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ON SUBORDINATED TELEGRAPH PROCESSES

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ABSTRACT. We study telegraph processes with time change implemented by means of an inverse α-stable subordinator. Two different types of subordination are proposed, namely, the time change in an integrated telegraph process and an integrated telegraph process with changed time. Explicit formulae for the first two moments are presented.

1. Introduction

The concept of random time change in continuous-time stochastic processes was first introduced in [2, 7] and a manuscript by W. Doeblin written around 1940 but published only in 2000, see [3]. See also [13] and the classic book [4]. In recent decades, time-changed (subordinated) random processes have become particularly popular for research as well as for numerous applications, see [9] for a historical overview. Due to memory effects, subordinated stochastic processes are useful for various model in the natural sciences, including models in biology, see e.g. [6].

For any stochastic process $X = X(t)$ (parent process) and an α-stable Lévy subordinator $\sigma_\alpha(t)$, the time-subordinated process $Y = Y(t)$ is defined by replacing the time in $X(t)$ with a positive non-decreasing process $E_\alpha(t)$ which is the inverse of $\sigma_\alpha(t)$.

In this paper, we study in detail the result of such replacement when the parent process X is a telegraph process $t \to c_{\varepsilon(t)}$ or, alternatively, an integrated telegraph process,

$$X(t) = \mathscr{T}(t) = \int_0^t c_{\varepsilon(s)}\mathrm{d}s. \tag{1.1}$$

Here $\varepsilon = \varepsilon(t)$ is a two-state Markov process and $c_0, c_1 \in (-\infty, \infty),\ c_0 > c_1$.

In other words, subordination is introduced in two ways: by direct replacement of time in the integrated telegraph process, $\mathscr{T}^\alpha(t) = \mathscr{T}(E_\alpha(t))$, or by replacing time in $c_{\varepsilon(t)}$ with subsequent integration, $\mathscr{T}_\alpha(t) = \int_0^t c_{\varepsilon(E_\alpha(s))}\mathrm{d}s$.

The paper is organised as follows. The next section collects some essentials and the basic properties of (inhomogeneous) telegraph processes and inverse α-stable subordinators. In Section 3 we explicitly express the first two moments of subordinated telegraph processes of both types.

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2. Preliminares

2.1. Telegraph process. Let $\varepsilon = \varepsilon(t) \in \{0,1\}$, $t \geq 0$, be a two-state càdlàg Markov process with the infinitesimal generator

$$\Lambda = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix}, \qquad \lambda_0, \lambda_1 > 0. \tag{2.1}$$

The transition matrix $\Pi(t) = (\pi_{ij}(t))_{i,j\in\{0,1\}}$, $\pi_{ij}(t) = \mathbb{P}_i\{\varepsilon(t) = j\}$, for the process $\varepsilon = \varepsilon(t)$ has the form

$$\Pi(t) = \exp(t\Lambda) = \frac{1}{2\lambda}\begin{pmatrix} \lambda_1 + \lambda_0 \mathrm{e}^{-2\lambda t} & \lambda_0 - \lambda_0 \mathrm{e}^{-2\lambda t} \\ \lambda_1 - \lambda_1 \mathrm{e}^{-2\lambda t} & \lambda_0 + \lambda_1 \mathrm{e}^{-2\lambda t} \end{pmatrix}, \qquad 2\lambda := \lambda_0 + \lambda_1,$$

see, for example, [12, (3.3.43)]. Here by $\mathbb{P}_i\{\cdot\}$, denotes the conditional probability given the initial state, $\mathbb{P}_i\{\cdot\} = \mathbb{P}\{\cdot \mid \varepsilon(0) = i\}$. The corresponding expectations are denoted as $\mathbb{E}_i[\cdot]$.

Explicit formulae for the means and covariances of the telegraph process $c_{\varepsilon(t)}$ are known. For completeness, we present these formulae here.

Proposition 2.1. *For any* $s > 0$,

$$\mathbb{E}_0[c_{\varepsilon(s)}] = \kappa + 2cp_1^* \mathrm{e}^{-2\lambda s}, \tag{2.2}$$

$$\mathbb{E}_1[c_{\varepsilon(s)}] = \kappa - 2cp_0^* \mathrm{e}^{-2\lambda s}. \tag{2.3}$$

For any $s_1, s_2,\ 0 < s_1 \leq s_2$,

$$\mathbb{E}_0\left[c_{\varepsilon(s_1)} \cdot c_{\varepsilon(s_2)}\right] = \kappa^2 + 2cp_1^*\kappa \mathrm{e}^{-2\lambda s_1} + 2cp_1^*\bar{\kappa}\mathrm{e}^{-2\lambda s_2} + 4c^2 p_0^* p_1^* \mathrm{e}^{-2\lambda(s_2 - s_1)}, \tag{2.4}$$

$$\mathbb{E}_1\left[c_{\varepsilon(s_1)} \cdot c_{\varepsilon(s_2)}\right] = \kappa^2 - 2cp_0^*\kappa \mathrm{e}^{-2\lambda s_1} - 2cp_0^*\bar{\kappa}\mathrm{e}^{-2\lambda s_2} + 4c^2 p_0^* p_1^* \mathrm{e}^{-2\lambda(s_2 - s_1)}. \tag{2.5}$$

The first and second moment for the integrated telegraph process $\mathscr{T}(t)$ are given by

$$\mathbb{E}_0[\mathscr{T}(t)] = \kappa t + p_1^* c \frac{1 - \mathrm{e}^{-2\lambda t}}{\lambda}, \tag{2.6}$$

$$\mathbb{E}_1[\mathscr{T}(t)] = \kappa t - p_0^* c \frac{1 - \mathrm{e}^{-2\lambda t}}{\lambda}, \tag{2.7}$$

and

$$\mathbb{E}_0[\mathscr{T}(t)]^2 = \kappa^2 t^2 + 2p_1^* c\bar{\kappa}\frac{1 - \mathrm{e}^{-2\lambda t}}{\lambda} t + 2p_1^*(2p_0^* - p_1^*)c^2 \frac{\mathrm{e}^{-2\lambda t} - 1 + 2\lambda t}{\lambda^2}, \tag{2.8}$$

$$\mathbb{E}_1[\mathscr{T}(t)]^2 = \kappa^2 t^2 - 2p_0^* c\bar{\kappa}\frac{1 - \mathrm{e}^{-2\lambda t}}{\lambda} t + 2p_0^*(2p_1^* - p_0^*)c^2 \frac{\mathrm{e}^{-2\lambda t} - 1 + 2\lambda t}{\lambda^2}. \tag{2.9}$$

Here $2c = c_0 - c_1,\ \kappa = p_0^* c_0 + p_1^* c_1,\ \bar{\kappa} = p_1^* c_0 + p_0^* c_1$;

$$p_0^* = \frac{\lambda_1}{\lambda_0 + \lambda_1}, \qquad p_1^* = \frac{\lambda_0}{\lambda_0 + \lambda_1}$$

are the stationary probabilities of the Markov process $\varepsilon = \varepsilon(t)$.

Proof. See [12] or [11]. □

The joint distribution of processes $\mathscr{T}(t)$ and $N(t)$ is characterised by probability density functions $(p_0(t,x;n), p_1(t,x;n))$, supported on $[c_1t,\ c_0t]$, and defined as

$$p_i(t,x;n)\mathrm{d}x = \mathbb{P}_i\{\mathscr{T}(t)\in\mathrm{d}x,\ N(t)=n\},\qquad n\ge 0,\ x\in(c_1t,\ c_0t).$$

In the case of no switching, $\mathbb{P}_i\{\mathscr{T}(t)\in\mathrm{d}x, N(t)=0\} = \exp(-\lambda_it)\delta_{c_it}(\mathrm{d}x),\ i\in\{0,1\}$.

In general, $(p_0(t,x;n),\ p_1(t,x;n))$ follow the Cattaneo system, [12, (3.8)],

$$\begin{cases} \dfrac{\partial p_0}{\partial t}(t,x;n)+c_0\dfrac{\partial p_0}{\partial x}(t,x;n) = -\lambda_0p_0(t,x;n)+\lambda_0p_1(t,x;n-1),\\ \dfrac{\partial p_1}{\partial t}(t,x;n)+c_1\dfrac{\partial p_1}{\partial x}(t,x;n) = \lambda_1p_0(t,x;n-1)-\lambda_1p_1(t,x;n),\\ \qquad\qquad\qquad\qquad\qquad\qquad n\ge 1,\quad t>0. \end{cases} \tag{2.10}$$

The probability density function $\mathbf{p}(t,x) = (p_0(t,x), p_1(t,x))$ of the telegraph process $\mathscr{T}(t)$ is determined by

$$p_0(t,x) = \sum_{n=0}^{\infty} p_0(t,x;n),\qquad p_1(t,x) = \sum_{n=0}^{\infty} p_1(t,x;n). \tag{2.11}$$

It is known, see e.g. [12], that

$$\begin{aligned} p_0(t,x) =& \mathrm{e}^{-\lambda_0t}\delta_{c_0t}(x)+\frac{1}{2c}\left[\lambda_0\theta(t,x)I_0\left(\frac{1}{c}\sqrt{\lambda_0\lambda_1(c_0t-x)(x-c_1t)}\right)\right.\\ &\left.+\sqrt{\lambda_0\lambda_1}\theta(t,x)\left(\frac{x-c_1t}{c_0t-x}\right)^{1/2}I_1\left(\frac{1}{c}\sqrt{\lambda_0\lambda_1(c_0t-x)(x-c_1t)}\right)\right],\\ p_1(t,x) =& \mathrm{e}^{-\lambda_1t}\delta_{c_1t}(x)+\frac{1}{2c}\left[\lambda_1\theta(t,x)I_0\left(\frac{1}{c}\sqrt{\lambda_0\lambda_1(c_0t-x)(x-c_1t)}\right)\right.\\ &\left.+\sqrt{\lambda_0\lambda_1}\theta(t,x)\left(\frac{x-c_1t}{c_0t-x}\right)^{-1/2}I_1\left(\frac{1}{c}\sqrt{\lambda_0\lambda_1(c_0t-x)(x-c_1t)}\right)\right]. \end{aligned} \tag{2.12}$$

Here $2c = c_0-c_1$, and $\theta(t,x) = \dfrac{1}{2c}\exp\left(-\lambda_0\dfrac{x-c_1t}{2c}-\lambda_1\dfrac{c_0t-x}{c_0-c_1}\right)$, and $I_0,\ I_1$ are the modified Bessel functions.

In the symmetric case, $c_0=-c_1=c,\ \lambda_0=\lambda_1=\lambda$, system (2.10) leads to the telegraph equation

$$\frac{\partial^2u}{\partial t^2}+2\lambda\frac{\partial u}{\partial t} = c^2\frac{\partial^2u}{\partial x^2}. \tag{2.13}$$

In this case, both the probability density functions $p_0(t,x),\ p_1(t,x)$ of $\mathscr{T}(t)$ and some other characteristics satisfy equation (2.13).

It is well known that under the so-called Kac scaling,

$$c,\ \lambda\to+\infty,\qquad\text{such that}\quad c^2/\lambda\to 1,$$

equation (2.13) becomes the heat equation, [5], and the symmetric telegraph process $\mathscr{T}(t)$ converges weakly to the standard Brownian motion, [10]. See also [12] for a proof of the corresponding functional limit theorem and its generalisations.

Note that $\mathscr{T}(t)$ is not a Markov process, whereas the pair $(\mathscr{T}(t),\ \varepsilon(t))$ is.

2.2. α-stable subordinator. Let $E_\alpha = E_\alpha(t)$ be the inverse process or the first hitting time process of α-stable subordinator σ_α, $0 < \alpha < 1$,

$$E_\alpha(t) = \inf\{u > 0 \mid \sigma_\alpha(u) > t\}.$$

It is known, see [1], that the Laplace-Stieltjes transform of $E_\alpha(t)$ is given by the Mittag-Leffler function

$$\psi(q) := \mathbb{E}\left[\mathrm{e}^{-qE_\alpha(t)}\right] = \mathscr{E}_\alpha(-qt^\alpha) = \sum_{n=0}^{\infty} \frac{(-qt^\alpha)^n}{\Gamma(1+\alpha n)}. \tag{2.14}$$

Further, the inverse α-stable subordinator E_α is a self-similar process: for $k > 0$

$$k^{-\alpha} E_\alpha(kt) \overset{d}{=} E_\alpha(t), \qquad \forall t > 0. \tag{2.15}$$

Furthermore, the mean values related to E_α are given in closed form:

$$\mathbb{E}\,[E_\alpha(t)] = -\,\psi'(q)|_{q=0} = \frac{t^\alpha}{\Gamma(1+\alpha)}, \tag{2.16}$$

$$\mathbb{E}\,[E_\alpha(t)]^2 = \psi''(q)|_{q=0} = \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)}, \tag{2.17}$$

$$\mathbb{E}\left[E_\alpha(t)\mathrm{e}^{-2\lambda E_\alpha(t)}\right] = -\,\psi'(q)|_{q=2\lambda} = \frac{t^\alpha}{\alpha}\mathscr{E}_{\alpha,\alpha}(-2\lambda t^\alpha) = \frac{t^\alpha}{\alpha}\sum_{n=0}^{\infty}\frac{(-2\lambda t^\alpha)^n}{\Gamma(\alpha+n\alpha)}. \tag{2.18}$$

3. Two types of subordination

Let $E_\alpha = E_\alpha(t)$ be the inverse α-stable subordinator independent of ε.
We study the two types of subordination of the integral telegraph process (1.1).

3.1. 1st type of subordination. Let us first consider the time-changed telegraph process $\mathscr{T}^\alpha(t)$, which is defined as

$$\mathscr{T}^\alpha(t) = \mathscr{T}(E_\alpha(t)) = \int_0^{E_\alpha(t)} c_{\varepsilon(s)}\mathrm{d}s, \qquad t > 0. \tag{3.1}$$

It is known that the increments of the inverse Lévy subordinators are not independent and stationary. Therefore, $\mathscr{T}(t)$ is not a Lévy process. In addition, this process is semi-Markov, but not strong Markov. See [8].

The distribution of the time-changed (subordinated) process $\mathscr{T}^\alpha(t)$ is given by the probability density function

$$\mathbf{p}_\alpha(t,x) = \int_0^\infty \mathbf{p}(s,x)\nu_\alpha(s,t)\mathrm{d}s,$$

where $\mathbf{p}(s,\cdot)$ is the probability density function $\mathscr{T}(s)$, see (2.11)-(2.12), and $\nu_\alpha(\cdot,t)$ is the probability density function of $E_\alpha(t)$.

Remark 3.1. Let $\mathbf{u}(t,x) = (\mathbb{E}_0[f(x+\mathscr{T}^\alpha(t)))],\ \mathbb{E}_1[f(x+\mathscr{T}^\alpha(t))])$ with $f(\cdot) \in C_0^1(\mathbb{R})$. Function $\mathbf{u}(t,x)$ is given by

$$\mathbf{u}(t,x) = \int_{-\infty}^{\infty} f(x+y)\mathbf{p}_\alpha(t,y)\mathrm{d}y,$$

and it satisfies the following time-fractional differential equation,

$${}^{C}\mathscr{D}_t^\alpha \mathbf{u}(t,x) = (\Lambda + \mathscr{L})\mathbf{u}(t,x),$$

see [1]. Here ${}^C\mathscr{D}_t^\alpha \mathbf{u}(t,x)$ is the Caputo derivative,

$$\left({}^C\mathscr{D}_t^\alpha \mathbf{u}\right)(t,x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha} \left(\mathbf{u}(s,x) - \mathbf{u}(0,x)\right) \mathrm{d}s,$$

and $\Lambda + \mathscr{L}$ is the infinitesimal generator of the Markov process $(\mathscr{T}(t),\ \varepsilon(t))$, where

$$\mathscr{L} = \begin{pmatrix} c_0 \dfrac{\partial}{\partial x} & 0 \\ 0 & c_1 \dfrac{\partial}{\partial x} \end{pmatrix}, \tag{3.2}$$

Λ is defined by (2.1), see [12].

By applying (2.6)-(2.9) (Proposition 2.1) and (2.18), the first two moments of $\mathscr{T}^\alpha(t)$ can be represented explicitly.

Theorem 3.2. *For* $0 < \alpha \le 1$, $t \ge 0$, *we have*

$$\mathbb{E}_0\left[\mathscr{T}^\alpha(t)\right] = \frac{\kappa t^\alpha}{\Gamma(1+\alpha)} + p_1^* c \frac{1 - \mathscr{E}_\alpha(-2\lambda t^\alpha)}{\lambda}, \tag{3.3}$$

$$\mathbb{E}_1\left[\mathscr{T}^\alpha(t)\right] = \frac{\kappa t^\alpha}{\Gamma(1+\alpha)} - p_0^* c \frac{1 - \mathscr{E}_\alpha(-2\lambda t^\alpha)}{\lambda}, \tag{3.4}$$

and

$$\begin{aligned} \mathbb{E}_0\left[\mathscr{T}^\alpha(t)\right]^2 = & \frac{2\kappa^2}{\Gamma(1+2\alpha)} t^{2\alpha} + \frac{2p_1^*\bar{\kappa}c}{\lambda}\left[\frac{1}{\Gamma(1+\alpha)} - \frac{1}{\alpha}\mathscr{E}_{\alpha,\alpha}(-2\lambda t^\alpha)\right] t^\alpha \\ & + \frac{2p_1^*(2p_0^* - p_1^*)c^2}{\lambda^2}\left[\mathscr{E}_\alpha(-2\lambda t^\alpha) - 1 + \frac{2\lambda}{\Gamma(1+\alpha)} t^\alpha\right], \end{aligned} \tag{3.5}$$

$$\begin{aligned} \mathbb{E}_1\left[\mathscr{T}^\alpha(t)\right]^2 = & \frac{2\kappa^2}{\Gamma(1+2\alpha)} t^{2\alpha} - \frac{2p_0^*\bar{\kappa}c}{\lambda}\left[\frac{1}{\Gamma(1+\alpha)} - \frac{1}{\alpha}\mathscr{E}_{\alpha,\alpha}(-2\lambda t^\alpha)\right] t^\alpha \\ & + \frac{2p_0^*(2p_1^* - p_0^*)c^2}{\lambda^2}\left[\mathscr{E}_\alpha(-2\lambda t^\alpha) - 1 + \frac{2\lambda}{\Gamma(1+\alpha)} t^\alpha\right]. \end{aligned} \tag{3.6}$$

Proof. By definition, (3.1), formulae (3.3)-(3.4) and (3.5)-(3.6) follow directly from (2.6)-(2.7) and (2.8)-(2.9), respectively, after applying (2.14)-(2.18). Indeed, since $\mathscr{T}$ and E_α are independent, by (2.6)-(2.7) we obtain

$$\begin{aligned} \mathbb{E}_0\left[\mathscr{T}_\alpha^{(1)}(t)\right] = & \mathbb{E}\left[\kappa E_\alpha(t) + \frac{p_1^* c}{\lambda}\left(1 - \exp(-2\lambda E_\alpha(t))\right)\right], \\ \mathbb{E}_1\left[\mathscr{T}_\alpha^{(1)}(t)\right] = & \mathbb{E}\left[\kappa E_\alpha(t) - \frac{p_0^* c}{\lambda}\left(1 - \exp(-2\lambda E_\alpha(t))\right)\right], \end{aligned}$$

which give (3.3)-(3.4). Formulae (3.5)-(3.6) follow similarly. □

3.2. 2d type of subordination. Let's consider the second type of subordination,

$$\mathscr{T}_\alpha(t) = \int_0^t c_{\varepsilon(E_\alpha(s))} \mathrm{d}s. \tag{3.7}$$

The representations of the first two moments of $\mathscr{T}_\alpha(t)$ follow from Proposition 2.1 and (2.18), as before.

Theorem 3.3.

$$\mathbb{E}_0\mathscr{T}_\alpha(t) = \kappa t + 2p_1^*c\delta_0(t), \tag{3.8}$$

$$\mathbb{E}_1\mathscr{T}_\alpha(t) = \kappa t - 2p_0^*c\delta_0(t), \tag{3.9}$$

and

$$\begin{aligned}\mathbb{E}_0\left[\mathscr{T}_\alpha(t)^2\right] = &(p_0^*c_0 + p_1^*c_1)^2t^2 \\ &+4p_1^*c\left(\kappa t\delta_0(t;\alpha) + (\bar{\kappa}-\kappa)\delta_1(t)\right) + 4p_0^*p_1^*c^2\delta_2(t;\alpha),\end{aligned} \tag{3.10}$$

$$\begin{aligned}\mathbb{E}_1\left[\mathscr{T}_\alpha(t)^2\right] = &(p_0^*c_0 + p_1^*c_1)^2t^2 \\ &-4p_0^*c\left(\kappa t\delta_0(t;\alpha) + (\bar{\kappa}-\kappa)\delta_1(t)\right) + 4p_0^*p_1^*c^2\delta_2(t;\alpha).\end{aligned} \tag{3.11}$$

Here

$$\delta_0(t;\alpha) = \int_0^t \mathscr{E}_\alpha(-2\lambda s^\alpha)\mathrm{d}s, \quad \delta_1(t;\alpha) = \int_0^t s\mathscr{E}_\alpha(-2\lambda s^\alpha)\mathrm{d}s,$$

$$\delta_2(t;\alpha) = \iint\limits_{[0,t]^2} sE_\alpha(-2\lambda|s_1^\alpha - s_2^\alpha|)\mathrm{d}s_1\mathrm{d}s_2.$$

Proof. By definition (3.7),

$$\mathbb{E}_0\left[\mathscr{T}_\alpha(t)\right] = \int_0^t \mathbb{E}_0\left[c_{\varepsilon(E_\alpha(s))}\right]\mathrm{d}s, \tag{3.12}$$

$$\mathbb{E}_0\left[\mathscr{T}_\alpha(t)\right]^2 = 2\int_0^t \mathrm{d}s_1\int_{s_1}^t \mathbb{E}_0\left[\mathscr{T}(s_1)\cdot\mathscr{T}(s_2)\right]\mathrm{d}s_2. \tag{3.13}$$

Formula (3.8) follows directly from (3.12) after applying (2.2) (Proposition 2.1) and (2.14).

By (2.4), from (2.8) we obtain

$$\begin{aligned}\mathbb{E}_0\left[\mathscr{T}_\alpha(t)\right]^2 = 2\int_0^t \mathrm{d}s_1\int_{s_1}^t \Big(&\kappa^2 + 2p_1^*c\kappa\mathscr{E}_\alpha(-2\lambda s_1^\alpha) + 2p_1^*c\bar{\kappa}\mathscr{E}_\alpha(-2\lambda s_2^\alpha) \\ &+4p_0^*p_1^*c^2\mathbb{E}_0\left[\mathrm{e}^{-2\lambda(E_\alpha(s_2)-E_\alpha(s_1))}\right]\Big)\mathrm{d}s_2.\end{aligned} \tag{3.14}$$

For $s_2 = ks_1$ with $k > 1$ by applying self-similarity property of E_α (see (2.15)) we get $E_\alpha(s_2) \overset{d}{=} k^\alpha E_\alpha(s_1)$. Therefore,

$$\begin{aligned}\mathbb{E}_0\left[\mathrm{e}^{-2\lambda(E_\alpha(s_2)-E_\alpha(s_1))}\right] = \mathbb{E}_0\left[\mathrm{e}^{-2\lambda(-1+k^\alpha)E_\alpha(s_1)}\right] &= \mathscr{E}_\alpha(-2\lambda(-1+k^\alpha)s_1^\alpha) \\ &= \mathscr{E}_\alpha(-2\lambda(s_2^\alpha - s_1^\alpha)).\end{aligned} \tag{3.15}$$

Formula (3.10) follows from (3.14)-(3.15).

Formulae (3.9) and (3.11) are obtained using symmetric reasoning. □

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