_s)^2}}, \quad as \quad t\to\infty. \tag{2.5}$$

Indeed, we take $a=0$ and $b=1$ in Eq. (2.4). When $\alpha\uparrow 1$ and using Remark 2.4 we obtain Eq. (2.5).

(2) When $\alpha\uparrow 1$, in Eq. (2.4) we have

$$\lim_{\alpha\to 1} E(L^{H_1H_2}_{T^\alpha_t}L^{H_1H_2}_{T^\alpha_s}) = \frac{a^2}{2}\left[t^{2H_1}+s^{2H_1}-(t-s)^{2H_1}\right] + \frac{b^2}{2}\left[t^{2H_2}+s^{2H_2}-(t-s)^{2H_2}\right].$$

Corollary 2.13. *The fractional mixed fractional Brownian motion has long range dependence for every $0<H_1<H_2<1$.*

The idea, used results for the time-changed process to obtain a results for the original one is already investigated in [11].

The fmfBm has been further generalized by Thäle in 2009 [31] to the generalized mixed fractional Brownian motion. A generalized mixed fractional Brownian motion of parameters $H = (H_1, H_2, ..., H_n)$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a stochastic process $Z = \{Z_t^{H,\alpha},\ t \geq 0\}$ defined by

$$Z_t^{H,\alpha} = \sum_{i=1}^{n} \alpha_i B_t^{H_i}$$

Forthcoming work, we will investigate the long range dependence property of the time-changed generalized mixed fractional Brownian motion by inverse α-stable subordinator [24].

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