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OPTIMAL CONTROL PROBLEM FOR ONE MATHEMATICAL MODEL OF HYDRODYNAMICS WITH RANDOM INITIAL DATA*

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ABSTRACT. The paper studies a stochastic mathematical model based on the improved modified Boussinesq equation (IMBq) with random initial data. This equation is used to describe wave propagation in shallow water with conservation of mass in layer and taking into account capillary effects, as well as to study deformation waves in thin elastic rods. The time derivative is understood in the sense of the Nelson–Gliklikh derivative. A theorem on the existence and uniqueness of a solution for an inhomogeneous equation with random initial data with zero mathematical expectations is proved. Sufficient conditions for solving the problem of optimal control in mathematical models with random initial data are found.

1. Introduction

Let $D \subset \mathbb{R}^n$ be a bounded domain with boundary $\mathcal{D}$ of class C^∞, $n \in \mathbb{N} \setminus \{1\}$, $T \in \mathbb{R}_+$. In the cylinder $D \times (0,T)$ we consider the modified Boussinesq equation

$$(\lambda - \Delta) \overset{\circ}{x}{}^{(2)} - \alpha^2 \Delta x - \Delta(x^3) = u(s,t), \tag{1.1}$$

with homogeneous Dirichlet boundary conditions

$$x(s,t) = 0, \quad (s,t) \in \mathcal{D} \times (0,T) \tag{1.2}$$

and Showalter – Sidorov initial conditions

$$(\lambda - \Delta)(x(s,0) - x_0(s)) = 0, \quad (\lambda - \Delta)(\overset{\circ}{x}(s,0) - x_1(s)) = 0, \quad s \in D, \tag{1.3}$$

where $\lambda, \alpha \in \mathbb{R}$, $\overset{\circ}{x}$ and $\overset{\circ}{x}{}^{(2)}$ are the Nelson – Gliklikh derivatives of the first and the second orders of the stochastic process x with respect to time. Equation (1) describes various wave processes in many subdiscipline of physics from hydrodynamics to quantum mechanics. For example, it models the propagation of long waves in shallow water taking into account capillary effects, and the nonlinear

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term is responsible for convective inertia. In this case, the function $x = x(s,t)$ determines the wave height.

In [1], a (modified) mathematical model of wave propagation in shallow water in a one-dimensional domain was investigated and a soliton solution of equation (1) was obtained. The existence of a unique global solution to the Cauchy problem for equation (1) was proved in the paper, with $\lambda = 1, \alpha = 1$. In [2], a similar solution was obtained to describe the interaction of shock waves. In [3], the Boussinesq equation was considered in a one-dimensional domain to describe the propagation of a longitudinal strain wave in an elastic rod. In [4], a soliton solution was found for the generalized Pochhammer – Chree equation and the interaction of two solitary wave solutions was numerically investigated.

The Boussinesq equation, obtained at the end of the 19th century, continues to be relevant and finds applications in other aspects; in addition to the works mentioned above, the following examples can be given. In [5], the solvability of natural (the first, the second and mixed) initial-boundary value problems for non-linear analogs of the Boussinesq equation is studied. In [6], the influence of gradient nonlinearity on the global solvability of initial-boundary value equations for the improved Boussinesq equation used to describe one-dimensional wave processes in media with dissipation and dispersion is investigated. Conditions for the blow-up of the solution are obtained.

The concept of mean derivatives was introduced by E. Nelson [7] for the needs of stochastic mechanics (a variant of quantum mechanics) in one-dimensional space. Then, Yu.E. Gliklikh developed this theory and applied it to solving problems of classical mechanics on nonlinear configuration spaces, statistical and quantum physics, and hydrodynamics [8, 9]. An idea arose to extend the concept of mean derivative to infinite-dimensional spaces and apply it to solving Sobolev type stochastic equations [10–14]. The purpose of such extension is to develop the theory of Sobolev type stochastic equations and to study applications of this theory to non-classical models of mathematical physics that have practical significance. In what follows, the mean derivative will be called the Nelson – Gliklikh derivative. Further in the text, all derivatives with respect to time should be understood in the Nelson – Gliklikh sense.

For the stochastic mathematical model (1)–(3), we pose the optimal control problem

$$J(x,u) \to \inf, \quad u \in \mathfrak{U}_{ad}. \tag{1.4}$$

For this, we introduce the control space $\mathfrak{U}$ and select in it a non-empty, closed and convex set $\mathfrak{U}_{ad}$, which we call the set of admissible controls. We will define the spaces and the specific form of the functional later.

In Sobolev type models, the optimal control problem was first considered in [15]. For first-order semilinear Sobolev type models, the optimal control problem was studied in [16, 17]. Problems of optimal control of oscillatory phenomena arise in such engineering problems as problems of calming the pitching of a ship, a crane boom, organizing vibration protection, and others. The importance of solving problems of optimal control of oscillatory processes has already been repeatedly noted in works [18, 19].

First, let us consider the operator-differential equation

$$L \overset{\circ}{\eta}{}^{(2)} + M\eta + N(\eta) = u(t), \tag{1.5}$$

with the Showalter – Sidorov initial condition

$$L(\eta(0) - \eta_0) = 0, \quad L(\overset{\circ}{\eta}(0) - \eta_1) = 0. \tag{1.6}$$

Here L, M are linear and continuous operators, $N(\eta)$ is a nonlinear operator, the conditions on which will be specified later. If L is not continuously invertible, then equation (5) is usually called a Sobolev type equation [20]. The initial data η_0, η_1 are random variables. In the deterministic case, problem (5), (6) was considered in [21, 22], and the optimal control problem (4)–(6) was considered in [23].

The first section provides preliminary information and constructs solution spaces. The second section is devoted to the solvability of an inhomogeneous semilinear Sobolev-type equation with random initial data with zero mathematical expectations; the results are applied to a mathematical model of wave propagation in shallow water with random initial states. The third section examines the problem of optimal control with random initial states with non-zero mathematical expectations. In conclusion, a remark on the Cauchy conditions is given and further directions for the study are indicated.

2. Spaces of Differentiable K-“Noises”

Let $\Omega \equiv (\Omega, A, P)$ denote the complete probability space. A measurable mapping $\xi : \Omega \to \mathbb{R}$ is called a random variable. The set of random variables whose mathematical expectations are zero (i.e. $\mathrm{E}\xi = 0$) and whose variances are finite (i.e. $\mathrm{D}\xi < +\infty$) form a Hilbert space $\mathbf{L}_2$ with inner product $(\xi_1, \xi_2) = \mathrm{E}\xi_1\xi_2$. Let A_0 denote the σ-subalgebra of the σ-algebra A and construct the space $\mathbf{L}_2^0$ of random variables measurable with respect to A_0. $\mathbf{L}_2^0$ is a subspace of the space $\mathbf{L}_2$. Let $\xi \in \mathbf{L}_2$, then $\Pi : \mathbf{L}_2 \to \mathbf{L}_2^0$ is the orthoprojector, and $\Pi\xi$ is the conditional mathematical expectation of the random variable ξ and is denoted by $\mathrm{E}(\xi|A_0)$.

Let $I = (0, T)$, $T \in \mathbb{R}_+$. Consider two mappings: $f : I \to \mathbf{L}_2$, which connects $t \in I$ with a random variable $\xi \in \mathbf{L}_2$, and $g : \mathbf{L}_2 \times \Omega \to \mathbb{R}$, which connects each pair (ξ, ω) with a point $\xi(\omega) \in \mathbb{R}$. The mapping $\eta : I \times \Omega \to \mathbb{R}$ of the form $\eta = \eta(t, \omega) = g(f(t), \omega)$ is called a (one-dimensional) stochastic process. If all trajectories of a stochastic process are a.s. continuous, then this process is called continuous. The set of continuous stochastic processes forms a Banach space, which we denote by $C(I; \mathbf{L}_2)$.

An example of a continuous stochastic process is the one-dimensional Wiener process $\beta = \beta(t)$, which can be represented as

$$\beta(t) = \sum_{k=0}^{\infty} \xi_k \sin\left(\frac{\pi}{2}(2k+1)t\right),$$

where ξ_k are uncorrelated Gaussian random variables such that $\mathrm{E}\xi_k = 0$, $D\xi_k = \left[\frac{\pi}{2}(2k+1)\right]^{-2}$.

Now let us fix an arbitrary continuous stochastic process $\eta(t) \in C(I;\mathbf{L}_2)$ and $t \in I$. Let N_t^η be the "present" σ-algebra generated by the random process $\eta(t)$, and $\mathrm{E}_t^\eta = \mathrm{E}(\cdot|N_t^\eta)$ be the conditional mathematical expectation. Then the right-hand mean derivative $D\eta(t,\cdot)$ (left-hand $D_*\eta(t,\cdot)$) of the stochastic process η at the point $t \in (\varepsilon,\tau)$ is the random variable

$$D\eta\,(t,\cdot) = \lim_{\Delta t\to 0+} \mathrm{E}_t^\eta \left(\frac{\eta\,(t+\Delta t,\cdot)-\eta(t,\cdot)}{\Delta t}\right)$$

$$\left(D_*\eta\,(t,\cdot) = \lim_{\Delta t\to 0+} \mathrm{E}_t^\eta \left(\frac{\eta\,(t,\cdot)-\eta\,(t-\Delta t,\cdot)}{\Delta t}\right)\right),$$

if the limit exists in the sense of the uniform metric on $\mathbb{R}$. A stochastic process η is called right (left) mean on I if there exists a right (left) mean derivative at each point $t \in I$. Let $\eta \in \mathbf{CL}_2$ be a stochastic process that is right and left mean-differentiable on I. The symmetric mean derivative is defined as $\overset{\circ}{\eta} = D_S\eta = \frac{1}{2}(D + D_*)\,\eta$. The symmetric mean-differentiable derivative will be called the Nelson–Gliklikh derivative. Let $\overset{\circ}{\eta}{}^{(l)}$, $l \in \mathbb{N}$ denote the l-th Nelson–Gliklikh derivative of the stochastic process η. Note that if $\eta(t)$ is a deterministic function, then the Nelson–Gliklikh derivative coincides with the classical derivative. In the case of a one-dimensional Wiener process $\beta = \beta(t)$ the following statements hold:

(i) $\overset{\circ}{\beta}(t) = \frac{\beta(t)}{2t}$ for any $t \in \mathbb{R}_+$;

(ii) $\overset{\circ}{\beta}{}^{(l)}(t) = (-1)^{l-1}\prod_{i=1}^{l-1}(2i-1)\dfrac{\beta(t)}{(2t)^l}$, $l \in \mathbb{N}$, $l \geq 2$.

We define the space of differentiable "noises" $C^l(I,\mathbf{L}_2)$, $l \in \mathbb{N}$, as the space of stochastic processes from $C(I;\mathbf{L}_2)$, whose trajectories are almost surely (a.s.) differentiable in the Nelson–Gliklikh sense up to the l-th order inclusive.

Let H be a separable Hilbert space with an orthonormal basis $\{\varphi_k\}$. Every element $\eta \in H$ can be written as

$$\eta = \sum_{k=1}^{\infty} \eta_k\varphi_k. \tag{2.1}$$

We choose a numerical sequence $K = \{\nu_k\}$ such that $\sum_{k=1}^{\infty}\nu_k^2 < +\infty$. Consider a sequence of independent random variables $\{\xi_k\} \subset \mathbf{L}_2$ with uniformly bounded variances, then $\sum_{k=1}^{\infty}\nu_k^2\mathrm{D}(\xi_k) < +\infty$.

We construct the Hilbert space $H_K\mathbf{L}_2$ of random K-variables, of the form

$$\xi = \sum_{k=1}^{\infty}\nu_k\xi_k\varphi_k. \tag{2.2}$$

Next, let $\{\xi_k(t)\}$ be a sequence of one-dimensional stochastic processes from the space $\mathbf{C}(I,\mathbf{L}_2)$. We define an H-valued stochastic K-process by the formula

$$\xi(t) = \sum_{k=1}^{\infty} \nu_k \xi_k(t) \varphi_k,$$

provided that the series converges uniformly on any compact set from I in the norm $H_K\mathbf{L}_2$. We introduce the Nelson–Gliklikh derivatives of the stochastic K-process

$$\overset{\circ}{\xi}{}^{(l)}(t) = \sum_{k=1}^{\infty} \nu_k \overset{\circ}{\xi}_k{}^{(l)}(t) \varphi_k.$$

The Nelson–Gliklikh derivatives exist up to the l-th order inclusive if the corresponding series converge uniformly on any compact set I in the norm $H_K\mathbf{L}_2$. The space $C^l(I; H_K\mathbf{L}_2)$ consists of continuous H-valued stochastic K-processes whose trajectories a.s. continuously differentiable in the Nelson–Gliklikh sense up to the l-th order inclusive. For brevity, we call the space $C^l(I; H_K\mathbf{L}_2)$ the space of differentiable K-"noises".

3. Inhomogeneous Equation

In this section we consider problem (5), (6). For this we will construct several functional spaces. Let $H = (H, \langle\cdot,\cdot\rangle) = W_2^{-1}(D)$ be a real, separable, Hilbert space; $\mathfrak{X} = L_2(D)$; $\mathfrak{N} = L_4(D)$. In the space H we define the inner product as follows

$$\langle \eta, v \rangle = \int_D \eta \tilde{v}\, ds, \quad \forall \eta, v \in W_2^{-1}(D),$$

where $\tilde{v}$ is a generalized solution of the Dirichlet problem (2) for the equation $-\Delta\tilde{v} = v$ in the domain D.

$$\mathfrak{N} \hookrightarrow \mathfrak{X} \hookrightarrow H \hookrightarrow \mathfrak{X}^* \hookrightarrow \mathfrak{N}^*. \tag{3.1}$$

Which is also true for their stochastic counterparts

$$\mathfrak{N}_K\mathbf{L}_2 \hookrightarrow \mathfrak{X}_K\mathbf{L}_2 \hookrightarrow H_K\mathbf{L}_2 \hookrightarrow (\mathfrak{X}_K\mathbf{L}_2)^* \hookrightarrow (\mathfrak{N}_K\mathbf{L}_2)^*. \tag{3.2}$$

Moreover, in the space $H_K\mathbf{L}_2$ the inner product takes the form

$$(\eta, v) = \mathrm{E}\int_D \eta\tilde{v}\, ds = \mathrm{E}\int_D \sum_{k=1}^{\infty} \nu_k \eta_k \varphi_k \sum_{k=1}^{\infty} \nu_k \tilde{v}_k \varphi_k\, ds = \mathrm{E}\sum_{k=1}^{\infty} \nu_k^2 \eta_k \tilde{v}_k. \tag{3.3}$$

Here and below, as φ_k we take the eigenfunctions of the Dirichlet problem in the domain D for the operator $(-\Delta)$, and by λ_k we denote the corresponding eigenvalues.

By analogy with the deterministic case in the space $H_K\mathbf{L}_2$, we introduce the norm

$$\|\eta\|^2_{H_K\mathbf{L}_2} = \sum_{k=1}^{\infty} \lambda_k^{-1} \nu_k^2 \mathrm{D}(\eta_k).$$

The space $\mathfrak{X}_K\mathbf{L}_2$ $((\mathfrak{X}_K\mathbf{L}_2)^*)$, consists of norm-bounded elements η

$$\|\eta\|^2_{\mathfrak{X}_K\mathbf{L_2}} = \mathrm{E}\int\limits_D \eta^2\, ds = \mathrm{E}\int\limits_D \left(\sum_{k=1}^{\infty}\nu_k\eta_k\varphi_k\right)^2 ds = \mathrm{E}\sum_{k=1}^{\infty}\nu_k^2\eta_k^2 = \sum_{k=1}^{\infty}\nu_k^2\mathrm{D}(\eta_k).$$

The space $\mathfrak{Y}_K\mathbf{L}_2$ is the closure of the set of random variables of the form (8) according to the norm

$$\|\eta\|^4_{\mathfrak{Y}_K\mathbf{L_2}} = \mathrm{E}\int\limits_D \eta^4\, ds = \mathrm{E}\int\limits_D \left(\sum_{k=1}^{\infty}\nu_k\eta_k\varphi_k\right)^4 ds =$$

$$\int\limits_D \sum_{k=1}^{\infty}\nu_k^4\mathrm{D}^2(\eta_k)\varphi_k^4\, ds + 6\int\limits_D \sum_{k,j=1}^{\infty}\nu_k^2\nu_j^2\mathrm{D}(\eta_k)\mathrm{D}(\eta_j)\varphi_k^2\varphi_j^2\, ds.$$

The space $(\mathfrak{Y}_K\mathbf{L}_2)^*$ is the closure of the set of random variables of the form (8) with respect to the norm

$$\|\eta\|^{\frac{4}{3}}_{\mathfrak{Y}^*{}_K\mathbf{L_2}} = \mathrm{E}\int\limits_D \eta^{\frac{4}{3}}\, ds.$$

We define the operator $L : \mathfrak{X}_K\mathbf{L}_2 \to (\mathfrak{X}_K\mathbf{L}_2)^*$ by the formula

$$(L\eta, v) = \mathrm{E}\int\limits_D (\eta v + \lambda\eta\tilde{v})\, ds.$$

For $\lambda \geq \lambda_1$ the operator L is self-adjoint, non-negative definite and Fredholm.

The operator $M : \mathfrak{X}_K\mathbf{L}_2 \to (\mathfrak{X}_K\mathbf{L}_2)^*$ given by the formula

$$(M\eta, v) = \alpha^2\mathrm{E}\int\limits_D \eta v\, ds$$

is self-adjoint, non-negative definite.

The operator $N(\eta) : \mathfrak{Y}_K\mathbf{L}_2 \to (\mathfrak{Y}_K\mathbf{L}_2)^*$, defined by the formula

$$(N(\eta), v) = \mathrm{E}\int\limits_D \eta^3 v\, ds$$

is an s-monotone, 4-coercive, and homogeneous operator of order 4. Its Frechet derivative

$$|(N'_\eta(v), w)| = 3\mathrm{E}\left|\int\limits_D \eta^2 vw\, ds\right| \leq C\|\eta\|_{\mathfrak{X}_K\mathbf{L_2}}\|v\|_{\mathfrak{Y}_K\mathbf{L_2}}\|w\|_{\mathfrak{Y}_K\mathbf{L_2}}$$

is symmetric and bounded by Hölder's inequality.

The operator N is s-monotone

$$(N'_\eta(v), v) = 3\mathrm{E}\int\limits_D \eta^2 v^2\, ds \geq 0$$

and 4-coercive

$$(N(\eta),\eta) = \mathrm{E}\int\limits_D \eta^3\eta\, ds = \|\eta\|^4_{\mathfrak{N}_K\mathbf{L}_2},$$

$$(N(\eta),v) = \mathrm{E}\int\limits_D \eta^3 v\, ds = \|\eta\|^3_{\mathfrak{N}_K\mathbf{L}_2}\|v\|_{\mathfrak{N}_K\mathbf{L}_2} = \|\eta^3\|_{\mathfrak{N}^*{}_K\mathbf{L}_2}\|v\|_{\mathfrak{N}_K\mathbf{L}_2}.$$

In a similar way to the construction of the space $C^l(I;H_K\mathbf{L}_2)$, we construct the spaces of differentiable K-"noises" $C^l(I;\mathfrak{X}_K\mathbf{L}_2)$, $C^l(I;\mathfrak{N}_K\mathbf{L}_2)$, where by $\mathfrak{X}_K\mathbf{L}_2$ and $\mathfrak{N}_K\mathbf{L}_2$ we denote the spaces of random K-variables of the form (8). Note that, due to the density and continuity of embeddings (9), the orthonormal basis φ_k of H will also be a basis for the spaces $\mathfrak{N}$, $\mathfrak{X}$, $\mathfrak{N}^*$, $\mathfrak{X}^*$.

Lemma 1. *(i) For all $\lambda \geq \lambda_1$, the operator $L \in \mathcal{L}(\mathfrak{X}_K\mathbf{L}_2;(\mathfrak{X}_K\mathbf{L}_2)^*)$ is self-adjoint, Fredholm, and non-negative definite.*
(ii) The operator $N(\eta) \in C^\infty(\mathfrak{N}_K\mathbf{L}_2;(\mathfrak{N}_K\mathbf{L}_2)^*)$ is an s-monotone, 4*-coercive, and homogeneous operator of order* 4.

Proof of Lemma 1. In the case $\lambda \geq \lambda_1$

$$\ker L = \begin{cases} \{0\}, & \lambda > \lambda_1; \\ \mathrm{span}\{\varphi_1\}, & \lambda = \lambda_1. \end{cases}$$

Then

$$\mathrm{im} L = \begin{cases} (\mathfrak{X}_K\mathbf{L}_2)^*, & \lambda > \lambda_1; \\ \xi \in (\mathfrak{X}_K\mathbf{L}_2)^* : \langle \xi,\varphi_1\rangle = 0, & \lambda = \lambda_1. \end{cases}$$

and

$$\mathrm{coim} L = \begin{cases} \mathfrak{X}_K\mathbf{L}_2, & \lambda > \lambda_1; \\ \xi \in \mathfrak{X}_K\mathbf{L}_2 : \langle \xi,\varphi_1\rangle = 0, & \lambda = \lambda_1. \end{cases}$$

By the construction of the spaces, the proof of this lemma is based on the idea of the proof for the deterministic case in [12]. □

Definition 1. A stochastic K-process $\eta \in C^\infty(I;\mathfrak{X}_K\mathbf{L}_2)$ is called a *solution of equation* (5) if a.s. all trajectories η satisfy equation (5) for all $t \in I$. The solution $\eta = \eta(t)$ of equation (5) is called a *solution of the Showalter–Sidorov problem* (6) if the solution satisfies condition (6) for some pair of random K-variables $\eta_0,\eta_1 \in \mathfrak{X}_K\mathbf{L}_2$.

Remark 1. Let $(\mathbb{I}-Q)u(t)$ be independent of t. Then, due to the degeneracy of equation (5), all its solutions $\eta = \eta(t)$ for all $t \in I$ lie in the phase manifold

$$\mathfrak{M} = \begin{cases} \mathfrak{X}_K\mathbf{L}_2, & \lambda > \lambda_1; \\ \eta \in \mathfrak{X}_K\mathbf{L}_2 : (\mathbb{I}-Q)(M\eta + N(\eta)) = (\mathbb{I}-Q)u(t), & \lambda = \lambda_1. \end{cases}$$

Here

$$Q = \begin{cases} \mathbb{I}, & \lambda > \lambda_1; \\ \mathbb{I} - \sum\limits_{k:\lambda_k=\lambda} \langle \cdot,\varphi_1\rangle, & \lambda = \lambda_1. \end{cases}$$

orthoprojector onto the image of the operator L.

Let us turn to the mathematical model (1)–(3). The phase manifold $\mathfrak{M}$ takes the form

$$\mathfrak{M} = \begin{cases} \mathfrak{X}_K \mathbf{L}_2, & \lambda > \lambda_1; \\ \eta \in \mathfrak{X}_K \mathbf{L}_2 : \mathrm{E} \int\limits_D \eta^3 \varphi_1 \, ds = \frac{1}{\lambda_1} \int\limits_D u(t) \varphi_1 \, ds, & \lambda = \lambda_1, \end{cases}$$

Moreover, we require that the expression $\frac{1}{\lambda_1} \int\limits_D u(t)\varphi_1 \, ds$ does not depend on t.

Define $\eta_0 \in \mathfrak{N}_K \mathbf{L}_2, \eta_1 \in \mathfrak{X}_K \mathbf{L}_2$ in the form

$$\eta_0 = \sum_{k=1}^{\infty} \nu_k \eta_{0k} \varphi_k, \quad \eta_1 = \sum_{k=1}^{\infty} \nu_k \eta_{1k} \varphi_k.$$

where η_{0k} and η_{1k} are sequences of random variables from $\mathbf{L}_2$.

Theorem 1. *Let $\lambda \geq \lambda_1$, $\mathfrak{U} = L_2(0,T;H)$, then for any sequences of random variables η_{0k} and η_{1k} from $\mathbf{L}_2$ and any $T \in \mathbb{R}^+$ there exists a solution $\eta \in C^\infty(I;\mathfrak{N}_K \mathbf{L}_2), \overset{\circ}{\eta} \in C^\infty(0,T;[\mathrm{coim} L]_K \mathbf{L}_2 \cap \mathfrak{X}_K \mathbf{L}_2)$ of the problem (1)–(3).*

Proof of Theorem 1. Given that the operator L is self-adjoint and Fredholm, we define $\mathfrak{X} \supset \ker L \equiv \mathrm{coker} L \subset \mathfrak{X}^*$. Using the subspace $\ker L$, we construct the subspaces $[\ker L]_K \mathbf{L}_2 \subset H_K \mathbf{L}_2$, and similarly, the subspaces $[\mathrm{coker} L]_K \mathbf{L}_2 \subset H_K \mathbf{L}_2$. Considering that the embeddings (10) are dense and continuous, we construct the spaces $(\mathfrak{X}_K \mathbf{L}_2)^* = [\mathrm{coker} L]_K \mathbf{L}_2 \oplus [\mathrm{im} L]_K \mathbf{L}_2$ and $(\mathfrak{N}_K \mathbf{L}_2)^* = [\mathrm{coker} L]_K \mathbf{L}_2 \oplus [\mathrm{im} L \cap \mathfrak{N}^*]_K \mathbf{L}_2$. Let us construct a subspace $\mathrm{coim} L \subset \mathfrak{X}$ such that the subspace $X_K \mathbf{L}_2 = [\ker L]_K \mathbf{L}_2 \oplus [\mathrm{coim} L]_K \mathbf{L}_2$. In view of $[\ker L]_K \mathbf{L}_2$ and the subspace $\mathrm{coim} L \cap \mathfrak{N}$, then $\mathfrak{N}_K \mathbf{L}_2 = [\ker L]_K \mathbf{L}_2 \oplus [\overline{\mathrm{coim} L}]_K \mathbf{L}_2$.

We fix $\omega \in \Omega$. Since the stochastic component in problem (1)–(3) is only in the initial condition (3), then for a fixed ω the Nelson – Gliklikh derivative coincides with the classical derivative. Thus, problem (1)–(3) is reduced to the deterministic case [21]. By virtue of the theorem on the existence of a unique solution [21], the existence of a trajectory solution to problem (1)–(3) is proven.

Since in the deterministic case there is a unique solution to problem (1)–(3). Therefore, each trajectory for a fixed ω is unique. □

4. Optimal Control

We seek a solution to problem (1)–(4) in the form:

$$x = y + \eta, \quad y = \mathrm{E}(x), \quad \mathrm{E}(\eta) = 0.$$

The optimal control problem (4)–(6) is divided into two problems. The first problem contains only deterministic functions:

$$Ly''_{tt} + My + N(y) = u(s,t), \quad (s,t) \in D \times (0,T), \tag{4.1}$$

$$y(s,t) = 0, \quad (s,t) \in \partial D \times (0,T), \tag{4.2}$$

$$L(y(s,0) - y_0(s)) = 0, \quad L(y'_t(s,0) - y_1(s)) = 0, \quad s \in D, \tag{4.3}$$

We define the functional as follows:

$$J(x,u)=J(y,u)=\beta\int\limits_0^T(\|y(s,t)-z(s,t)\|^4_{L_4}+\|y'_t(s,t)-z'_t(s,t)\|^2_{L_2})dt+$$
$$+(1-\beta)\int\limits_0^T\|u(s,t)\|^2_{L_2}\,dt\to\inf,\quad u\in\mathfrak{U}_{ad},\tag{4.4}$$

i.e. the control function affects the mathematical expectation of the stochastic process describing the state of the system, in other words, the useful part of the signal. This type of functional was chosen due to its universality. By choosing the weighting coefficient $\beta\in(0,1)$, it takes into account both the control costs and the proximity between the desired $z(s,t)$ and the current state of the system [23]. The norms in the functional are determined by the theorem of existence of a solution to the corresponding initial-boundary value problem. For the mathematical model (1)–(3), the functional (15) can be interpreted as follows: it is required to bring the wave to a given shape in a limited period of time with the least control costs.

To solve problem (12)–(15), we construct the space $\mathfrak{U}=L_2(0,T;H)$ and define in it a non-empty closed and convex set $\mathfrak{U}_{ad}$ and the space $\mathfrak{Y}_{det}=\{y|y\in C^\infty(0,T;L_4),\ y'_t\in C^\infty(0,T;[\mathrm{coim}L]\cap L_2)$. The existence theorem for a solution to problem (12)–(15) is proved in [24, Theorem 3.2].

Knowing the control function u and the mathematical expectation y, we find the stochastic process x, performing the inverse substitution $x=\eta+y$ we return to following problem

$$L\overset{\circ}{x}^{(2)}+Mx-\Delta(x^3)=u(t),\quad(s,t)\in D\times(0,T),\tag{4.5}$$

$$x(s,t)=0,\quad(s,t)\in\partial D\times(0,T),\tag{4.6}$$

$$L(x(s,0)-x_0)=0,\quad L(\overset{\circ}{x}(s,0)-x_1)=0,\quad s\in D.\tag{4.7}$$

In order to apply the theory from paragraph 3, we will construct special spaces

$$x\in L_4\oplus\mathfrak{N}_K\mathbf{L}_2,\quad\text{and}\quad x'_t\in L_2\oplus\mathfrak{X}_K\mathbf{L}_2,$$

the spaces conjugate to them are constructed with respect to the inner product in $H\oplus\mathfrak{H}_K\mathbf{L}_2$,

$$(x_1,x_2)_{H\oplus H_K\mathbf{L}_2}=(x_1,x_2)_H+(x_1,x_2)_{H_K\mathbf{L}_2}.$$

There is a chain of embeddings

$$L_4\oplus\mathfrak{N}_K\mathbf{L}_2\hookrightarrow L_2\oplus\mathfrak{X}_K\mathbf{L}_2\hookrightarrow H\oplus H_K\mathbf{L}_2\hookrightarrow(L_2\oplus\mathfrak{X}_K\mathbf{L}_2)^*\hookrightarrow(L_4\oplus\mathfrak{N}_K\mathbf{L}_2)^*.$$

Denote $X_{st}=\{x\in C^\infty(0,T;\mathfrak{N}_K\mathbf{L}_2\oplus L_4),\overset{\circ}{x}\in C^\infty(0,T;[\mathrm{coim}L]_K\mathbf{L}_2\cap\mathfrak{X}_K\mathbf{L}_2\oplus L_2)\}$. The solution to problem (16)–(18) follows from Theorem 1. **Corollary 1.** Let $\lambda\geq\lambda_1$, then for any sequences of random variables x_{0k} and x_{1k} from $\mathbf{L}_2$ and any $T\in\mathbb{R}^+$ there exists a solution $x\in X_{st}$ of problem (16)–(18).

Definition 2. A pair (x, u) is called a *solution to the optimal control problem* (4)–(6) if $(y, u) \in \mathfrak{Y}_{det} \times \mathfrak{U}_{ad}$ satisfies problem (12)–(15), and $x \in \mathfrak{X}_{st}$ satisfies problem (16)–(18) in the sense of Definition 1.

By [24, Theorem 3.2] and Corollary 1, we have

Theorem 2. Let $\lambda \geq \lambda_1$, then for any $x_0 = y_0 + \eta_0$, $x_1 = y_1 + \eta_1$, where $\eta_0 \in \mathfrak{N}_K\mathbf{L}_2, \eta_1 \in \mathfrak{X}_K\mathbf{L}_2$, $y_0 = E(x_0) \in \mathfrak{N}$, $y_1 = E(x_1) \in \mathfrak{X} \cap \text{coim}L$ and any $T \in \mathbb{R}^+$ there exists a solution to the optimal control problem (1)–(4) of the form $(x, u) = (y + \eta, u)$, where (y, u) is a solution to problem (12)–(15), and x is a solution to problem (16)–(18).

5. Conclusions

Similar results can be easily obtained for the mathematical model (1), (2) with the Cauchy initial conditions

$$x(0) = x_0, \quad \overset{\circ}{x}(0) = x_1,$$

instead of the Showalter – Sidorov conditions (3). However, it is necessary to check whether the initial data belong to the tangent bundle of the phase space, i.e. $(x_0, x_1) \in T_{x_0}\mathfrak{M}$. The next logical step seems to be the study of the structure of the phase space [26, 27] and obtaining a numerical solution as well as the study of the stability of solutions [28]. The results obtained in the article can be used in the development of application software for identifying the ultimate loads causing deformation waves for newly developed structural materials.

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