Global and Stochastic Analysis
Vol. 7 No. 2 (July-December, 2020

MUK PUBLICATIONS
Open Access Publisher

EXISTENCE, BOUNDEDNESS AND STABILITY OF SOLUTIONS OF TIME-VARYING SEMILINEAR DIFFERENTIAL-ALGEBRAIC EQUATIONS

MARIA FILIPKOVSKA (FILIPKOVSKAYA)

Abstract. Theorems on the existence and uniqueness of global solutions, Lagrange stability and instability (solutions have finite escape time), ultimate boundedness (dissipativity), Lyapunov stability, instability and asymptotic stability (including asymptotic stability in the large) for time-varying semilinear differential-algebraic equations are proved. Various useful remarks concerning the application of the obtained theorems are given.

1. Introduction

Differential algebraic equations (DAEs) are a convenient abstract form for the representation of many dynamic models of real objects and processes in radio-electronics, robotics, control theory, economics, chemistry and ecology [1, 2, 3, 4, 5]. Differential equations (DEs) of this type have been studied in many works (see [1, 3, 2, 4, 5, 6, 7] and references therein). For instance, the global solvability of semilinear DAEs was studied in [6, 1, 2, 4], and in [2, 4, 5, 8] the Lyapunov stability of semilinear and nonlinear DAEs was studied using various approaches to the research.

Ultimately bounded systems of ordinary differential equations (ODEs), which are also called dissipative systems and D-systems, were studied in [14, 15]. Asymptotic stability in the large or complete stability for ODE systems was considered in [16, 14]. The theorems from [15] and [16] are applied in this paper. Also, the results from [14] relative to the continuation of solutions and the Lagrange stability of ODEs are used. These results are based on the application of differential inequalities and functions of the Lyapunov function type. In general, Lyapunov's second method and the methods based on it are used, as well as time-varying spectral projectors and the operator $G(t)$, constructed using them, from [13, 1] are applied (see Section 2).

Date: Date of Submission June 02, 2020; Date of Acceptance June 30, 2020, Communicated by Yuri E. Gliklikh .

2010 *Mathematics Subject Classification.* Primary 34A09, 34D23; Secondary 34D20, 34A12.

Key words and phrases. implicit differential equation, differential-algebraic equation, non-invertible operator, time-varying, global solvability, Lyapunov stability, Lagrange stability, instability.

The research is supported by the grant of the National Academy of Sciences of Ukraine (state registration number 0119U102376, project "Qualitative, asymptotic and numerical analysis of various classes of differential equations and dynamical systems, their classification, and practical application").

In this paper, time-varying semilinear DAEs with the characteristic pencils regular for each t are considered. The conditions of global solvability, Lagrange stability and instability, Lyapunov stability, instability and asymptotic stability (including asymptotic stability in the large) are obtained. This work extends and generalizes the results obtained in [9, 10, 11, 12].

The following classical notation is used in the paper: δ_{ij} is the Kronecker delta, I_X is the identity operator in the space X; $\operatorname{Ker} A$ and $\mathcal{R}(A)$ are the kernel (the null-space) and the range (the image) of the operator A.

The following classical definitions are also used in the paper. Let $D \subset \mathbb{R}^n$ be a region containing the origin. A function $W \in C(D, \mathbb{R})$ is called *positive definite* if $W(x) > 0$ for all $x \neq 0$ and $W(0) = 0$. A function $V \in C([t_+, \infty) \times D, \mathbb{R})$ is called *positive definite* if $V(t, 0) \equiv 0$ and there exists a positive definite function $W \in C(D, \mathbb{R})$ such that $V(t, x) \geq W(x)$ for all $x \neq 0$, $t \in [t_+, \infty)$.

2. Problem setting and preliminaries

Consider implicit differential equations

$$\frac{d}{dt}[A(t)x(t)] + B(t)x(t) = f(t, x(t)), \tag{2.1}$$

$$A(t)\frac{d}{dt}x(t) + B(t)x(t) = f(t, x(t)), \tag{2.2}$$

with the initial condition

$$x(t_0) = x_0, \tag{2.3}$$

where $A, B\colon [t_+, \infty) \to L(\mathbb{R}^n)$, $f\colon [t_+, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $t_0 \geq t_+ \geq 0$. The operators $A(t)$, $B(t)$ may be degenerate. Equations of the type (2.1), (2.2) with the degenerate operator $A(t)$ are called time-varying (nonautonomous) *differential-algebraic equations* (*DAEs*) and are also called *degenerate differential equations*, *algebraic-differential systems* and *descriptor systems*.

In the terminology of DAEs, equations of the form (2.1), (2.2) are commonly referred to as *semilinear*, but they are sometimes called nonlinear (since the function f is nonlinear).

A function $x \in C([t_0, t_1), \mathbb{R}^n)$ is a *solution of the initial value problem* (the Cauchy problem) (2.1), (2.3) on an interval $[t_0, t_1) \subseteq [t_+, \infty)$ if $Ax \in C^1([t_0, t_1), \mathbb{R}^n)$, $x(t)$ satisfies the equation (2.1) on $[t_0, t_1)$ and the initial condition (2.3). A function $x \in C^1([t_0, t_1), \mathbb{R}^n)$ is a *solution of the initial value problem* (2.2), (2.3) on $[t_0, t_1) \subseteq [t_+, \infty)$ if $x(t)$ satisfies (2.2) on $[t_0, t_1)$ and (2.3).

The operator pencil $\lambda A(t) + B(t)$ (λ is a complex parameter) corresponds to the left side of the equations (2.1), (2.2). Let for each $t \geq t_+$ the pencil be regular ($\det(\lambda A(t) + B(t)) \not\equiv 0$), i.e., for each $t \geq t_+$ the set of its regular points is not empty (the set of regular points of the pencil $\lambda A(t) + B(t)$ is the set of regular points of its complex extension) . For the regular points λ of the pencil, there exists the resolvent $R(\lambda, t) = (\lambda A(t) + B(t))^{-1}$. It is assumed that for each $t \geq t_+$ the pencil is regular and the following condition is satisfied: there exist functions $C_1\colon [t_+, \infty) \to (0, \infty)$, $C_2\colon [t_+, \infty) \to (0, \infty)$ such that

$$\|R(\lambda, t)\| \leq C_1(t), \quad |\lambda| \geq C_2(t), \tag{2.4}$$

for all $t \in [t_+, \infty)$. Thus, for each $t \geq t_+$ the pencil $\lambda A(t) + B(t)$ is a regular pencil of index not higher than 1 (see the definition, for example, in [17, Section 2]). The condition (2.4) means that either the point $\mu = 0$ is a simple pole of the resolvent $(A(t)+\mu B(t))^{-1}$ (this is equivalent to the fact that $\lambda = \infty$ is a removable singular point of the resolvent $R(\lambda, t)$), or $\mu = 0$ is a regular point of the pencil $A(t) + \mu B(t)$.

The definitions and facts presented in this section will be used hereinafter.

Below is the information from [1, Subsection 3.3], [13], which will be used hereinafter. If the regular pencil satisfies (2.4), then for each $t \in [t_+, \infty)$ there exist the two pairs of mutually complementary projectors

$$\begin{aligned} P_1(t) &= \frac{1}{2\pi i} \oint\limits_{|\lambda|=C_2(t)} R(\lambda, t)\, d\lambda\, A(t), \quad P_2(t) = I_{\mathbb{R}^n} - P_1(t), \\ Q_1(t) &= \frac{1}{2\pi i} \oint\limits_{|\lambda|=C_2(t)} A(t)\, R(\lambda, t)\, d\lambda, \quad Q_2(t) = I_{\mathbb{R}^n} - Q_1(t) \end{aligned} \tag{2.5}$$

($P_i(t)P_j(t)=\delta_{ij}P_i(t)$, $P_1(t)+P_2(t)=E_{\mathbb{R}^n}$, $Q_i(t)Q_j(t)=\delta_{ij}Q_i(t)$, $Q_1(t)+Q_2(t)=E_{\mathbb{R}^n}$, $i, j = 1, 2$), which generate the direct decompositions of spaces

$$\begin{aligned} \mathbb{R}^n &= X_1(t)\dot{+}X_2(t), \quad X_j(t) = P_j(t)\mathbb{R}^n, \\ \mathbb{R}^n &= Y_1(t)\dot{+}Y_2(t), \quad Y_j(t) = Q_j(t)\mathbb{R}^n, \quad j = 1, 2, \end{aligned} \tag{2.6}$$

such that the pairs of subspaces $X_1(t)$, $Y_1(t)$ and $X_2(t)$, $Y_2(t)$ are invariant under $A(t)$, $B(t)$ (i.e., $A(t)$, $B(t)\colon X_j(t) \to Y_j(t)$, $j = 1, 2$). The restricted operators $A_j(t) = A(t)|_{X_j(t)} : X_j(t) \to Y_j(t)$, $B_j(t) = B(t)|_{X_j(t)} : X_j(t) \to Y_j(t)$ $(j = 1, 2)$ are such that $A_2(t) = 0$ and $A_1^{-1}(t)$, $B_2^{-1}(t)$ exist (if $X_1(t) \neq \{0\}$, $X_2(t) \neq \{0\}$ respectively). The subspaces $X_i(t)$, $Y_j(t)$ are such that $Y_1(t) = \mathcal{R}(A(t))$, $X_2(t) = \operatorname{Ker} A(t)$, $Y_2(t) = B(t)X_2(t)$ and $X_1(t) = R(\lambda, t)Y_1(t)$, $|\lambda| \geq C_2(t)$. The projectors are real (since $A(t)$ and $B(t)$ are real) and satisfy the properties:

$$\begin{aligned} A(t)P_1(t) = Q_1(t)A(t) = A(t), \quad A(t)P_2(t) = Q_2(t)A(t) = 0, \\ B(t)P_j(t) = Q_j(t)B(t), \quad j = 1, 2. \end{aligned} \tag{2.7}$$

Using the spectral projectors, for each $t \in [t_+, \infty)$ we obtain the auxiliary operator

$$G(t) = A(t) + B(t)P_2(t) = A(t) + Q_2(t)B(t) \in L(\mathbb{R}^n) \tag{2.8}$$

such that $G(t)X_j(t) = Y_j(t)$ [1, Subsection 3.3], [13]. This operator has the inverse $G^{-1}(t) = A_1^{-1}(t)Q_1(t) + B_2^{-1}(t)Q_2(t) \in L(\mathbb{R}^n)$ $(G^{-1}(t)\colon Y_j(t) \to X_j(t))$ with the properties $G^{-1}(t)A(t)P_1(t) = G^{-1}(t)A(t) = P_1(t)$, $G^{-1}(t)B(t)P_2(t) = P_2(t)$, $A(t)G^{-1}(t)Q_1(t) = A(t)G^{-1}(t) = Q_1(t)$, $B(t)G^{-1}(t)Q_2(t) = Q_2(t)$.

The projectors $P_i(t)$, $Q_i(t)$ $(i = 1, 2)$ and the operators $G(t)$, $G^{-1}(t)$ as operator functions have the same degree of smoothness as the operator functions $A(t)$, $B(t)$ and the function $C_2(t)$ [1, Subsection 3.3]. Suppose that $A, B \in C^1([t_+, \infty), L(\mathbb{R}^n))$ and $C_2 \in C^1([t_+, \infty), (0, \infty))$, then $P_i, Q_i, G, G^{-1} \in C^1([t_+, \infty), L(\mathbb{R}^n))$, i.e., $P_i(t)$, $Q_i(t)$, $G(t)$ and $G^{-1}(t)$ are continuously differentiable as operator functions on $[t_+, \infty)$.

For each t any vector $x \in \mathbb{R}^n$ can be uniquely represented (with respect to the decomposition (2.6)) in the form

$$x = P_1(t)x + P_2(t)x = x_{p_1}(t) + x_{p_2}(t), \quad x_{p_i}(t) = P_i(t)x \in X_i(t). \tag{2.9}$$

Note that the DAE (2.1) is equivalent to $A(t)[P_1(t)x(t)]' + A'(t)[P_1(t)x(t)] + B(t)x(t) = f(t, x(t))$. Using the projectors $Q_1(t)$, $Q_2(t)$ and taking into account (2.7), we reduce the DAE (2.1) to the equivalent system

$$\begin{aligned} A(t)P_1(t)[P_1(t)x(t)]' + Q_1(t)A'(t)P_1(t)x(t) + B(t)P_1(t)x(t) = \\ = Q_1(t)f(t, x(t)), \end{aligned} \tag{2.10}$$

$$B(t)P_2(t)x(t) = Q_2(t)[f(t, x(t)) - A'(t)P_1(t)x(t)]. \tag{2.11}$$

Using the operator $G^{-1}(t)$ and the equality $P_1(t)[P_1(t)x(t)]' = [P_1(t)x(t)]' - P_1'(t)P_1(t)x(t)$, we transform (2.10), (2.11) to the following system equivalent to the DAE (2.1):

$$\begin{aligned} [P_1(t)x(t)]' = \big[P_1'(t) - G^{-1}(t)Q_1(t)[A'(t) + B(t)]\big]P_1(t)x(t) + \\ + G^{-1}(t)Q_1(t)f(t, P_1(t)x(t) + P_2(t)x(t)), \end{aligned} \tag{2.12}$$

$$G^{-1}(t)Q_2(t)[f(t, P_1(t)x(t) + P_2(t)x(t)) - A'(t)P_1(t)x(t)] - P_2(t)x(t) = 0. \tag{2.13}$$

Taking into account the notation $x_{p_i}(t) = P_i(t)x(t)$ from (2.9), the system (2.12), (2.13) can be rewritten in the form

$$\begin{aligned} x_{p_1}'(t) = \big[P_1'(t) - G^{-1}(t)Q_1(t)[A'(t) + B(t)]\big]x_{p_1}(t) + \\ + G^{-1}(t)Q_1(t)f(t, x_{p_1}(t) + x_{p_2}(t)), \end{aligned} \tag{2.14}$$

$$G^{-1}(t)Q_2(t)[f(t, x_{p_1}(t) + x_{p_2}(t)) - A'(t)x_{p_1}(t)] - x_{p_2}(t) = 0. \tag{2.15}$$

Similarly, the DAE (2.2) is reduced to the equivalent system

$$A(t)P_1(t)x'(t) + B(t)P_1(t)x(t) = Q_1(t)f(t, x(t)), \tag{2.16}$$

$$B(t)P_2(t)x(t) = Q_2(t)f(t, x(t)), \tag{2.17}$$

and then is reduced to the equivalent system

$$\begin{aligned} &[P_1(t)x(t)]' = G^{-1}(t)[-B(t)P_1(t)x(t) + Q_1(t)f(t, x(t))] + P_1'(t)x(t), \\ &G^{-1}(t)Q_2(t)f(t, x(t)) - P_2(t)x(t) = 0, \end{aligned}$$

which can be rewritten in the form

$$\begin{aligned} x_{p_1}'(t) = G^{-1}(t)[-B(t)x_{p_1}(t) + Q_1(t)f(t, x_{p_1}(t) + x_{p_2}(t))] + \\ + P_1'(t)(x_{p_1}(t) + x_{p_2}(t)), \end{aligned} \tag{2.18}$$

$$G^{-1}(t)Q_2(t)f(t, x_{p_1}(t) + x_{p_2}(t)) - x_{p_2}(t) = 0. \tag{2.19}$$

Remark 2.1. Introduce the manifolds

$$L_{t_+} = \{(t, x) \in [t_+, \infty) \times \mathbb{R}^n \mid Q_2(t)[A'(t)P_1(t)x + B(t)P_2(t)x - f(t, x)] = 0\}, \tag{2.20}$$

$$\widehat{L}_{t_+} = \{(t, x) \in [t_+, \infty) \times \mathbb{R}^n \mid Q_2(t)[B(t)P_2(t)x - f(t, x)] = 0\}. \tag{2.21}$$

The consistency condition $(t_0, x_0) \in L_{t_+}$ ($(t_0, x_0) \in \widehat{L}_{t_+}$) for the initial values t_0, x_0 is one of the necessary conditions for the existence of a solution of the initial value problem (2.1), (2.3) (the initial value problem (2.2), (2.3)). The initial point

(t_0, x_0) (the initial values t_0, x_0) satisfying this condition is called a *consistent initial point* (*consistent initial values*).

Generally, in the formulas (2.20), (2.21) the number t_+ is a parameter. In particular, in what follows we will use the notation L_T for a manifold having the form L_{t_+}, where $t_+ = T$.

It is clear that the graphs of the solutions $x(t)$ of the DAEs (2.1) and (2.2) (i.e., the sets of points $(t, x(t))$, where t from the domain of definition of the solution $x(t)$) must lie in the manifolds L_{t_+} and $\widehat{L}_{t_+}$, respectively.

Remark 2.1 is obvious, given the equivalence of the DAE (2.1) and DAE (2.2) to the systems (2.10), (2.11) and (2.16), (2.17) , respectively.

Consider the differential inequalities

$$v' \le \chi(t, v), \qquad v' \ge \chi(t, v), \tag{2.22}$$

where $\chi \in C([t_+, \infty) \times (0, \infty), \mathbb{R})$ (in what follows, we are only interested in positive scalar functions $v \in C^1([t_+, \infty), (0, \infty))$ satisfying one of the inequalities). Assume that $\chi(t, v) = k(t)U(v)$, where $U \in C((0, \infty), \mathbb{R})$ is a positive function, $k \in C([t_+, \infty), \mathbb{R})$, then the inequalities (2.22) take the form

$$v' \le k(t)U(v), \tag{2.23}$$

$$v' \ge k(t)U(v), \tag{2.24}$$

and the following cases are possible [14, p. 109]: if $\int\limits_c^\infty \dfrac{dv}{U(v)} = \infty$ ($c > 0$ is some constant), then the inequality (2.23) has no positive solutions $v(t)$ with finite escape time (i.e., $v(t)$ exists on some finite interval and is unbounded [14, p. 107]); if $\int\limits_c^\infty \dfrac{dv}{U(v)} = \infty$ and $\int\limits_{t_0}^\infty k(t)dt < \infty$ ($t_0 \ge t_+$ is some number), then the inequality (2.23) has no unbounded positive solutions for $t \ge t_+$; if $\int\limits_c^\infty \dfrac{dv}{U(v)} < \infty$ and $\int\limits_{t_0}^\infty k(t)dt = \infty$, then the inequality (2.24) has no positive solutions defined in the future (i.e., defined for all $t \ge t_+$ [14, p. 107]).

Definition 2.2. Consider an operator function $H\colon J \to L(X)$, where X is a finite-dimensional linear space or Hilbert space and $J \subseteq \mathbb{R}$ is an interval. The following definitions are introduced by analogy with that of [18, p. 50–51], [18, p. 209]. We call an operator $H(t) \in L(X)$ ($t \in J$), self-adjoint for every $t \in J$, simply self-adjoint. The self-adjoint operator $H(t) \in L(X)$ ($t \in J$) is called *positive* if $(H(t)x, x) > 0$ for all $t \in J$, $x \neq 0$. The self-adjoint operator $H(t)$ is called *positive definite* or *uniformly positive* if there exists a constant $H_0 > 0$ such that $(H(t)x, x) \ge H_0\|x\|^2$ for all t, x. An operator function $H\colon J \to L(X)$ is called *self-adjoint* if the operator $H(t)$ is self-adjoint. The self-adjoint operator function $H\colon J \to L(X)$ is called *positive* if the operator $H(t)$ is positive, and *positive definite* or *uniformly positive* if $H(t)$ is positive definite.

If X is a finite-dimensional linear space and a self-adjoint operator $H \in L(X)$ is time-invariant and positive ($(Hx, x) > 0$ for all $x \neq 0$), then it is also positive definite. Clearly, $(Hx,x) \geq H_0\|x\|^2$, where $H_0 = \inf\limits_{\|x\|=1}(Hx,x) > 0$ [18]. For a time-varying operator $H(t)$, $H_0 = \inf\limits_{\|x\|=1,\, t\in[t_+,\infty)}(H(t)x,x)$ can be taken.

3. Lagrange stability and global solvability of the DAEs

The definitions of solutions defined in the future and with finite escape time, and the Lagrange stability of an equation are given for an explicit ODE in [14]. Similar definitions for DAEs are given below [11].

A solution $x(t)$ of the initial value problem (2.1), (2.3) is called *global* or *defined in the future* if it exists on $[t_0,\infty)$.

A solution $x(t)$ of (2.1), (2.3) is called *Lagrange stable* if it is global and bounded, i.e., $\sup\limits_{t\in[t_0,\infty)}\|x(t)\| < \infty$.

A solution $x(t)$ of the initial value problem (2.1), (2.3) has a *finite escape time* if it exists on some finite interval $[t_0,T)$ and is unbounded, i.e., there exists $T > t_0$ $(T < \infty)$ such that $\lim\limits_{t\to T-0}\|x(t)\| = \infty$.

A solution $x(t)$ of (2.1), (2.3) is called *Lagrange unstable* if it has a finite escape time. If the solution is Lagrange unstable, then it is also said to be blow-up in finite time.

The *equation* (2.1) *is Lagrange stable for the initial point* (t_0,x_0) if for this initial point the solution of the initial value problem (2.1), (2.3) is Lagrange stable. The equation (2.1) is called *Lagrange stable* if every solution of the initial value problem (2.1), (2.3) is Lagrange stable (i.e., the equation is Lagrange stable for every consistent initial point).

The *equation* (2.1) *is Lagrange unstable for the initial point* (t_0,x_0) if for this initial point the solution of the initial value problem (2.1), (2.3) is Lagrange unstable. The equation (2.1) is called *Lagrange unstable* if every solution of the initial value problem (2.1), (2.3) is Lagrange unstable.

Similar definitions hold for the initial value problem (2.2), (2.3).

3.1. Existence and uniqueness of global solutions.

Theorem 3.1 (Global solvability of the DAE (2.1))**.**

Let $f\in C([t_+,\infty)\times\mathbb{R}^n,\mathbb{R}^n)$, $\dfrac{\partial f}{\partial x}\in C([t_+,\infty)\times\mathbb{R}^n, L(\mathbb{R}^n))$, $A,\ B\in C^1([t_+,\infty), L(\mathbb{R}^n))$, *the pencil* $\lambda A(t)+B(t)$ *satisfy* (2.4), *where* $C_2 \in C^1([t_+,\infty),(0,\infty))$, *and the following conditions be satisfied:*

1) *for each* $t \in [t_+,\infty)$ *and each* $x_{p_1}(t) \in X_1(t)$ *there exists a unique* $x_{p_2}(t)\in X_2(t)$ *such that*
$$(t, x_{p_1}(t) + x_{p_2}(t)) \in L_{t_+}; \tag{3.1}$$
2) *for any fixed* $t_* \in [t_+,\infty)$, $x^*_{p_1}(t_*) \in X_1(t_*)$, $x^*_{p_2}(t_*) \in X_2(t_*)$ *such that* $(t_*, x^*_{p_1}(t_*) + x^*_{p_2}(t_*)) \in L_{t_+}$ *the operator* $\Phi_{t_*,x^*_{p_1}(t_*),x^*_{p_2}(t_*)}$ *defined by*

$$\Phi_{t_*,x^*_{p_1}(t_*),x^*_{p_2}(t_*)} = \\ = \left[\frac{\partial}{\partial x}\big[Q_2(t_*)f(t_*, x^*_{p_1}(t_*) + x^*_{p_2}(t_*))\big] - B(t_*)\right] P_2(t_*)\colon X_2(t_*) \to Y_2(t_*) \tag{3.2}$$

is invertible;

3) *there exist a number $R > 0$, a positive definite function $V \in C^1([t_+,\infty) \times U_R^c(0),\mathbb{R})$, where $U_R^c(0) = \{z \in \mathbb{R}^n \mid \|z\| \geq R\}$, and a function $\chi \in C([t_+,\infty) \times (0,\infty),\mathbb{R})$ such that:*
   3.1) *$V(t,z) \to \infty$ uniformly in t on every finite time interval $[a,b) \subset [t_+,\infty)$ as $\|z\| \to \infty$,*
   3.2) *for all $t \in [t_+,\infty)$, $x_{p_1}(t) \in X_1(t)$, $x_{p_2}(t) \in X_2(t)$ such that $(t, x_{p_1}(t) + x_{p_2}(t)) \in L_{t_+}$, $\|x_{p_1}(t)\| \geq R$ the following inequality holds:*

$$V'_{(2.14)}(t, x_{p_1}(t)) \leq \chi\big(t, V(t, x_{p_1}(t))\big), \tag{3.3}$$

where

$$V'_{(2.14)}(t,x_{p_1}(t)) = \frac{\partial V}{\partial t}(t,x_{p_1}(t)) + \bigg(\frac{\partial V}{\partial z}(t,x_{p_1}(t)),\ \big[P_1'(t) - G^{-1}(t)Q_1(t)[A'(t) + \\ + B(t)]\big]x_{p_1}(t) + G^{-1}(t)Q_1(t)f(t, x_{p_1}(t) + x_{p_2}(t))\bigg) \tag{3.4}$$

is the derivative of V along the trajectories of the equation (2.14) *(where $x_{p_1}(t) = z(t)$),*

   3.3) *the differential inequality*

$$v' \leq \chi(t,v), \qquad t \geq t_+, \tag{3.5}$$

has no positive solutions $v(t)$ with finite escape time.

Then for each initial point $(t_0,x_0) \in L_{t_+}$ there exists a unique global solution of the initial value problem (2.1), (2.3).

Proof. As shown above, the DAE (2.1) is equivalent to the system (2.12), (2.13) or (2.14), (2.15). Consider the mappings Π, $F \in C([t_+,\infty) \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ defined as

$$\Pi(t,z,u) = \big[P_1'(t) - G^{-1}(t)Q_1(t)[A'(t) + B(t)]\big]P_1(t)z + \\ + G^{-1}(t)Q_1(t)f(t, P_1(t)z + P_2(t)u), \tag{3.6}$$

$$F(t,z,u) = G^{-1}(t)Q_2(t)\big[f(t, P_1(t)z + P_2(t)u) - A'(t)z\big] - u. \tag{3.7}$$

They have continuous partial derivatives with respect to z, u on $[t_+,\infty) \times \mathbb{R}^n \times \mathbb{R}^n$. Write the partial derivatives of $F(t,z,u)$ with respect to z, u:

$$\frac{\partial}{\partial z}F(t,z,u) = G^{-1}(t)\bigg[\frac{\partial}{\partial x}\big[Q_2(t)f(t,P_1(t)z+P_2(t)u)\big] - Q_2(t)A'(t)\bigg]P_1(t), \tag{3.8}$$

$$\frac{\partial}{\partial u}F(t,z,u) = G^{-1}(t)\frac{\partial}{\partial x}\big[Q_2(t)f(t,P_1(t)z + P_2(t)u)\big]P_2(t) - I_{\mathbb{R}^n} = \\ = G^{-1}(t)\Phi_{t,P_1(t)z,P_2(t)u}P_2(t) - P_1(t), \tag{3.9}$$

where $\Phi_{t,P_1(t)z,P_2(t)u}$ is the operator (3.2), and denote $\tilde{\Phi}_{t,z,u} = \Phi_{t,P_1(t)z,P_2(t)u}$. Consider the system

$$z'(t) = \Pi(t, z(t), u(t)), \tag{3.10}$$

$$F(t, z(t), u(t)) = 0. \tag{3.11}$$

Lemma 3.2. *If a function $x(t)$ is a solution of the DAE* (2.1) *on* $[t_0, t_1)$ *and satisfies the initial condition* (2.3)*, then the functions $z(t) = P_1(t)x(t)$, $u(t) = P_2(t)x(t)$ are a solution of the system* (3.10), (3.11) *on* $[t_0, t_1)$ *and satisfy the initial conditions $z(t_0) = P_1(t_0)x_0$, $u(t_0) = P_2(t_0)x_0$, and $z \in C^1([t_0, t_1), \mathbb{R}^n)$, $u \in C([t_0, t_1), \mathbb{R}^n)$.*

Conversely, if functions $z \in C^1([t_0, t_1), \mathbb{R}^n)$, $u \in C([t_0, t_1), \mathbb{R}^n)$ are a solution of the system (3.10), (3.11) *on* $[t_0, t_1)$ *and satisfy the initial conditions $z(t_0) = P_1(t_0)x_0$, $u(t_0) = P_2(t_0)x_0$, then $P_1(t)z(t) = z(t)$, $P_2(t)u(t) = u(t)$ and the function $x(t) = z(t)+u(t)$ is a solution of the DAE* (2.1) *on* $[t_0, t_1)$ *and satisfies* (2.3).

Proof. Let $x(t)$ be a solution of the DAE (2.1) on $[t_0, t_1)$ and satisfy (2.3). Notice that $(t_0, x_0) \in L_{t_+}$ since (2.1) is equivalent to the system (2.10), (2.11) and $x(t_0) = x_0$ satisfies (2.11) at $t = t_0$ (see Remark 2.1). Since (2.1) is equivalent to (2.12), (2.13), then $z(t) = P_1(t)x(t)$, $u(t) = P_2(t)x(t)$ are a solution of the system (2.12), (2.13) on $[t_0, t_1)$ and consequently are a solution of the system (3.10), (3.11). It is clear that $z(t_0) = P_1(t_0)x_0$ and $u(t_0) = P_2(t_0)x_0$. The smoothness of $z(t)$, $u(t)$ follows from the smoothness of $x(t)$ and the projectors $P_i(t)$.

Now let $z \in C^1([t_0, t_1), \mathbb{R}^n)$, $u \in C([t_0, t_1), \mathbb{R}^n)$ be a solution of the system (3.10), (3.11) on $[t_0, t_1)$ and $z(t_0) = P_1(t_0)x_0$, $u(t_0) = P_2(t_0)x_0$. Obviously, $(t_0, x_0) \in L_{t_+}$. Multiplying (3.11) by $P_1(t)$ and $P_2(t)$, we get that $P_1(t)u(t) \equiv 0$ and $P_2(t)u(t) \equiv u(t)$. Multiplying (3.10) by $P_2(t)$, we get that $z(t)$ satisfies the equation $P_2(t)z'(t) = P_2(t)P_1'(t)P_1(t)z(t)$. Since $P_2(t)z'(t) = [P_2(t)z(t)]' - P_2'(t)z(t)$, $P_2(t)P_1'(t) = -P_2'(t)P_1(t)$ and $z(t_0) \in X_1(t_0)$, then $P_2(t)z(t)$ satisfies the equation $[P_2(t)z(t)]' = P_2'(t)[P_2(t)z(t)]$ and the initial condition $P_2(t_0)z(t_0) = 0$. Consequently, $P_2(t)z(t) \equiv 0$ and therefore $P_1(t)z(t) \equiv z(t)$. Thus, the function $x(t) = z(t) + u(t)$ is such that $P_1(t)x(t) = z(t)$ and $P_2(t)x(t) = u(t)$. Therefore, the function $x(t) = z(t) + u(t)$ is a solution of the system (2.12), (2.13) on $[t_0, t_1)$ and $x(t_0) = x_0$. Hence, it is a solution of (2.1) on $[t_0, t_1)$ and satisfies (2.3). □

As shown in the proof of Lemma 3.2, if $u(t) \in \mathbb{R}^n$ satisfies (3.11), i.e., $F(t, z(t), u(t)) = 0$, then $u(t) = P_2(t)u(t)$, i.e., $u(t) \in X_2(t)$.

Lemma 3.3. *For each $t \in [t_+, \infty)$ and each $z \in \mathbb{R}^n$ there exists a unique $u \in X_2(t)$ such that*

$$F(t, z, u) = 0. \tag{3.12}$$

Proof. Notice that $F(t, z, u) = F(t, P_1(t)z, u)$ for any $z \in \mathbb{R}^n$, since $Q_2(t)A'(t) = Q_2(t)A'(t)P_1(t)$, and that $(t, x_{p_1}(t) + x_{p_2}(t))$ belongs to L_{t_+} if and only if t, $x_{p_1}(t)$, $x_{p_2}(t)$ satisfy (2.15) or the equivalent condition $F(t, x_{p_1}(t), x_{p_2}(t)) = 0$. Therefore, by virtue of the condition 1), for each $t \in [t_+, \infty)$ and each $z \in \mathbb{R}^n$ there exists a unique $u = x_{p_2}(t) \in X_2(t)$ such that $(t, P_1(t)z + u) \in L_{t_+}$ (i.e., $F(t, z, u) = 0$). □

Take any initial point $(t_0, x_0) \in L_{t_+}$ and any fixed $t_* \in [t_0, \infty)$, $z_* \in \mathbb{R}^n$, where $z_* = P_1(t_*)x_0$ for $t_* = t_0$. By Lemma 3.3, there exists a unique $u_* \in X_2(t_*)$ ($u_* = P_2(t_*)x_0$ for $t_* = t_0$) such that $F(t_*, z_*, u_*) = 0$. Since the operator $\tilde{\Phi}_{t,z,u}$ is invertible for each point $(t, z, u) = (t_*, z_*, u_*)$ such that $u_* \in X_2(t_*)$ and $F(t_*, z_*, u_*) = 0$ (i.e., $(t_*, P_1(t_*)z_* + u_*) \in L_{t_0}$), then for such points $(t, z, u) = (t_*, z_*, u_*)$ the operator

$$\Psi_{t,z,u} = \frac{\partial}{\partial u} F(t, z, u) = G^{-1}(t)\tilde{\Phi}_{t,z,u}P_2(t) - P_1(t) \in L(\mathbb{R}^n) \tag{3.13}$$

has the inverse

$$[\Psi_{t,z,u}]^{-1} = \left[\tilde{\Phi}_{t,z,u}\right]^{-1} G(t)P_2(t) - P_1(t) \in L(\mathbb{R}^n).$$

Using the implicit function theorems, we obtain the following statement: There exist an interval $U_{\delta_1}(t_*) = \{t \in (t_0, \infty) \mid |t - t_*| < \delta_1\}$ ($U_{\delta_1}(t_0) = [t_0, t_0 + \delta_1)$ for $t_* = t_0$), neighborhoods $U_{\delta_2}(z_*), U_{\delta_3}(u_*) \subset \mathbb{R}^n$ and a unique function $\nu(t, z) \in C(U_{\delta_1}(t_*) \times U_{\delta_2}(z_*), U_{\delta_3}(u_*))$ which is continuously differentiable in z and such that $F(t, z, \nu(t, z)) = 0$ for $(t, z) \in U_{\delta_1}(t_*) \times U_{\delta_2}(z_*)$, and $\nu(t_*, z_*) = u_*$. Since $F(t, z, \nu(t, z)) = 0$, i.e., $u = \nu(t, z)$ is a solution of (3.12), then $\nu(t, z) = P_2(t)\nu(t, z) \in X_2(t)$ for each $(t, z) \in U_{\delta_1}(t_*) \times U_{\delta_2}(z_*)$. Thus, it is proved that in some neighborhood $U(t_*, z_*)$ of each $(t_*, z_*) \in [t_0, \infty) \times \mathbb{R}^n$ (where $z_* = P_1(t_*)x_0$ for $t_* = t_0$) there exists a unique solution $u = \nu_{t_*,z_*}(t, z)$ of the equation (3.12), continuous in (t, z), continuously differentiable in z and such that $\nu_{t_*,z_*}(t, z) \in X_2(t)$ for each $(t, z) \in U(t_*, z_*)$. Introduce the function $\eta\colon [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ defined by $\eta(t, z) = \nu_{t_*,z_*}(t, z)$ at the point $(t, z) = (t_*, z_*)$ for each $(t_*, z_*) \in [t_0, \infty) \times \mathbb{R}^n$. Then the function $u = \eta(t, z)$ is continuous in (t, z), continuously differentiable in z, a solution of (3.12) and $\eta(t, z) \in X_2(t)$ for $(t, z) \in [t_0, \infty) \times \mathbb{R}^n$. Prove the uniqueness of $u = \eta(t, z)$. Suppose there exists a function $u = \mu(t, z)$ having the same properties as $u = \eta(t, z)$ at some point $(\tilde{t}, \tilde{z}) \in [t_0, \infty) \times \mathbb{R}^n$. By Lemma 3.3, there exists a unique $\tilde{u} \in X_2(\tilde{t})$ such that $(t, z, u) = (\tilde{t}, \tilde{z}, \tilde{u})$ satisfies (3.12). Consequently, $\eta(\tilde{t}, \tilde{z}) = \mu(\tilde{t}, \tilde{z}) = \tilde{u}$. It is similarly proved that if the point $(\tilde{t}, \tilde{z})$ belongs to the intersection of neighborhoods $U^1(t^1, z^1)$, $U^2(t^2, z^2)$ of some points (t^1, z^1), $(t^2, z^2) \in [t_0, \infty) \times \mathbb{R}^n$ in which the solutions $u = \nu_{t^1,z^1}(t, z)$ and $u = \nu_{t^2,z^2}(t, z)$ of (3.12) are defined respectively, then $\nu_{t^1,z^1}(\tilde{t}, \tilde{z}) = \nu_{t^2,z^2}(\tilde{t}, \tilde{z}) = \eta(\tilde{t}, \tilde{z}) = \tilde{u}$. Since this holds for any $(\tilde{t}, \tilde{z}) \in [t_0, \infty) \times \mathbb{R}^n$, there exists the unique function $u = \eta(t, z)$ with the above properties.

Substitute the function $u = \eta(t, z)$ in (3.6) and denote $\widetilde{\Pi}(t, z) = \Pi(t, z, \eta(t, z))$. Then (3.10) takes the form

$$z'(t) = \widetilde{\Pi}(t, z(t)). \tag{3.14}$$

By the properties of η and Π, the function $\widetilde{\Pi}$ is continuous in (t, z) and continuously differentiable in z on $[t_0, \infty) \times \mathbb{R}^n$. Hence, there exists a unique solution $z = \zeta(t)$ of (3.14) satisfying the initial condition $\zeta(t_0) = z_0$, where $z_0 = P_1(t_0)x_0$, on some interval $[t_0, \alpha)$. By the extension theorems (continuation theorems), there exists the maximal interval of existence $[t_0, \omega)$ ($\omega \le \infty$) for the solution $z = \zeta(t)$ of the initial value problem (3.14), $\zeta(t_0) = z_0$, and the solution $\zeta(t)$ is unique on this interval. Since the functions $z = \zeta(t)$, $u = \eta(t, \zeta(t))$ are the solution of the system (3.10), (3.11) on $[t_0, \omega)$ and satisfy the initial conditions $\zeta(t_0) = z_0 = P_1(t_0)x_0$,

$\eta(t_0, z_0) = P_2(t_0)x_0$, then, by Lemma 3.2, $\zeta(t) = P_1(t)\zeta(t) \in X_1(t)$, $\eta(t, \zeta(t)) = P_2(t)\eta(t, \zeta(t)) \in X_2(t)$ for all $t \in [t_0, \omega)$, and the function $x(t) = \zeta(t) + \eta(t, \zeta(t))$ is a solution of (2.1) on $[t_0, \omega)$ and satisfies (2.3). The uniqueness of the solution $z = \zeta(t)$, $u = \gamma(t)$ of the system (3.10), (3.11) and, accordingly, the solution $x(t)$ of the equation (2.1) on $[t_0, \omega)$ follows from the uniqueness of the solution $u = \eta(t, z)$ of (3.12) and the solution $z = \zeta(t)$ of (3.14).

By virtue of the extension theorems, either $\omega = \infty$, i.e., the maximal interval of existence for the solution $z = \zeta(t)$ of (3.14) coincides with $[t_0, \infty)$, or $\omega < \infty$ and $\lim\limits_{t\to\omega-0} \|\zeta(t)\| = \infty$, i.e., the solution has the finite escape time $[t_0, \omega)$. We prove that $\omega = \infty$. Recall that $\zeta(t) = P_1(t)x(t) = x_{p_1}(t)$, $\eta(t, \zeta(t)) = P_2(t)x(t) = x_{p_2}(t)$, where $x(t) = \zeta(t) + \eta(t, \zeta(t))$ is a solution of the DAE (2.1), the DAE (2.1) is equivalent to the system (2.14), (2.15), and, accordingly, $x_{p_1}(t) = \zeta(t)$, $x_{p_2}(t) = \eta(t, \zeta(t))$ is a solution of (2.14), (2.15) ($z = \zeta(t)$ is a solution of the equation (3.14) which coincides with (2.14) for $x_{p_1}(t) = P_1(t)z(t)$, $x_{p_2}(t) = \eta(t, z(t))$). Assume that $\omega < \infty$ (then $\lim\limits_{t\to\omega-0} \|\zeta(t)\| = \infty$). Then there exists $t_1 \in (t_0, \omega)$ such that for each $t \in [t_1, \omega)$ the solution $\zeta(t)$ belongs to $U^c_{1R_0} = \{x \in \mathbb{R}^n \mid (t, x) \in L_{t_0},\ \|P_1(t)x\| \geq R_0 > R\} \subset U^c_R(0)$. By virtue of the condition 3), for all $t \geq t_1$ the derivative of V along the trajectories of the equation (3.14) satisfy

$$V'_{(3.14)}(t, \zeta(t)) = \frac{\partial V}{\partial t}(t, \zeta(t)) + \left(\frac{\partial V}{\partial z}(t, \zeta(t)), \widetilde{\Pi}(t, \zeta(t))\right) \leq \chi\big(t, V(t, \zeta(t))\big). \quad (3.15)$$

Therefore, for $t \geq t_1$ the function $v(t) = V(t, \zeta(t))$ is a positive solution of the differential inequality (3.5), which has the finite escape time (since $\zeta(t)$ has the finite escape time). This contradicts the condition 3). Consequently, $\omega = \infty$ and the solution $\zeta(t)$ is global.

Thus, it is proved that the function $x(t) = \zeta(t) + \eta(t, \zeta(t))$ is a unique solution of the initial value problem (2.1), (2.3) on $[t_0, \infty)$. Since (t_0, x_0) is an arbitrary point from L_{t_+}, the existence of a unique global solution is proved for each initial point $(t_0, x_0) \in L_{t_+}$. □

Denote the right-hand side of the equation (2.14) by

$$\begin{aligned}\widehat{\Pi}(t, x_{p_1}(t) + x_{p_2}(t)) = \big[P'_1(t) - G^{-1}(t)Q_1(t)[A'(t) + B(t)]\big]x_{p_1}(t) +\\ + G^{-1}(t)Q_1(t)f(t, x_{p_1}(t) + x_{p_2}(t))\end{aligned}$$

(recall that any element $x \in \mathbb{R}^n$ can be uniquely represented in the form $x = x_{p_1}(t) + x_{p_2}(t)$ (2.9), where $x_{p_i}(t) = P_i(t)x$). Then the equation (2.14) or (2.12) is written as $x'_{p_1}(t) = \widehat{\Pi}(t, x_{p_1}(t) + x_{p_2}(t))$ or $[P_1(t)x(t)]' = \widehat{\Pi}(t, x(t))$.

Consider the equation

$$\begin{aligned}x'_{p_1}(t) =& \widehat{\Pi}_T(t, x_{p_1}(t) + x_{p_2}(t)),\\ &\widehat{\Pi}_T(t, x_{p_1}(t) + x_{p_2}(t)) = \begin{cases}\widehat{\Pi}(t, x_{p_1}(t) + x_{p_2}(t)), & t \in [t_+, T],\\ \widehat{\Pi}(T, x_{p_1}(T) + x_{p_2}(T)), & t > T,\end{cases}\end{aligned} \quad (3.16)$$

where $T > t_+$ is a parameter. The function $\widehat{\Pi}_T(t, x)$ is called the *truncation* of $\widehat{\Pi}(t, x)$ over t.

Remark 3.4. Theorem 3.1 remains valid if the condition 3) is replaced by the following:

3) there exists a positive definite function $V \in C^1([t_+,\infty) \times U_R^c(0),\mathbb{R})$ (where $U_R^c(0) = \{z \in \mathbb{R}^n \mid \|z\| \geq R\}$, $R > 0$ is some number) and for each $T > 0$ there exist a number $R_T \geq R$ and a function $\chi_T \in C([t_+,\infty) \times (0,\infty),\mathbb{R})$ such that:
   3.1) $V(t,z) \to \infty$ uniformly in t on every finite time interval $[a,b) \subset [t_+,\infty)$ as $\|z\| \to \infty$,
   3.2) for all $t \in [t_+,\infty)$, $x_{p_1}(t) \in X_1(t)$, $x_{p_2}(t) \in X_2(t)$ such that $(t, x_{p_1}(t) + x_{p_2}(t)) \in L_{t_+}$, $\|x_{p_1}(t)\| \geq R_T$, the following inequality holds:
$$V'_{(3.16)}(t,x_{p_1}(t)) \leq \chi_T\big(t, V(t,x_{p_1}(t))\big), \tag{3.17}$$
   where $V'_{(3.16)}(t,x_{p_1}(t)) = \dfrac{\partial V}{\partial t}(t,x_{p_1}(t)) + \left(\dfrac{\partial V}{\partial z}(t,x_{p_1}(t)), \widehat{\Pi}_T(t,x_{p_1}(t) + x_{p_2}(t))\right)$ is the derivative of V along the trajectories of the equation (3.16),
   3.3) the differential inequality $v' \leq \chi_T(t,v)$, $t \geq t_+$, has no positive solutions $v(t)$ with finite escape time.

Since the solution of the equation with the truncation coincides with the solution of the original equation with the same initial values on the interval $[t_+,T]$, where the right-hand sides of the equations coincide, then the proof of the remark is easily derived from the proof of the theorem if we consider $T = \omega$ in it.

A system of s pairwise disjoint projectors $\{\Theta_k\}_{k=1}^s$ (the projectors are one-dimensional), $\Theta_k \in L(Z)$, the sum of which is the identity operator I_Z in an s-dimensional linear space Z, i.e., $\Theta_i\,\Theta_j = \delta_{ij}\,\Theta_i$ and $I_Z = \sum\limits_{k=1}^{s} \Theta_k$, is called an *additive resolution of the identity* in Z [10]. The additive resolution of the identity generates the decomposition $Z = Z_1 \dot{+} \cdots \dot{+} Z_s$ into the direct sum of the one-dimensional subspaces $Z_k = \Theta_k\,Z$, and the system $\{z_k \in Z\}_{k=1}^s$ of the vectors such that $z_k \neq 0$ and $z_k = \Theta_k\,z_k$ forms a basis of Z. Note that the property of basis invertibility does not depend on the choice of an additive resolution of the identity or a basis of Z.

The following definition agrees with the one introduced in [10].

Definition 3.5. An operator function $\Phi\colon D \to L(W,Z)$, where W, Z are s-dimensional linear spaces and $D \subset W$, is called *basis invertible* on an interval $J \subset D$, if for some additive resolution of the identity $\{\Theta_k\}_{k=1}^s$ in the space Z and for any set $\{w^k\}_{k=1}^s$ of elements $w^k \in J$ the operator $\Lambda = \sum\limits_{k=1}^{s} \Theta_k \Phi(w^k) \in L(W,Z)$ has the inverse $\Lambda^{-1} \in L(Z,W)$.

It follows from the basis invertibility of the mapping Φ on the interval J that Φ is invertible on this interval, but the converse is not true (see [11, Example 2.1]) unless the spaces W, Z are one-dimensional.

The global solvability theorem is given below, taking into account the following remark. If in Theorem 3.1 we replace the requirement of invertibility by the requirement of basis invertibility in the condition 2), then the uniqueness of $x_{p_2}(t)$ in the condition 1) is not required.

Remark 3.6. Notice that $\operatorname{rank} P_j(t) = \operatorname{rank} Q_j(t) = \dim X_j(t) = \dim Y_j(t)$, $j = 1, 2$, and $\dim Y_1(t) = \operatorname{rank} A(t)$, and that, by virtue of the smoothness of $P_j(t)$ and $Q_j(t)$, the dimensions of the subspaces $X_j(t) = P_j(t)\mathbb{R}^n$ and $Y_j(t) = Q_j(t)\mathbb{R}^n$ are constant (see [20, p. 34, Lemma 4.10]): $\dim X_1(t) = \dim Y_1(t) = const$ and $\dim X_2(t) = \dim Y_2(t) = const$ for all $t \in [t_+, \infty)$. Denote $\dim X_2(t) = \dim Y_2(t) = d$, then $\dim X_1(t) = \dim Y_1(t) = n - d$, $t \in [t_+, \infty)$.

Theorem 3.7 (Global solvability of the DAE (2.1))**.**

Let $f \in C([t_+, \infty) \times \mathbb{R}^n, \mathbb{R}^n)$, $\dfrac{\partial f}{\partial x} \in C([t_+, \infty) \times \mathbb{R}^n, L(\mathbb{R}^n))$, $A, B \in C^1([t_+, \infty), L(\mathbb{R}^n))$, *the pencil* $\lambda A(t) + B(t)$ *satisfy* (2.4), *where* $C_2 \in C^1([t_+, \infty), (0, \infty))$, *and the following conditions be satisfied:*

1) *for each* $t \in [t_+, \infty)$, $x_{p_1}(t) \in X_1(t)$ *there exists* $x_{p_2}(t) \in X_2(t)$ *such that* (3.1)*;*
2) *for any fixed* $t_* \in [t_+, \infty)$, $x^*_{p_1}(t_*) \in X_1(t_*)$, $x^i_{p_2}(t_*) \in X_2(t_*)$ *such that* $(t_*, x^*_{p_1}(t_*) + x^i_{p_2}(t_*)) \in L_{t_+}$, $i = 1, 2$, *the operator function* $\Phi_{t_*, x^*_{p_1}(t_*)}(x_{p_2}(t_*))$ *defined by*

$$\begin{aligned} &\Phi_{t_*, x^*_{p_1}(t_*)} \colon X_2(t_*) \to L(X_2(t_*), Y_2(t_*)),\\ &\Phi_{t_*, x^*_{p_1}(t_*)}(x_{p_2}(t_*)) = \left[\frac{\partial}{\partial x}\big[Q_2(t_*) f(t_*, x^*_{p_1}(t_*) + x_{p_2}(t_*))\big] - B(t_*)\right] P_2(t_*), \end{aligned} \tag{3.18}$$

is basis invertible on $[x^1_{p_2}(t_*), x^2_{p_2}(t_*)]$*;*

3) *it coincides with the condition 3) of Theorem 3.1.*

Then for each initial point $(t_0, x_0) \in L_{t_+}$ *there exists a unique global solution of the initial value problem* (2.1), (2.3)*.*

Proof. As in the proof of Theorem 3.1, consider the mappings (3.6), (3.7) and the system (3.10), (3.11). The partial derivative of the mapping $F(t, z, u)$ (3.7) with respect to z, u have the form (3.8), (3.9), where in (3.9) the operator $\Phi_{t, P_1(t)z, P_2(t)u}$ is replaced by the operator function $\Phi_{t, P_1(t)z}(P_2(t)u)$ (3.18), i.e.,

$$\begin{aligned} \frac{\partial}{\partial u} F(t, z, u) = G^{-1}(t) \frac{\partial}{\partial x}\big[Q_2(t) f(t, P_1(t)z + P_2(t)u)\big] P_2(t) - I_{\mathbb{R}^n} = \\ = G^{-1}(t) \Phi_{t, P_1(t)z}(P_2(t)u) P_2(t) - P_1(t), \end{aligned}$$

Denote $\tilde{\Phi}_{t,z}(u) = \Phi_{t, P_1(t)z}(P_2(t)u)$ and introduce the operator function

$$\Psi_{t,z} \colon \mathbb{R}^n \to L(\mathbb{R}^n),\ \Psi_{t,z}(u) = \frac{\partial}{\partial u} F(t, z, u) = G^{-1}(t) \tilde{\Phi}_{t,z}(u) P_2(t) - P_1(t). \tag{3.19}$$

By virtue of the basis invertibility of (3.18) for any fixed $t_* \in [t_+, \infty)$, $z_* \in \mathbb{R}^n$, $u^i_* \in X_2(t_*)$ $(i = 1, 2)$ such that $F(t_*, z_*, u^i_*) = 0$ (i.e., $(t_*, P_1(t_*)z_* + u^i_*) \in L_{t_+}$), the operator function $\tilde{\Phi}_{t_*, z_*}$ is basis invertible on $[u^1_*, u^2_*]$. This property is needed to prove Lemma 3.3 (see below). It also follows from the basis invertibility of

$\tilde{\Phi}_{t_*,z_*}(u)$ that there exists the inverse operator $\left[\tilde{\Phi}_{t_*,z_*}(u_*)\right]^{-1}$ for any fixed $t_* \in [t_+,\infty)$, $z_* \in \mathbb{R}^n$, $u_* \in X_2(t_*)$ such that $F(t_*,z_*,u_*)=0$ and, consequently, for such points (t_*,z_*,u_*) the operator $\Psi_{t_*,z_*}(u_*)$ has the inverse $[\Psi_{t_*,z_*}(u_*)]^{-1} = \left[\tilde{\Phi}_{t_*,z_*}(u_*)\right]^{-1} G(t_*)P_2(t_*) - P_1(t_*) \in L(\mathbb{R}^n)$.

Lemmas 3.2 and 3.3 remain valid, but the proof of Lemma 3.3 changes.

Lemma (Lemma 3.3). For each $t \in [t_+,\infty)$ and each $z \in \mathbb{R}^n$ there exists a unique $u \in X_2(t)$ such that (3.12).

Proof. By virtue of the condition 1) (here the existence of a unique $x_{p_2}(t)$ is not required, unlike the condition 1) of Theorem 3.1), for each $t \in [t_+,\infty)$ and each $z \in \mathbb{R}^n$ there exists $u \in X_2(t)$ such that $(t, P_1(t)z+u) \in L_{t_+}$, i.e., $F(t,z,u)=0$. We prove the uniqueness of such u. Consider arbitrary fixed $t_* \in [t_+,\infty)$, $z_* \in \mathbb{R}^n$, $u_*^i \in X_2(t_*)$, $i=1,2$, such that $F(t_*,z_*,u_*^i)=0$. The basis invertibility of $\tilde{\Phi}_{t_*,z_*}(u)$ on $[u_*^1,u_*^2]$ means that for any set of points $\{u_k\}_{k=1}^d \subset [u_*^1,u_*^2]$, the operator

$$\Lambda_1 = \sum_{k=1}^{d} \tilde{\Theta}_k(t_*)\tilde{\Phi}_{t_*,z_*}(u_k) \in L(X_2(t_*),Y_2(t_*)), \tag{3.20}$$

where $\{\tilde{\Theta}_k(t_*)\}_{k=1}^d$ is some additive resolution of the identity in $Y_2(t_*)$ ($d = \dim Y_2(t) = \dim X_2(t)$, $t \in [t_+,\infty)$, see Remark 3.6), has the inverse $\Lambda_1^{-1} \in L(Y_2(t_*),X_2(t_*))$. Since $Q_2(t_*)$ (restricted to $Y_2(t_*)$) is the identity in $Y_2(t_*)$ (because $Q_2(t_*)y_* = y_*$ for any $y_* \in Y_2(t_*)$), we choose $\{\tilde{\Theta}_k(t_*)\}_{k=1}^d$ such that $\sum\limits_{k=1}^{d} \tilde{\Theta}_k(t_*) = Q_2(t_*)|_{Y_2(t_*)}$, i.e., $\{\tilde{\Theta}_k(t_*)\}_{k=1}^d$ is an additive resolution of the identity $Q_2(t_*)|_{Y_2(t_*)}$ in $Y_2(t_*)$. Then the system $\{\Theta_k(t_*)\}_{k=1}^d$ of the projectors $\Theta_k(t_*) = G^{-1}(t_*)\tilde{\Theta}_k(t_*)G(t_*)\Big|_{X_2(t_*)}$ is an additive resolution of the identity $P_2(t_*)|_{X_2(t_*)}$ in $X_2(t_*)$ ($\sum\limits_{k=1}^{d} \Theta_k(t_*) = P_2(t_*)|_{X_2(t_*)}$). Note that $F(t_*,z_*,u_*) = P_2(t_*)F(t_*,z_*,u_*)$ for any $t_* \in [t_+,\infty)$, $z_* \in \mathbb{R}^n$, $u_* \in X_2(t_*)$. The projections $F_k(t_*,z_*,u_*) = \Theta_k(t_*)F(t_*,z_*,u_*) = \Theta_k(t_*)P_2(t_*)F(t_*,z_*,u_*)$, where $u_* \in X_2(t_*)$, are the functions with values in the one-dimensional spaces $\Theta_k(t_*)X_2(t_*)$ isomorphic to $\mathbb{R}$. By the formula of finite increments [19], there exists a point $u_k \in [u_*^1,u_*^2]$ such that $F_k(t_*,z_*,u_*^2) - F_k(t_*,z_*,u_*^1) = \frac{\partial}{\partial u}F_k(t_*,z_*,u_k)(u_*^2-u_*^1) = \Theta_k(t_*)P_2(t_*)\frac{\partial}{\partial u}F(t_*,z_*,u_k)(u_*^2-u_*^1) = \Theta_k(t_*)P_2(t_*)\Psi_{t_*,z_*}(u_k)(u_*^2-u_*^1)$, $k=\overline{1,d}$. By summing the obtained expressions over k and taking into account that $F(t_*,z_*,u_*^i)=0$ ($i=1,2$), we obtain that $\sum\limits_{k=1}^{d} \Theta_k(t_*)P_2(t_*)\Psi_{t_*,z_*}(u_k)(u_*^2-u_*^1) = G^{-1}(t_*)\sum\limits_{k=1}^{d} \tilde{\Theta}_k(t_*)\tilde{\Phi}_{t_*,z_*}(u_k)(u_*^2-u_*^1) = G^{-1}(t_*)\Lambda_1(u_*^2-u_*^1) = 0$. Since there exists Λ_1^{-1}, then $u_*^2 = u_*^1$. □

The further proof of the theorem coincides with the proof of Theorem 3.1 located below Lemma 3.3. □

3.2. Lagrange stability.

Theorem 3.8 (Lagrange stability of the DAE (2.1))**.**

Let $f \in C([t_+,\infty)\times\mathbb{R}^n,\mathbb{R}^n)$, $\dfrac{\partial f}{\partial x} \in C([t_+,\infty)\times\mathbb{R}^n, L(\mathbb{R}^n))$, $A, B \in C^1([t_+,\infty), L(\mathbb{R}^n))$, *the pencil* $\lambda A(t) + B(t)$ *satisfy* (2.4), *where* $C_2 \in C^1([t_+,\infty),(0,\infty))$, *the requirements 1), 2) of Theorem 3.1 or 1), 2) of Theorem 3.7 be fulfilled, and*

3) *there exists a number* $R > 0$, *a positive definite function* $V \in C^1([t_+,\infty)\times U_R^c(0),\mathbb{R})$, *where* $U_R^c(0) = \{z \in \mathbb{R}^n \mid \|z\| \ge R\}$, *and a function* $\chi \in C([t_+,\infty)\times(0,\infty),\mathbb{R})$ *such that:*
   3.1) $V(t,z)\to\infty$ *uniformly in* t *on* $[t_+,\infty)$ *as* $\|z\|\to\infty$;
   3.2) *for all* $t \in [t_+,\infty)$, $x_{p_1}(t) \in X_1(t)$, $x_{p_2}(t) \in X_2(t)$ *such that* $(t, x_{p_1}(t) + x_{p_2}(t)) \in L_{t_+}$, $\|x_{p_1}(t)\| \ge R$, *the inequality* (3.3) *holds;*
   3.3) *the differential inequality* (3.5) *has no unbounded positive solutions* $v(t)$ *for* $t \in [t_+,\infty)$.

Let one of the following conditions be also satisfied:

4.a) *for all* $(t, x_{p_1}(t) + x_{p_2}(t)) \in L_{t_+}$, $\|x_{p_1}(t)\| \le M < \infty$, $M = const > 0$ *(M is an arbitrary constant), there exists a constant* $K_M = K(M) > 0$, *independent of* t, $x_{p_2}(t)$, *such that* $\|G^{-1}(t)Q_2(t)[f(t, x_{p_1}(t) + x_{p_2}(t)) - A'(t)x_{p_1}(t)]\| \le K_M < \infty$;

4.b) *for all* $(t, x_{p_1}(t) + x_{p_2}(t)) \in L_{t_+}$, $\|x_{p_1}(t)\| \le M < \infty$, $M = const > 0$, *there exists a constant* $K_M = K(M) > 0$, *independent of* t, $x_{p_2}(t)$, *such that* $\|x_{p_2}(t)\| \le K_M < \infty$;

4.c) *for each* $t_* \in [t_+,\infty)$ *there exists* $\tilde{x}_{p_2}(t_*) \in X_2(t_*)$ *such that for any fixed* $x^*_{p_i}(t_*) \in X_i(t_*)$ *satisfying* $(t_*, x^*_{p_1}(t_*) + x^*_{p_2}(t_*)) \in L_{t_+}$ *the operator function* $\Phi_{t_*, x^*_{p_1}(t_*)}(x_{p_2}(t_*))$ (3.18) *is basis invertible on* $(\tilde{x}_{p_2}(t_*), x^*_{p_2}(t_*)]$ *and the corresponding inverse operator is bounded uniformly in* t_*, $x_{p_2}(t_*)$ *(i.e., the operator* $\Lambda_1^{-1} = \left[\sum\limits_{k=1}^{d} \tilde{\Theta}_k(t_*)\Phi_{t_*, x^*_{p_1}(t_*)}(x_{p_2,k}(t_*))\right]^{-1}$ *inverse to* (3.20), *where* $z_* = x^*_{p_1}(t_*)$ *and* $\{u_k = x_{p_2,k}(t_*)\}_{k=1}^d$ *is an arbitrary set of the elements from* $(\tilde{x}_{p_2}(t_*), x^*_{p_2}(t_*)]$, *is bounded uniformly in* t_*, $x_{p_2,k}(t_*)$ *on* $[t_+,\infty)$, $(\tilde{x}_{p_2}(t_*), x^*_{p_2}(t_*)]$ *), and, in addition,* $\sup\limits_{t_* \in [t_+,\infty)} \|\tilde{x}_{p_2}(t_*)\| < \infty$,

$$\sup_{t \ge t_+,\, \|x_{p_1}(t)\| \le M < \infty,\, M = const} \|G^{-1}(t)Q_2(t)[f(t, x_{p_1}(t) + \tilde{x}_{p_2}(t_*)) - A'(t)x_{p_1}(t)]\| < \infty. \tag{3.21}$$

Then the DAE (2.1) *is Lagrange stable.*

Proof. As in the proof of Theorem 3.1, we prove that $z = \zeta(t)$ and $u = \eta(t,\zeta(t))$ ($\zeta(t) \in X_1(t)$, $\eta(t,\zeta(t)) \in X_2(t)$, $(t, \zeta(t) + \eta(t,\zeta(t))) \in L_{t_0}$ for all $t \in [t_0,\infty)$) are the unique solution of the system (3.10), (3.11) on $[t_0,\infty)$ satisfying the initial conditions $\zeta(t_0) = P_1(t_0)x_0$, $\eta(t_0,\zeta(t_0)) = P_2(t_0)x_0$ (where $(t_0, x_0) \in L_{t_+}$), and $x(t) = \zeta(t) + \eta(t,\zeta(t))$ is the unique solution of the initial value problem (2.1), (2.3).

By virtue of 3) the solution $z = \zeta(t)$ of (3.14) is bounded on $[t_0, \infty)$. Indeed, if $\zeta(t)$ is unbounded on $[t_0, \infty)$, then there exists a sequence $\{t_k\}_{k=1}^{\infty} \subset [t_0, \infty)$ such that $t_k \to \infty$ as $k \to \infty$ and $\|\zeta(t_k)\| \to \infty$ as $k \to \infty$. Then $v(t) = V(t, \zeta(t))$ is the unbounded positive solution of the inequality (3.5) for $t \geq t_1$, which contradicts the condition 3). Thus, there exists a constant $M > 0$ such that

$$\|\zeta(t)\| \leq M, \qquad t \in [t_0, \infty). \tag{3.22}$$

Since the equation (3.11) can be rewritten as $u(t) = G^{-1}(t)Q_2(t)\big[f(t, P_1(t)z(t) + P_2(t)u(t)) - A'(t)z(t)\big]$, then

$$\eta(t, \zeta(t)) = G^{-1}(t)Q_2(t)\big[f(t, \zeta(t) + \eta(t, \zeta(t))) - A'(t)\zeta(t)\big]. \tag{3.23}$$

Hence, by (3.22) and the condition 4.a) there exists a constant $K_M > 0$ such that for all $t \in [t_0, \infty)$ the following estimate holds:

$$\|\eta(t, \zeta(t))\| \leq K_M < \infty. \tag{3.24}$$

By (3.22) and 4.b) we also obtain that there exists a constant $K_M > 0$ such that (3.24) for any $t \in [t_0, \infty)$.

Now we prove the boundedness of $\|\eta(t, \zeta(t))\|$ using the condition 4.c). Take arbitrary fixed $t_* \in [t_+, \infty)$, $z_* \in \mathbb{R}^n$, $u_* \in X_2(t_*)$ satisfying the condition $F(t_*, z_*, u_*) = 0$ (i.e., $(t_*, P_1(t_*)z_* + u_*) \in L_{t_+}$). By virtue of the condition 4.c), there exists an element $\tilde{u}_* = \tilde{u}(t_*) \in X_2(t_*)$ such that the operator function $\tilde{\Phi}_{t_*, z_*}(u) = \Phi_{t_*, P_1(t_*)z_*}(P_2(t_*)u)$ (3.18) is basis invertible on $(\tilde{u}_*, u_*]$ and the corresponding inverse operator, i.e., the operator $\Lambda_1^{-1} = \left[\sum\limits_{k=1}^{d} \tilde{\Theta}_k(t_*)\tilde{\Phi}_{t_*, z_*}(u_k)\right]^{-1} = \Lambda_1^{-1}(t_*, z_*, u_k) \in L(Y_2(t_*), X_2(t_*))$ inverse to the operator (3.20), where $\{u_k\}_{k=1}^{d}$ is an arbitrary set of the elements $u_k \in (\tilde{u}_*, u_*]$ $(d = \dim X_2(t_*))$ and $\{\tilde{\Theta}_k(t_*)\}_{k=1}^{d}$ is an additive resolution of the identity in $Y_2(t_*)$, is bounded uniformly in t_*, u_k on $[t_+, \infty)$, $(\tilde{u}_*, u_*]$. As in the proof of Lemma (see p. 13), we choose $\{\tilde{\Theta}_k(t_*)\}_{k=1}^{d}$ such that $\sum\limits_{k=1}^{d} \tilde{\Theta}_k(t_*) = Q_2(t_*)\big|_{Y_2(t_*)}$ and take the additive resolution of the $\big\{\Theta_k(t_*) = G^{-1}(t_*)\tilde{\Theta}_k(t_*)G(t_*)\big|_{X_2(t_*)}\big\}_{k=1}^{d}$ in $X_2(t_*)$. Also, consider the projections $F_k(t_*, z_*, u) = \Theta_k(t_*)F(t_*, z_*, u) = \Theta_k(t_*)P_2(t_*)F(t_*, z_*, u)$, where $u \in X_2(t_*)$. By the formula of finite increments, there exists a point $u_k \in (\tilde{u}_*, u_*]$ such that $F_k(t_*, z_*, u_*) - F_k(t_*, z_*, \tilde{u}_*) = \frac{\partial}{\partial u}F_k(t_*, z_*, u_k)(u_* - \tilde{u}_*) = \Theta_k(t_*)P_2(t_*)\Psi_{t_*, z_*}(u_k)(u_* - \tilde{u}_*)$, where the operator function $\Psi_{t,z}$ is defined in (3.19), $k = \overline{1, d}$. Since $F_k(t_*, z_*, u_*) = 0$, then, by summing the obtained equalities over k, we obtain that there exists a set $\{u_k\}_{k=1}^{d} \subset (\tilde{u}_*, u_*]$ such that $-F(t_*, z_*, \tilde{u}_*) = G^{-1}(t_*)\Lambda_1(u_* - \tilde{u}_*)$. Since there exists Λ_1^{-1}, then $u_* = \tilde{u}_* - \Lambda_1^{-1}G(t_*)F(t_*, z_*, \tilde{u}_*) = \tilde{u}_* - \Lambda_1^{-1}\big(Q_2(t_*)\big[f(t_*, P_1(t_*)z_* + P_2(t_*)\tilde{u}_*) - A'(t_*)z_*\big] - G(t_*)\tilde{u}_*\big)$. This holds for any fixed $t_* \in [t_+, \infty)$, $z_* \in \mathbb{R}^n$, $u_* \in X_2(t_*)$ satisfying $F(t_*, z_*, u_*) = 0$. Therefore, for each $t_* \in [t_0, \infty)$ the equality $\eta(t_*, \zeta(t_*)) = \tilde{u}_* - \Lambda_1^{-1}G(t_*)F(t_*, z_*, \tilde{u}_*) = \tilde{u}_* - \Lambda_1^{-1}G(t_*)\big(G^{-1}(t_*)Q_2(t_*)\big[f(t_*, \zeta(t_*) + P_2(t_*)\tilde{u}_*) - A'(t_*)\zeta(t_*)\big] - \tilde{u}_*\big)$ is fulfilled. By virtue of the condition 4.c), the set of the elements $\tilde{u}_* = \tilde{u}(t_*)$ is bounded, i.e., there exists a constant $\tilde{M} < \infty$ such that

$\|\tilde{u}_*\| = \|\tilde{u}(t_*)\| \le \tilde{M}$ for each $t_* \in [t_+,\infty)$. From the continuity of the nonlinear mapping $\Lambda_1^{-1} = \Lambda_1^{-1}(t_*, z_*, u_k)$ in t_*, z_* and the compactness of the ball $\|\zeta(t_*)\| \le M$, $t_* \in [t_0,\infty)$, where $z = \zeta(t) \in C([t_0,\infty),\mathbb{R}^n)$ $(\zeta(t) \in X_1(t))$, it follows that Λ_1^{-1} is uniformly continuous in z_* (in $P_1(t_*)z_*$) and is bounded on $\|z_*\| = \|\zeta(t_*)\| \le M$. By the condition 4.c), $\Lambda_1^{-1} = \Lambda_1^{-1}(t_*, z_*, u_k) \in L(Y_2(t_*), X_2(t_*))$ is bounded uniformly in t_*, u_k on $[t_+,\infty)$, $(\tilde{u}_*, u_*]$. Therefore, there exists a constant $N > 0$, independent of t_*, z_*, u^k, such that $\|\Lambda_1^{-1}\| \le N$ for each $t_* \in [t_+,\infty)$, each $z_* \in \mathbb{R}^n$ and each $u_* \in X_2(t_*)$ satisfying $F(t_*, z_*, u_*) = 0$ and for any set $\{u_k\}_{k=1}^d \subset (\tilde{u}_*, u_*]$. Thus, $\|\eta(t_*, \zeta(t_*))\| \le \tilde{M}(1+N\|G(t_*)\|)+\|G^{-1}(t_*)Q_2(t_*)\big[f(t_*,\zeta(t_*)+P_2(t_*)\tilde{u}_*)-A'(t_*)\zeta(t_*)\big]\|$ for each $t_* \in [t_+,\infty)$. Then it follows from (3.22), (3.21) that there exists a constant $K_M > 0$ such that $\|\eta(t_*, \zeta(t_*))\| \le K_M$ for all $t_* \in [t_+,\infty)$.

It follows from the above that $\|x(t)\| = \|\zeta(t) + \gamma(t)\| \le M + K_M$ for all $t \in [t_0,\infty)$, i.e., the solution $x(t)$ is bounded on $[t_0,\infty)$ and therefore Lagrange stable. Since for every consistent initial point (t_0, x_0) (i.e., for $(t_0, x_0) \in L_{t_+}$) there exists a unique solution of the initial value problem (2.1), (2.3) which is Lagrange stable, then every solution of (2.1), (2.3) is Lagrange stable (recall that the initial value problem (2.1), (2.3) has a solution only for the initial points $(t_0, x_0) \in L_{t_+}$). Thus, the equation (2.1) is Lagrange stable. □

Remark 3.9. The condition 4.a) is a corollary of the condition 4.b), since the equation $Q_2(t)[A'(t)P_1(t)x + B(t)P_2(t)x - f(t,x)] = 0$ determining L_{t_+} can be rewritten as $G^{-1}(t)Q_2(t)[f(t, x_{p_1}(t) + x_{p_2}(t)) - A'(t)x_{p_1}(t)] = x_{p_2}(t)$ (see (2.15)). However, it can occur that $x_{p_2}(t)$ needs to be expressed from this equation in a different way to get the estimate $\|x_{p_2}(t)\| \le K_M$.

3.3. Lagrange instability.

Theorem 3.10 (Lagrange instability of the DAE (2.1))**.**

Let $f \in C([t_+,\infty) \times \mathbb{R}^n, \mathbb{R}^n)$, $\dfrac{\partial f}{\partial x} \in C([t_+,\infty) \times \mathbb{R}^n, L(\mathbb{R}^n))$, $A, B \in C^1([t_+,\infty), L(\mathbb{R}^n))$ and the pencil $\lambda A(t) + B(t)$ satisfy (2.4), where $C_2 \in C^1([t_+,\infty),(0,\infty))$. Let the requirements 1), 2) of Theorem 3.1 or 1), 2) of Theorem 3.7 be fulfilled, and

3) *there exists a region $\Omega \subset \mathbb{R}^n$ such that $0 \notin \Omega$ and the component $P_1(t)x(t)$ of each existing solution $x(t)$ with the initial point $(t_0, x_0) \in L_{t_+}$, where $P_1(t_0)x_0 \in \Omega$, remains all the time in Ω;*
4) *there exist a positive definite function $V \in C^1([t_+,\infty)\times\Omega, \mathbb{R})$ and a function $\chi \in C([t_+,\infty) \times (0,\infty), \mathbb{R})$ such that:*
   4.1) *for all $t \in [t_+,\infty)$, $x_{p_1}(t) \in X_1(t)$, $x_{p_2}(t) \in X_2(t)$ such that $(t, x_{p_1}(t) + x_{p_2}(t)) \in L_{t_+}$, $x_{p_1}(t) \in \Omega$, the inequality*
   $$V'_{(2.14)}(t, x_{p_1}(t)) \ge \chi\big(t, V(t, x_{p_1}(t))\big) \tag{3.25}$$
   holds ($V'_{(2.14)}(t, x_{p_1}(t))$ has the form (3.4)),
   4.2) *the differential inequality*
   $$v' \ge \chi(t, v), \qquad t \ge t_+, \tag{3.26}$$

has no positive solutions defined in the future (i.e., defined for all $t \geq t_+$*).*

Then for each initial point $(t_0, x_0) \in L_{t_+}$ *such that* $P_1(t_0)x_0 \in \Omega$*, there exists a unique global solution of the initial value problem* (2.1), (2.3) *and this solution is Lagrange unstable.*

Proof. It is proved in the same way as in Theorem 3.1 that there exists the unique solution $z = \zeta(t)$ of (3.14) on $[t_0, \omega)$ which satisfies the initial condition $\zeta(t_0) = P_1(t_0)x_0$, where $[t_0, \omega)$ is the maximal interval of existence. Further, as in the proof of Theorem 3.1 (see p. 9) we obtain that there exists the unique solution $x(t) = \zeta(t) + \eta(t, \zeta(t))$ of (2.1) on $[t_0, \omega)$ which satisfies (2.3). Recall that $z = \zeta(t)$ and $u = \eta(t, \zeta(t))$ ($\zeta(t) \in X_1(t)$, $\eta(t, \zeta(t)) \in X_2(t)$) are the unique solution of the system (3.10), (3.11) on $[t_0, \omega)$ which satisfies the initial conditions $\zeta(t_0) = P_1(t_0)x_0$, $\eta(t_0, \zeta(t_0)) = P_2(t_0)x_0$.

Prove that the solution $x(t)$ is Lagrange unstable, i.e., has a finite escape time ($\omega < \infty$). By the condition 3) there exists the region $\Omega \subset \mathbb{R}^n$ such that $0 \notin \Omega$ and the component $P_1(t)x(t) = x_{p_1}(t)$ of each existing solution $x(t)$ with the initial point $(t_0, x_0) \in L_{t_+}$, where $P_1(t_0)x_0 \in \Omega$, remains all the time in Ω. Since $\zeta(t) = P_1(t)x(t)$, each solution $\zeta(t)$ of the equation (3.14) starting in Ω remains all the time in it. By virtue of the condition 4), for all $t \geq t_0$, $\zeta(t) \in \Omega$ the inequality

$$V'_{(3.14)}(t, \zeta(t)) \geq \chi\big(t, V(t, \zeta(t))\big)$$

holds. Therefore, for $t \geq t_0$ the function $v(t) = V(t, \zeta(t))$ is a positive solution of (3.26). Since, by the condition 4), the inequality (3.26) has no positive solution defined in the future, then, as in the proof the theorem [14, p. 109, Theorem XIV], we obtain that the solution $\zeta(t)$ has a finite escape time, i.e., $\omega < \infty$ and $\lim\limits_{t \to \omega - 0} \|\zeta(t)\| = \infty$. Hence, the solution $x(t) = \zeta(t) + \eta(t, \zeta(t))$ of the initial value problem (2.1), (2.3) also has the finite escape time $[t_0, \omega)$. □

The Lagrange instability theorem gives conditions under which the DAE has no global solutions (for consistent initial points (t_0, x_0), where the component $P_1(t_0)x_0$ from a certain region Ω).

Note that the Lagrange instability of a solution implies its Lyapunov instability, but in general the Lyapunov instability of a solution does not imply its Lagrange instability.

4. Dissipativity or ultimate boundedness of the DAE

Definition 4.1. Solutions of the equation (2.1) are called *ultimately bounded*, if there exists a constant $K > 0$ (the constant is independent of the choice of a solution, i.e. the choice of t_0, x_0) and for each solution $x(t)$ with an initial point (t_0, x_0) there exists a number $\tau = \tau(t_0, x_0) \geq t_0$ such that $\|x(t)\| < K$ for all $t \in [t_0 + \tau, \infty)$.

The equation (2.1) is called *ultimately bounded* or *dissipative*, if for any consistent initial point (t_0, x_0) there exists a global solution of the initial value problem (2.1), (2.3) and all solutions are ultimately bounded.

Definition 4.2. If in Definition 4.1 the number τ does not depend on the choice of t_0, i.e., $\tau = \tau(x_0)$, then the solutions of the equation (2.1) are called *uniformly ultimately bounded* and, accordingly, the equation (2.1) is called *uniformly ultimately bounded* or *uniformly dissipative.*

Analogous definitions hold for the DAE (2.2).

Theorem 4.3 (uniform ultimate boundedness (dissipativity) of the DAE (2.1))**.** *Let* $f \in C([t_+,\infty)\times\mathbb{R}^n,\mathbb{R}^n)$, $\dfrac{\partial f}{\partial x} \in C([t_+,\infty)\times\mathbb{R}^n,L(\mathbb{R}^n))$, $A, B \in C^1([t_+,\infty),L(\mathbb{R}^n))$, *the pencil* $\lambda A(t)+B(t)$ *satisfy* (2.4), *where* $C_2 \in C^1([t_+,\infty),(0,\infty))$, *and the requirements 1), 2) of Theorem 3.1 or 1), 2) of Theorem 3.7 be fulfilled. Let the following conditions be also fulfilled:*

3) *there exist a number* $R>0$, *a positive definite function* $V\in C^1([t_+,\infty)\times U_R^c(0),\mathbb{R})$ ($U_R^c(0)=\{z\in\mathbb{R}^n \mid \|z\|\ge R\}$) *and functions* $U_j\in C([0,\infty))$, $j=0,1,2$, *such that* $U_0(r)$ *is non-decreasing and* $U_0(r)\to+\infty$ *as* $r\to+\infty$, $U_1(r)$ *is increasing,* $U_2(r)>0$ *for* $r>0$, *and for all* $t\in[t_+,\infty)$, $x_{p_1}(t)\in X_1(t)$, $x_{p_2}(t)\in X_2(t)$ *such that* $(t,x_{p_1}(t)+x_{p_2}(t))\in L_{t_+}$, $\|x_{p_1}(t)\|\ge R$ *the condition* $U_0(\|x_{p_1}(t)\|)\le V(t,x_{p_1}(t))\le U_1(\|x_{p_1}(t)\|)$ *and one of the following inequalities hold:*
   3.a) $V'_{(2.14)}(t,x_{p_1}(t))\le -U_2\big(\|x_{p_1}(t)\|\big)$ ($V'_{(2.14)}(t,x_{p_1}(t))$ *has the form* (3.4))*;*
   3.b) $V'_{(2.14)}(t,x_{p_1}(t))\le -U_2\big((H(t)x_{p_1}(t),x_{p_1}(t))\big)$, *where* $H\in C([t_+,\infty),L(\mathbb{R}^n))$ *is some positive definite self-adjoint operator function such that* $\sup\limits_{t\in[t_+,\infty)}\|H(t)\|<\infty$*;*
   3.c) $V'_{(2.14)}(t,x_{p_1}(t))\le -C\,V(t,x_{p_1}(t))$, *where* $C>0$ *is some constant;*
4) *there exist a constant* $c>0$ *and a number* $T>t_+$ *such that* $\|G^{-1}(t)Q_2(t)[f(t,x_{p_1}(t)+x_{p_2}(t))-A'(t)x_{p_1}(t)]\|\le c\,\|x_{p_1}(t)\|$ *for all* $(t,x_{p_1}(t)+x_{p_2}(t))\in L_T$.

Then the DAE (2.1) *is uniformly ultimately bounded (uniformly dissipative).*

Proof. As in the proof of Theorem 3.1 or 3.7 we obtain that for each initial point $(t_0,x_0)\in L_{t_+}$ there exists the unique global solution $x(t)=\zeta(t)+\eta(t,\zeta(t))$ of the initial value problem (2.1), (2.3), where $\zeta(t)=P_1(t)x(t)=x_{p_1}(t)$, $\eta(t,\zeta(t))=P_2(t)x(t)=x_{p_2}(t)$. Indeed, since, By virtue of 3), the inequality $V'_{(3.14)}(t,\zeta(t))\le -U_2\big(\|\zeta(t)\|\big)$, or $V'_{(3.14)}(t,\zeta(t))\le -U_2\big((H(t)\zeta(t),\zeta(t))\big)$, or $V'_{(3.14)}(t,\zeta(t))\le -C\,V(t,\zeta(t))$ holds instead of (3.15), then the inequality $v'\le 0$ which also has no positive solutions with finite escape time holds instead of (3.5). By virtue of the condition 3) with the inequalities 3.a) and 3.c), as in the proofs of Yoshizawa's theorem [15, Theorem 10.4] and its corollary, we obtain that solutions of the equation (3.14) are uniformly ultimately bounded, i.e., there exists a constant $N>0$ and for each solution $z=\zeta(t)$ satisfying $\zeta(t_0)=P_1(t_0)x_0$, there exists a number $\tau_1=\tau_1(x_0)\ge t_0$ such that $\|\zeta(t)\|<N$ for all $t\ge t_0+\tau_1$. It is easy to verify that from the condition 3) with the inequality 3.b) it also follows that solutions of (3.14) are uniformly ultimately bounded. Note that, due to

the properties of the operator $H(t)$, there exist constants $H_0, H_1 > 0$ such that $H_0\|z\|^2 \le (H(t)z,z) \le H_1\|z\|^2$ for all $t \in [t_+,\infty)$, $z \in \mathbb{R}^n$. By virtue of the condition 4) and the equality (3.23) there exists a constant $c > 0$ and a number $\tau_2 = \tau_2(x_0) > t_0$ such that $\|\eta(t,\zeta(t))\| \le c\,\|\zeta(t)\| < c\,N$ for all $t \ge \tau_2$. Thus, for each solution with the initial point (t_0,x_0) there exists a number $\tau = \tau(x_0) \ge t_0$ such that $\|x(t)\| \le \|\zeta(t)\| + \|\eta(t,\zeta(t))\| < (1+c)N$ for all $t \in [t_0+\tau,\infty)$, where the constant $(1+c)N > 0$ is independent of t_0, x_0. Hence, the DAE (2.1) is uniformly ultimately bounded. □

5. Lyapunov stability, asymptotic stability and instability

Consider the DAEs (2.1) and (2.2), where $f(t,0) \equiv 0$. They are called DAEs *of perturbed motion* and have the equilibrium state (stationary solution) $x_*(t) \equiv 0$. Recall that $(t_0,x_0) \in L_{t_+}$ ($(t_0,x_0) \in \widehat{L}_{t_+}$) is called a consistent initial point for the initial value problem (2.1), (2.3) ((2.2), (2.3)) (see Remark 2.1). Obviously, the point $(t,0)$ belongs to L_{t_+} and $\widehat{L}_{t_+}$ for each $t \in [t_+,\infty)$ (if $f(t,0) \equiv 0$).

Let $f\colon [t_+,\infty) \times U_R^x(0) \to \mathbb{R}^n$, where $U_R^x(0) = \{x \in \mathbb{R}^n \mid \|x\| < R\}$.

Definition 5.1. The equilibrium state $x_*(t) \equiv 0$ of the DAE (2.1), where $f(t,0) \equiv 0$, is called *Lyapunov stable* or simply *stable* if for any $\varepsilon > 0$ ($\varepsilon < R$), $t_0 \in [t_+,\infty)$ there exists a number $\delta = \delta(\varepsilon,t_0) > 0$ ($\delta \le \varepsilon$) such that for any consistent initial point (t_0,x_0) satisfying the condition $\|x_0\| < \delta$ there exists a global solution $x(t)$ of the initial value problem (2.1), (2.3) and this solution satisfies the inequality $\|x(t)\| < \varepsilon$ for all $t \in [t_0,\infty)$. If, in addition, there exists $\widetilde{\delta} = \widetilde{\delta}(t_0) > 0$ ($\widetilde{\delta} \le \delta$) such that for each solution $x(t)$ with an initial point (t_0,x_0) satisfying the condition $\|x_0\| < \widetilde{\delta}$ the requirement $\lim\limits_{t\to\infty} x(t) = 0$ is satisfied, then the equilibrium state $x_*(t) \equiv 0$ is called *asymptotically Lyapunov stable* or simply *asymptotically stable*.

If in Definition 5.1 δ is independent of t_0, i.e., $\delta = \delta(\varepsilon)$, then the equilibrium state is called *uniformly Lyapunov stable* or *uniformly stable* (on $[t_+,\infty)$).

Definition 5.2. The equilibrium state $x_*(t) \equiv 0$ of the DAE (2.1), where $f(t,0) \equiv 0$, is called *Lyapunov unstable* or simply *unstable* if for some $\varepsilon > 0$ ($\varepsilon < R$), $t_0 \in [t_+,\infty)$ and any $\delta > 0$ there exist a solution $x_\delta(t)$ of the initial value problem (2.1), (2.3) and a time moment $t_1 > t_0$ such that $\|x_0\| < \delta$ and $\|x_\delta(t_1)\| \ge \varepsilon$.

Definition 5.3. Consider the DAE (2.1), where $f(t,x)$ is defined on $[t_+,\infty) \times \mathbb{R}^n$ and $f(t,0) \equiv 0$. If the equilibrium state $x_*(t) \equiv 0$ of the DAE is asymptotically stable and, moreover, for each point $(t_0,x_0) \in L_{t_+}$ (i.e. for each consistent initial point) there exists a global solution $x(t)$ of the initial value problem (2.1), (2.3) and $\lim\limits_{t\to\infty} x(t) = 0$, then the equilibrium state $x_*(t) \equiv 0$ is called *asymptotically stable in the large*, and the DAE is called *completely stable* or *asymptotically stable*.

Similar definitions hold for the DAE (2.2) ($f(t,0) \equiv 0$).

Theorem 5.4 (Lyapunov stability and asymptotic stability of the equilibrium state of the DAE (2.1))**.**

Let $f \in C([t_+,\infty) \times U_R^x(0),\mathbb{R}^n)$ $(U_R^x(0) = \{x \in \mathbb{R}^n \mid \|x\| < R\})$, $f(t,0) \equiv 0$, $\dfrac{\partial f}{\partial x} \in C([t_+,\infty) \times U_R^x(0), L(\mathbb{R}^n))$, $A, B \in C^1([t_+,\infty), L(\mathbb{R}^n))$ *and the pencil* $\lambda A(t) + B(t)$ *satisfy* (2.4), *where* $C_2 \in C^1([t_+,\infty),(0,\infty))$. *Let for each* $t_* \in [t_+,\infty)$ *and* $x_{p_1}^*(t_*) = 0$, $x_{p_2}^*(t_*) = 0$ *the operator* (3.2) *has the inverse. Then the following statements are true:*

1. *Let there exist numbers* $r_1, r_2 > 0$, $r_1 + r_2 < R$, *and a positive definite function* $V \in C^1([t_+,\infty) \times B_{r_1}(0),\mathbb{R})$, *where* $B_{r_1}(0) = \{z \in \mathbb{R}^n \mid \|z\| \le r_1\}$, *such that for all* $t \in [t_+,\infty)$ *and* $x \in B_{r_1,r_2}^{x_{p_1},x_{p_2}}(0) = \{x \in \mathbb{R}^n \mid \|x_{p_i}(t)\| \le r_i,\ x_{p_i}(t) = P_i(t)x,\ i = 1,2\}$ *the inequality*
$$V'_{(2.14)}(t, x_{p_1}(t)) \le 0 \tag{5.1}$$
holds ($V'_{(2.14)}(t, x_{p_1}(t))$ *has the form* (3.4)).
Then the equilibrium state $x_*(t) \equiv 0$ *of the DAE* (2.1) *is Lyapunov stable.*
2. *Let there exist numbers* $r_1, r_2 > 0$, $r_1 + r_2 < R$, *and positive definite functions* $V \in C^1([t_+,\infty) \times B_{r_1}(0),\mathbb{R})$, $W \in C(B_{r_1}(0),\mathbb{R})$, $U \in C(B_{r_1}(0),\mathbb{R})$ *such that* $V(t,z) \le W(z)$ *for all* $t \in [t_+,\infty)$, $z \in B_{r_1}(0)$, *and for all* $t \in [t_+,\infty)$, $x \in B_{r_1,r_2}^{x_{p_1},x_{p_2}}(0)$, $x_{p_1}(t) \ne 0$, *the following inequality holds:*
$$V'_{(2.14)}(t, x_{p_1}(t)) \le -U\big(x_{p_1}(t)\big). \tag{5.2}$$
Let for some $T > t_+$,
$$G^{-1}(t)Q_2(t)[f(t, P_1(t)x + P_2(t)x) - A'(t)P_1(t)x] \to 0 \text{ uniformly in } t \text{ on } [T,\infty) \text{ as } x \to 0. \tag{5.3}$$
Then the equilibrium state $x_*(t) \equiv 0$ *of the DAE* (2.1) *is asymptotically stable.*

Proof. The proof of the statement 1.

Recall that the DAE (2.1) is equivalent to the system (2.12), (2.13) or (2.14), (2.15). Consider the mappings (3.6), (3.7) and the system (3.10), (3.11). Obviously, $f(t,0) \equiv 0$ if and only if $\Pi(t,0,0) \equiv 0$ and $F(t,0,0) \equiv 0$. We will assume that $A(t)$ is not equal to zero or invertible for all t, because in the case when $A(t)$ is invertible (for all t), the DAE (2.1) can be reduced to an explicit ODE and then the classical Lyapunov theorems can be used, and in the case when $A(t)$ is identically equal to zero, the DAE becomes an algebraic equation (in the sense that it does not contain a derivative) and the implicit function theorems as well as the constructions similar to those given below are used. Thus, the theorem remains true for these special cases, but the proof of the theorem is of interest precisely to the DAE (degenerate DE). Therefore, in what follows, it is assumed that $X_1(t) \ne \{0\}$ and $X_2(t) \ne \{0\}$. Recall that the dimensions of the subspaces $X_1(t)$ and $X_2(t)$ are constant for all $t \in [t_+,\infty)$ (see Remark 3.6).

It is clear that there exist some regions D^z, $D^u \subset \mathbb{R}^n$ containing the origin for which the mappings Π, F are defined, i.e., $P_1(t)D^z + P_2(t)D^u \subset U_R^x(0)$ and $\Pi(t,z,u)\colon [t_+,\infty) \times D^z \times D^u \to \mathbb{R}^n$, $F(t,z,u)\colon [t_+,\infty) \times D^z \times D^u \to \mathbb{R}^n$. The mappings $\Pi, F \in C([t_+,\infty) \times D^z \times D^u,\mathbb{R}^n)$ are continuously differentiable in z, u and the partial derivatives of $F(t,z,u)$ have the form (3.8), (3.9), where $\Phi_{t,P_1(t)z,P_2(t)u}$ is the operator (3.2). Denote $\tilde{\Phi}_{t,z,u} = \Phi_{t,P_1(t)z,P_2(t)u}$ as in Theorem 3.1.

Clearly, Lemma 3.2 remains valid. Note that if $u(t) \in \mathbb{R}^n$ satisfy (3.11), i.e., $F(t, z(t), u(t)) = 0$, then $u(t) \in X_2(t)$.

By the theorem condition, for each $t_* \in [t_+, \infty)$ the operator $\tilde{\Phi}_{t_*,0,0} = \Phi_{t_*,0,0}$ is invertible. Therefore, for each point $(t, z, u) = (t_*, 0, 0)$ the operator $\Psi_{t,z,u} = \frac{\partial}{\partial u} F(t, z, u)$ (3.13) is invertible. Let $t_* \in [t_+, \infty)$ be an arbitrary fixed element. Since $F(t_*, 0, 0) = 0$ and the conditions of the implicit function theorems are satisfied, then there exist neighborhoods $U_{\sigma_1}(t_*) \times U^z_{\delta_1}(0) \subset [t_+, \infty) \times D^z$ $(U_{\sigma_1}(t_+) = [t_+, t_+ + \sigma_1))$, $U^u_{\gamma_1}(0) \subset D^u$ and a unique function $u = \mu(t, z) \in C(U_{\sigma_1}(t_*) \times U^z_{\delta_1}(0), U^u_{\gamma_1}(0))$ which is continuously differentiable in z on $U_{\sigma_1}(t_*) \times U^z_{\delta_1}(0)$, satisfies the equation (3.12) (i.e., $F(t, z, \mu(t, z)) = 0$ for $(t, z) \in U_{\sigma_1}(t_*) \times U^z_{\delta_1}(0)$) and $\mu(t_*, 0) = 0$. Since $u = \mu(t, z)$ satisfies (3.12) for $(t, z) \in U_{\sigma_1}(t_*) \times U^z_{\delta_1}(0)$, then $\mu(t, z) \in X_2(t)$ and $(t, P_1(t)z + \mu(t, z)) \in L_{t_+}$ for each $(t, z) \in U_{\sigma_1}(t_*) \times U^z_{\delta_1}(0)$. Thus, the following statement similar to Lemma 3.3 is proved: For each $t \in [t_+, \infty)$ and each z from the sufficiently small neighborhood $U^z_{\delta_1}(0)$ there exists a unique u from the sufficiently small neighborhood $U^u_{\gamma_1}(0)$, satisfying (3.12). Since the obtained implicit function $u = \mu(t, z)$ is continuous at the point $(t_*, 0)$, then for every $\varepsilon_1 > 0$ there are $\tilde{\sigma}_1 = \tilde{\sigma}_1(\varepsilon_1, t_*) > 0$ $(\tilde{\sigma}_1 \le \sigma_1)$, $\tilde{\delta}_1 = \tilde{\delta}_1(\varepsilon_1, t_*) > 0$ $(\tilde{\delta}_1 \le \delta_1)$ such that $\|\mu(t, z)\| < \varepsilon_1$ for $(t, z) \in U_{\tilde{\sigma}_1}(t_*) \times U^z_{\tilde{\delta}_1}(0)$ and therefore $\|u\| < \varepsilon_1$ for $u = \mu(t, z)$. Thus, the following lemma is proved.

Lemma 5.5. *For any $\varepsilon_u > 0$, $t \in [t_+, \infty)$ and any $z \in U^z_{\delta_*}(0)$, where $\delta_* > 0$ is sufficiently small, there exists a unique $u \in U^u_{\varepsilon_u}(0)$ satisfying* (3.12) *and this u belongs to $X_2(t)$ (i.e., $\|u\| < \varepsilon_u$, $F(t, z, u) = 0$ and $u = P_2(t)u$).*

Let $\varepsilon > 0$ $(\varepsilon < R)$ is an arbitrary number. We represent it as the sum $\varepsilon = \varepsilon_z + \varepsilon_u$ of numbers $\varepsilon_z > 0$, $\varepsilon_u > 0$ which will be indicated below.

Using the implicit function theorems and Lemma 5.5, we obtain the following statement. For any fixed $t_* \in [t_0, \infty)$ there exist an interval $U_{\sigma_2}(t_*) \subset [t_+, \infty)$ $(\sigma_2 = \sigma_2(\varepsilon_u, t_*), U_{\sigma_2}(t_+) = [t_+, t_+ + \sigma_2))$, a neighborhood $U^z_{\delta_2}(0)$ $(\delta_2 = \delta_2(\varepsilon_u, t_*) \le \varepsilon_z)$ and a unique function $\nu_{t_*}(t, z) \in C(U_{\sigma_2}(t_*) \times U^z_{\delta_2}(0), U^u_{\varepsilon_u}(0))$ which is a solution of the equation (3.12) with respect to u (i.e., $F(t, z, \nu_{t_*}(t, z)) = 0$ for $(t, z) \in U_{\sigma_2}(t_*) \times U^z_{\delta_2}(0)$), is continuously differentiable in z and belongs to $X_2(t)$ for each $(t, z) \in U_{\sigma_2}(t_*) \times U^z_{\delta_2}(0)$, as well as satisfies the equality $\nu_{t_*}(t_*, 0) = 0$. Introduce the function $u = \eta(t, z)\colon [t_+, \infty) \times U^z_{\delta_2}(0) \to U^u_{\varepsilon_u}(0)$ and define by $\eta(t, z) = \nu_{t_*}(t, z)$ at the point $(t, z) = (t_*, z_*)$ for each point $(t_*, z_*) \in [t_+, \infty) \times U^z_{\delta_2}(0)$. Then the function $u = \eta(t, z)$ is continuous in (t, z), continuously differentiable in z, a unique solution of the equation (3.12) and $\eta(t, z) \in X_2(t)$ for each $(t, z) \in [t_+, \infty) \times U^z_{\delta_2}(0)$. Obviously, $\eta(t, 0) \equiv 0$.

Substitute the introduced function $u = \eta(t, z)$ in (3.6) and denote $\widetilde{\Pi}(t, z) = \Pi(t, z, \eta(t, z))$. Then the equation (3.10) takes the form (3.14). By the properties of η and Π, the function $\widetilde{\Pi}$ is continuous in (t, z) and continuously differentiable in z on $[t_+, \infty) \times U^z_{\delta_2}(0)$, and $\widetilde{\Pi}(t, 0) \equiv 0$. Clearly, for each initial point $(t_0, z_0) \in [t_+, \infty) \times U^z_{\delta_2}(0)$ there exists a unique local solution of (3.14).

Take any initial value $t_0 \in [t_+, \infty)$ and choose any consistent initial value x_0, i.e., $(t_0, x_0) \in L_{t_+}$ $(F(t_0, P_1(t_0)x_0, P_2(t_0)x_0) = 0)$, satisfying the condition $\|x_0\| < \delta \le \varepsilon$, where $\delta = \delta(\varepsilon, t_0) > 0$ is chosen so that $\|P_1(t_0)x_0\| < \delta_z \le$

$\min\{\varepsilon_z, \delta_2\}$, δ_z is a sufficiently small number which will be determined below, and $\|P_2(t_0)x_0\| < \varepsilon_u$. Denote $z_0 = P_1(t_0)x_0$ and $u_0 = P_2(t_0)x_0$. Then $\eta(t_0, z_0) = u_0$ since $F(t_0, z_0, u_0) = 0$. For the chosen initial point (t_0, z_0) there exists a unique local solution $z = \zeta(t)$ of (3.14) satisfying the initial condition $\zeta(t_0) = z_0$. Then the functions $z = \zeta(t)$, $u = \eta(t, \zeta(t))$ are a unique local solution of the system (3.10), (3.11) satisfying the initial conditions $\zeta(t_0) = z_0$, $\eta(t_0, \zeta(t_0)) = u_0$, and by Lemma 3.2 the function $x(t) = \zeta(t)+\eta(t, \zeta(t))$ $(\zeta(t) = P_1(t)x(t) = x_{p_1}(t) \in X_1(t)$, $\eta(t, \zeta(t)) = P_2(t)x(t) = x_{p_2}(t) \in X_2(t))$ is a unique local solution of (2.1) satisfying the initial condition (2.3), where $x_0 = z_0 + u_0$.

Without loss of generality, we can assume that $\delta_2 \le r_1$ and $\varepsilon_u \le r_2$, where the numbers r_1, r_2 are defined in the statement 1. By virtue of (5.1), for any $t \in [t_0, \infty)$ and $z \in X_1(t)$ such that $\|z\| < \delta_2$ the derivative of V along the trajectories of (3.14) (see (3.15)) satisfy the inequality

$$V'_{(3.14)}(t, z) \le 0. \tag{5.4}$$

Recall that $\|z_0\| < \delta_z \le \min\{\varepsilon_z, \delta_2\}$, where $z_0 = P_1(t_0)x_0 = \zeta(t_0)$. As in the proof of the classical Lyapunov theorem on stability, we obtain that the number $\delta_z = \delta_z(\varepsilon_z, t_0) > 0$ can be chosen such that the solution $z = \zeta(t)$ has an extension to $[t_0, \infty)$ (i.e., is global) and $\|\zeta(t)\| < \varepsilon_z$ for all $t \in [t_0, \infty)$. This holds for any $\varepsilon_z > 0$. Choose δ_z, ε_z and ε_u such that $\varepsilon_z + \varepsilon_u = \varepsilon$, $\|\zeta(t)\| < \varepsilon_z$ for $t \in [t_0, \infty)$ and $\|\eta(t, \zeta(t))\| < \varepsilon_u$ for $\|\zeta(t)\| < \varepsilon_z$, $t \in [t_0, \infty)$. Then $\|x(t)\| = \|\zeta(t) + \eta(t, \zeta(t))\| < \varepsilon_z + \varepsilon_u = \varepsilon$ for all $t \in [t_0, \infty)$. Since $\varepsilon > 0$ and $t_0 \in [t_+, \infty)$ were chosen arbitrarily, the statement 1 is proved.

The proof of the statement 2.

The Lyapunov stability of the zero solution is proved in the same way as above. We show that the solution $x(t) = \zeta(t) + \eta(t, \zeta(t))$ with the initial point (t_0, x_0), $x_0 = z_0 + u_0$, constructed in the proof of the statement 1, satisfies the requirement $\lim\limits_{t\to\infty} x(t) = 0$ for $\|x_0\| < \delta$ and a sufficiently small $\delta = \delta(t_0) > 0$. As above, δ is chosen so that $\|z_0\| = \|P_1(t_0)x_0\| < \delta_z$, where δ_z is sufficiently small number which will be defined below. The mentioned δ and δ_z are different from those chosen in the proof of the statement 1, but for convenience we retain the previous notation.

Since, by the condition of the statement 2, there exists $W \in C(B_{r_1}(0), \mathbb{R})$ such that $W(0) = 0$ and $0 \le V(t, z) \le W(z)$ for all $t \in [t_+, \infty)$, $z \in B_{r_1}(0)$, then $V(t, z)$ has an infinitely small upper limit in $B_{r_1}(0)$ (see the definition in [16, p. 11, Def. 1.7]). Since, by virtue of (5.2), the inequality $V'_{(3.14)}(t, z) \le -U(z)$, the scalar function $U(z)$ is continuous and positive definite, holds instead of (5.4), then, as in the proof of the classical Lyapunov theorem on asymptotic stability, we obtain that the number $\delta_z = \delta_z(t_0) > 0$ can be chosen such that $\lim\limits_{t\to\infty} \zeta(t) = 0$. Then, taking into account the condition (5.3) and the equalities (3.23) and $\eta(t, 0) \equiv 0$, we obtain that $\lim\limits_{t\to\infty} \eta(t, \zeta(t)) = 0$. Consequently, $\lim\limits_{t\to\infty} x(t) = 0$, and the statement 2 is proved.

Theorem 5.6 (Complete stability of the DAE (2.1) (asymptotic stability in the large))**.**

Let $f \in C([t_+,\infty)\times\mathbb{R}^n,\mathbb{R}^n)$, $f(t,0)\equiv 0$, $\dfrac{\partial f}{\partial x} \in C([t_+,\infty)\times\mathbb{R}^n, L(\mathbb{R}^n))$, $A, B \in C^1([t_+,\infty), L(\mathbb{R}^n))$ *and the pencil* $\lambda A(t) + B(t)$ *satisfy* (2.4), *where* $C_2 \in C^1([t_+,\infty),(0,\infty))$. *Let the requirements 1), 2) of Theorem 3.1 or 1), 2) of Theorem 3.7 be satisfied. Let also* (5.3) *hold, and there exist positive definite functions* $V \in C^1([t_+,\infty)\times\mathbb{R}^n,\mathbb{R})$ *and* $W \in C(\mathbb{R}^n,\mathbb{R})$, $U \in C(\mathbb{R}^n,\mathbb{R})$ *such that:*

1) $V(t,z) \le W(z)$ *for all* $t \in [t_+,\infty)$, $z \in \mathbb{R}^n$*;*
2) $V(t,z) \to \infty$ *uniformly in* t *on* $[t_+,\infty)$ *as* $\|z\| \to \infty$*;*
3) *for all* $(t, x_{p_1}(t) + x_{p_2}(t)) \in L_{t_+}$, $x_{p_1}(t) \ne 0$ $(x_{p_i}(t) = P_i(t)x,\ i = 1,2)$, *the inequality* (5.2) *holds.*

Then the equilibrium state $x_*(t) \equiv 0$ *of the DAE* (2.1) *is asymptotically stable in the large* (*the DAE is completely stable*).

Proof. Since the theorem conditions include the conditions of the statement 2 of Theorem 5.4, the equilibrium state is asymptotically stable. As in the proof of Theorem 3.1 or 3.7, where $v' \le 0$ holds instead of (3.5), we obtain that for each consistent initial point (t_0, x_0) there exists the unique global solution $x(t) = \zeta(t) + \eta(t,\zeta(t))$ of the initial value problem (2.1), (2.3), where $\zeta(t) = P_1(t)x(t) = x_{p_1}(t)$, $\eta(t,\zeta(t)) = P_2(t)x(t) = x_{p_2}(t)$. Prove that $\lim\limits_{t\to\infty} x(t) = 0$.

Since $f(t,0)\equiv 0$, then, as in Theorem 5.4, $\eta(t,0) \equiv 0$. Note that $V'_{(3.14)}(t,\zeta(t)) \le -U(\zeta(t))$, where $U(z)$ is continuous and positive definite, holds for $t \ge t_0$, $\zeta(t) \ne 0$ (since (5.2)), and $V'_{(3.14)}(t,0)\equiv 0$. From the properties of the function $V(t,z)$ and $W(z)$ it follows that $V(t,z)$ has an infinitely small upper limit in $\mathbb{R}^n$ (see the definition [16, p. 11, Def. 1.7]). Taking into account the properties of $V(t,z)$, $W(z)$ and $U(z)$, as in the proof of the Barbashin-Krasovsky theorem on asymptotic stability in the large [16, p. 36, Theorem 5.2], we obtain that $\lim\limits_{t\to\infty}\zeta(t) = 0$. Then, as in the proof of the statement 2 of Theorem 5.4, we obtain that $\lim\limits_{t\to\infty}\eta(t,\zeta(t)) = 0$. Therefore, $\lim\limits_{t\to\infty} x(t) = 0$, and the theorem is proved. □

Notice that in general for a semilinear DAE (of unperturbed motion), as well as in the ODE case, the Lyapunov stability of a non-stationary solution does not imply its Lagrange stability. Also, in general, the Lagrange stability of a solution of a semilinear DAE does not imply its Lyapunov stability.

Theorem 5.7 (Lyapunov instability of the equilibrium state of the DAE (2.1))**.**

Let $f \in C([t_+,\infty)\times U_R^x(0),\mathbb{R}^n)$ $(U_R^x(0) = \{x \in \mathbb{R}^n \mid \|x\| < R\})$, $f(t,0)\equiv 0$, $\dfrac{\partial f}{\partial x} \in C([t_+,\infty)\times U_R^x(0), L(\mathbb{R}^n))$, $A, B \in C^1([t_+,\infty), L(\mathbb{R}^n))$ *and the pencil* $\lambda A(t) + B(t)$ *satisfy* (2.4), *where* $C_2 \in C^1([t_+,\infty),(0,\infty))$. *Let for each* $t_* \in [t_+,\infty)$ *the operator* (3.2), *where* $x^*_{p_1}(t_*) = 0$ *and* $x^*_{p_2}(t_*) = 0$, *has the inverse. Let there exist numbers* $T \ge t_+$ *and* $r_1, r_2 > 0$, $r_1 + r_2 < R$, *and a function* $V \in C^1([T,\infty)\times B_{r_1}(0),\mathbb{R})$ $(B_{r_1}(0) = \{z \in \mathbb{R}^n \mid \|z\| \le r_1\})$ *such that:*

1) $V(t,z) \to 0$ *uniformly in* t *on* $[T,\infty)$ *as* $\|z\| \to 0$*;*

2) *there exists a positive function* $U \in C(B_{r_1}(0),[0,\infty))$ *such that* $V'_{(2.14)}(t,x_{p_1}(t)) \ge U\big(x_{p_1}(t)\big) > 0$ *or* $V'_{(2.14)}(t,x_{p_1}(t)) \le -U\big(x_{p_1}(t)\big) < 0$ *for all* $t \in [T,\infty)$, $x \in B^{x_{p_1},x_{p_2}}_{r_1,r_2}(0)$, $x_{p_1}(t) \ne 0$ ($V'_{(2.14)}(t,x_{p_1}(t))$ *has the form* (3.4), $B^{x_{p_1},x_{p_2}}_{r_1,r_2}(0) = \{x \in \mathbb{R}^n \mid \|x_{p_i}(t)\| \le r_i,\ x_{p_i}(t) = P_i(t)x,\ i = 1,2\}$).
3) *for any* $\Delta_1, \Delta_2 > 0$, $\Delta_i \le r_i$, *there exist* $x_{p_1}(T) \ne 0$, $x_{p_2}(T)$ *such that* $\|x_{p_i}(T)\| < \Delta_i$, $i = 1,2$, *and* $V(T,x_{p_1}(T))\, V'_{(2.14)}(T,x_{p_1}(T)) > 0$ (*i.e., the sign of* V *coincides with the sign of* $V'_{(2.14)}$ *at the point* $(T,x_{p_1}(T))$).

Then the equilibrium state $x_*(t) \equiv 0$ *of the DAE* (2.1) *is Lyapunov unstable.*

Proof. Let $\varepsilon_u > 0$ be an arbitrary number satisfying $\varepsilon_u \le r_2$, where r_2 is defined in the theorem conditions. As in the proof of the statement 1 of Theorem 5.4 (where $\varepsilon_z = r_1$) we construct the function $\eta(t,z) \in C([t_+,\infty) \times U^z_{\delta_2}(0), U^u_{\varepsilon_u}(0))$, where $0 < \delta_2 \le r_1$ (r_1 is defined in the conditions of the present theorem) such that $u = \eta(t,z)$ is continuously differentiable in z, belongs to $X_2(t)$ for each $(t,z) \in [t_+,\infty) \times U^z_{\delta_2}(0)$, satisfy $\eta(t,0) \equiv 0$ and is a unique solution of (3.12). Substituting the obtained function $u = \eta(t,z)$ in (3.6) and denoting $\widetilde{\Pi}(t,z) = \Pi(t,z,\eta(t,z))$, we get the equation (3.14). By the properties of $\widetilde{\Pi}$, for each initial point $(t_0,z_0) \in [t_+,\infty) \times U^z_{\delta_2}(0)$ there exists a unique local solution of this equation.

As in the proof of the statement 1 of Theorem 5.4 we obtain that for any consistent initial point (t_0,x_0) satisfying the condition $\|x_0\| < \Delta$, where $\Delta = \delta_2 + \varepsilon_u > 0$ is chosen so that $\|P_1(t_0)x_0\| < \delta_2$ and $\|P_2(t_0)x_0\| < \varepsilon_u$, there exists a unique local solution $z = \zeta(t)$, $u = \eta(t,\zeta(t))$ of the system (3.10), (3.11) satisfying the initial conditions $\zeta(t_0) = z_0 = P_1(t_0)x_0$, $\eta(t_0,\zeta(t_0)) = u_0 = P_2(t_0)x_0$. Then by Lemma 3.2 the function $x(t) = \zeta(t) + \eta(t,\zeta(t))$ ($\zeta(t) = P_1(t)x(t) = x_{p_1}(t)$, $\eta(t,\zeta(t)) = P_2(t)x(t) = x_{p_2}(t)$) is a unique local solution of (2.1) satisfying the initial condition (2.3), where $x_0 = z_0 + u_0$.

It follows from the condition 1) that for some numbers $M > 0$ and $\delta_2' > 0$ the inequality $|V(t,z)| < M$ holds for all $t \in [T,\infty)$, $\|z\| \le \delta_2' < \delta_2$. Let $\delta_z > 0$, $\delta_u > 0$ ($\delta_z < \delta_2'$, $\delta_u < \varepsilon_u$) be arbitrary (arbitrarily small) numbers. Take the initial value $t_0 = T$, where T satisfy the theorem conditions. Assume that in the condition 2) $V'_{(2.14)}(t,x_{p_1}(t)) \ge U\big(x_{p_1}(t)\big) > 0$. Then, by the condition 3), one can always find a consistent initial value x_0 (i.e., $(t_0,x_0) \in L_{t_+}$) satisfying the conditions $\|x_0\| < \delta = \delta_z + \delta_u$, $0 < \|P_1(t_0)x_0\| < \delta_z$ and $\|P_2(t_0)x_0\| < \delta_u$, such that $V(t_0,P_1(t_0)x_0) = m > 0$, where m is some number. Therefore, as in the proof of the classical Lyapunov theorem on instability, we obtain that for the solution $z = \zeta(t)$ of (3.14) satisfying the initial condition $\zeta(t_0) = z_0 = P_1(t_0)x_0$, where $t_0 = T$, $0 < \|z_0\| < \delta_z$, there exists $t_1 > t_0$ such that $\|\zeta(t_1)\| > \delta_2'$. Hence, for the corresponding solution $x(t) = \zeta(t) + \eta(t,\zeta(t))$ with the initial point (t_0,x_0) the inequalities $\|x_0\| < \delta$ and $\|x(t_1)\| > \varepsilon = \delta_2'/\|P_1(t_1)\| > 0$ hold (since $\|\zeta(t_1)\| = \|P_1(t_1)x(t_1)\|$). This proves the theorem. □

Remark 5.8. Since the Lagrange instability of a solution implies its Lyapunov instability, *the theorems on the Lagrange instability of DAEs can also be considered as the theorems on the Lyapunov instability.*

6. Remarks on the application of the proved theorems, and changes in the conditions of the theorems for the DAE (2.2)

6.1. Changes in the theorem conditions for the equation (2.2)**.** To obtain the theorems for the DAE (2.2), it is necessary to make the following changes to the formulations of the corresponding theorems for the DAE (2.1):

- the manifold L_{t_+} is replaced by $\widehat{L}_{t_+}$, it is additionally assumed that $f(t,x)$ is continuously differentiable in t, and the derivative $V'_{(2.14)}(t,x_{p_1}(t))$ is replaced by

$$V'_{(2.18)}(t,x_{p_1}(t)) = \frac{\partial V}{\partial t}(t,x_{p_1}(t)) + \Bigg(\frac{\partial V}{\partial z}(t,x_{p_1}(t)),\, G^{-1}(t)[-B(t)x_{p_1}(t)+ Q_1(t)f(t,x_{p_1}(t)+x_{p_2}(t))] + P_1'(t)[x_{p_1}(t)+x_{p_2}(t)]\Bigg)$$

  everywhere;
- in the condition 4.a) of Theorem 3.8, the inequality $\|G^{-1}(t)Q_2(t)[f(t,x_{p_1}(t)+x_{p_2}(t)) - A'(t)x_{p_1}(t)]\| \le K_M$ is replaced by $\|G^{-1}(t)Q_2(t)f(t,x_{p_1}(t)+x_{p_2}(t))\| \le K_M$;
- the requirement (3.21) of Theorem 3.8 is replaced by $\sup\limits_{t\ge t_+,\, \|x_{p_1}(t)\|\le M<\infty,\, M=const} \|G^{-1}(t)Q_2(t)f(t,x_{p_1}(t)+\tilde{x}_{p_2}(t_*))\| < \infty$;
- in the condition 4) of Theorem 4.3, the inequality $\|G^{-1}(t)Q_2(t)[f(t,x_{p_1}(t)+x_{p_2}(t))-A'(t)x_{p_1}(t)]\| \le c\,\|x_{p_1}(t)\|$ is replaced by $\|G^{-1}(t)Q_2(t)f(t,x_{p_1}(t)+x_{p_2}(t))\| \le c\,\|x_{p_1}(t)\|$, and L_T is replaced by $\widehat{L}_T$;
- in the condition (5.3) of Theorems 5.4, 5.6, the limit $G^{-1}(t)Q_2(t)[f(t,P_1(t)x + P_2(t)x) - A'(t)P_1(t)x] \to 0$ is replaced by $G^{-1}(t)Q_2(t)f(t,P_1(t)x+P_2(t)x) \to 0$.

The proofs of the theorems for (2.2) are carried out in the same way as the proofs of the corresponding theorems for (2.1).

6.2. Remarks on the form of the functions χ and V. The main difficulty in applying the obtained theorems lies in constructing suitable functions χ, V and then in proving that these functions satisfy the theorem conditions.

It is usually convenient to choose the function $\chi \in C([t_+,\infty)\times(0,\infty),\mathbb{R})$, which is present in Theorems 3.1–3.10 on the global solvability and Lagrange stability and instability of the DAE, in the form

$$\chi(t,v) = k(t)U(v),$$

where $U \in C(0,\infty)$ (i.e., $U \in C((0,\infty),\mathbb{R})$ is a positive function) and $k \in C([t_+,\infty),\mathbb{R})$. Then the inequalities (3.3) and (3.25) take the form $V'_{(2.14)}(t,x_{p_1}(t)) \le k(t)\,U\big(V(t,x_{p_1}(t))\big)$ and $V'_{(2.14)}(t,x_{p_1}(t)) \ge k(t)\,U\big(V(t,x_{p_1}(t))\big)$ respectively, and the theorem conditions can be changed as follows (see explanations in Section 2):

- in Theorems 3.1, 3.7 on the global solvability, it suffices to require that $\int\limits_c^\infty \dfrac{dv}{U(v)} = \infty$ ($c>0$ is some constant) instead of the condition 3.3);

- in Theorem 3.8 on the Lagrange stability, it suffices to require that $\int\limits_c^\infty \frac{dv}{U(v)} = \infty$ and $\int\limits_{t_0}^\infty k(t)dt < \infty$ ($t_0 \ge t_+$ is some number) instead of the condition 3.3);
- in Theorem 3.10 on the Lagrange instability, it suffices to require that $\int\limits_c^\infty \frac{dv}{U(v)} < \infty$ and $\int\limits_{t_0}^\infty k(t)dt = \infty$ instead of the condition 4.2).

It is often convenient to choose the positive definite scalar function $V(t,z)$, which we will call a Lyapunov function if it satisfies the theorems on the Lyapunov stability (asymptotic stability, instability, and asymptotic stability in the large), and a Lyapunov type function if it satisfies the remaining theorems, in the form

$$V(t,z) = (H(t)z, z), \tag{6.1}$$

where $H \in C^1([t_+,\infty), L(\mathbb{R}^n))$ is a positive definite self-adjoint operator function. By the properties of the operator function $H(t)$, the function $V(t,z)$ (6.1) satisfies the conditions (except for the conditions on the derivative of the function V along the trajectories of (2.14), which remain in the theorems) of Theorems 3.1, 3.7, 3.8 and 3.10 on the global solvability, Lagrange stability and Lagrange instability, and the statement 1 on the Lyapunov stability from Theorem 5.4. If, additionally, $\sup\limits_{t\in[t_+,\infty)} \|H(t)\| < \infty$, then the function (6.1) also satisfies the conditions (except for the conditions on the derivative of the function V along the trajectories of (2.14), which remain in the theorems) of Theorems 4.3, 5.7 and 5.6 on the ultimate boundedness, Lyapunov instability and asymptotic stability in the large, and the statement 2 on the asymptotic stability from Theorem 5.4.

If the time-invariant self-adjoint operator $H \in L(\mathbb{R}^n)$ is taken in (6.1) (i.e., $V(t,z) \equiv V(z) = (Hz,z)$), it suffices to require that it be positive (in all theorems).

The derivative (3.4) of the function V (6.1) along the trajectories of (2.14) has the form

$$V'_{(2.14)}(t, x_{p_1}(t)) = \Big(H'(t)x_{p_1}(t), x_{p_1}(t)\Big) + 2\Big(H(t)x_{p_1}(t),\ \big[P_1'(t)- \\ - G^{-1}(t)Q_1(t)[A'(t)+B(t)]\big]x_{p_1}(t) + G^{-1}(t)Q_1(t)f(t, x_{p_1}(t)+x_{p_2}(t))\Big).$$

The derivative of V (6.1) along the trajectories of (2.18) has the form

$$V'_{(2.18)}(t, x_{p_1}(t)) = \Big(H'(t)x_{p_1}(t), x_{p_1}(t)\Big) + 2\Big(H(t)x_{p_1}(t), G^{-1}(t)[-B(t)x_{p_1}(t)+ \\ + Q_1(t)f(t, x_{p_1}(t)+x_{p_2}(t))] + P_1'(t)[x_{p_1}(t)+x_{p_2}(t)]\Big).$$

References

[1] Vlasenko, L. A.: *Evolution models with implicit and degenerate differential equations*, Sistemnye Tekhnologii, Dniepropetrovsk, 2006.

[2] Chistyakov, V.F. and Shcheglova, A.A.: *Selected chapters of the theory of algebraic-differential systems*, Nauka, Novosibirsk, 2003.

[3] Boyarintsev, Yu.E.: *Methods for solving continuous and discrete problems for singular systems of equations*, Nauka, Novosibirsk, 1996.

[4] Lamour, R., März, R. and Tischendorf, C.: *Differential-Algebraic Equations: A Projector Based Analysis*, Springer-Verlag Berlin, Heidelberg, 2013.
[5] Riaza, R.: *Differential-Algebraic Systems: Analytical Aspects and Circuit Applications*, World Scientific, Hackensack, NJ, 2008.
[6] Gliklikh, Yu.E.: On global in time solutions for differential-algebraic equations, *Vestnik YuUrGU. Ser. Mat. Model. Progr.*, **7**, (3) (2014), 33–39.
[7] Shcheglova A.A. and Kononov A.D.: Stability of differential-algebraic equations under uncertainty, *Differential Equations*, **54** (7) (2018) 860–869.
[8] Tuan, V. and Viet, P.V.: Stability of solutions of a quasilinear index-2 tractable DAE by the Lyapunov second method, *Ukrainian Mathematical Journal*, **56**, (10) (2004), 1574–1593.
[9] Filipkovska M.S.: Continuation of solutions of semilinear differential-algebraic equations and applications in nonlinear radiotechnics, *Visn. Kharkiv. Nats. Univ. Mat. Model. Inform. Tekh. Avt. Syst. Upr.*, **19** (1015) (2012), 306–319.
[10] Rutkas A.G. and Filipkovska M.S.:, Extension of solutions of one class of differential-algebraic equations, *Zh. Obchysl. Prykl. Mat.*, (1) (2013), 135–145.
[11] Filipkovska M.S.: Lagrange stability of semilinear differential-algebraic equations and application to nonlinear electrical circuits, *Journal of Mathematical Physics, Analysis, Geometry*, **14** (2) (2018), 169–196.
[12] Filipkovskaya M.S.: Lagrange stability and instability of nonregular semilinear differential-algebraic equations and applications, *Ukrainian Mathematical Journal*, **70** (6) (2018), 947–979.
[13] Rutkas; A.G. and Vlasenko L.A.: Existence of solutions of degenerate nonlinear differential operator equations, *Nonlinear Oscillations*, **4** (2) (2001), 252–263.
[14] La Salle J. and Lefschetz S.: *Stability by Liapunov's Direct Method with Applications*, Academic Press, New York, 1961.
[15] Yoshizawa, T.: *Stability theory by Liapunov's second method*, The Mathematical Society of Japan, Tokyo, 1966.
[16] Krasovsky, N.N.: *Some problems of the theory of stability of motion*, Fizmatgiz, Moscow, 1959.
[17] Filipkovska M.S. Two combined methods for the global solution of implicit semilinear differential equations with the use of spectral projectors and Taylor expansions, *International Journal of Computing Science and Mathematics*, DOI: 10.1504/IJCSM.2019.10025236 (in press)
[18] Daletskii, Yu.L. and Krein, M.G.: *Stability of solutions of differential equations in Banach space*, Nauka, Moscow, 1970.
[19] Schwartz L.: *Analyse Mathématique*, **I**, Hermann, Paris, 1967.
[20] Kato T.: *Perturbation theory for linear operators*, Springer, Berlin, 1966.

Maria Filipkovska: B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, Kharkiv, 61103, Ukraine;, V.N. Karazin Kharkiv National University, Kharkiv, 61022, Ukraine

Email address: `filipkovskaya@ilt.kharkov.ua`