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ON A CERTAIN STOCHASTIC ALGEBRAIC-DIFFERENTIAL INCLUSION WITH MEAN DERIVATIVES, SATISFYING THE RANK-DEGREE CONDITION

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Abstract. In this paper we study a Leontief-type inclusion with symmetric first-order mean derivative whose matrix pencil is regular and satisfies the rank-degree condition. We prove the existence of a perfect solution and a solution that minimizes some quality functional for such inclusion.

Introduction

The concept of mean derivatives was introduced by E. Nelson (see [1, 2, 3]) for the needs of his Stochastic Mechanics, which is an analogue of quantum mechanics. The equation of motion in Stochastic Mechanics was called the Newton-Nelson equation, this is a second-order stochastic equation with mean derivatives, where a special second-order mean derivative was used.

In [4, 5], a new method for studying dynamically distorted signals in electronic devices was developed based on algebraic differential equations called Leontief-type equations. Later, in the works of G.A. Sviridyuk and his school, as well as Yu.E. Gliklikh, E.Yu. Mashkov, etc., noise was taken into account, which was represented in terms of symmetric Nelson's mean derivatives (current velocities).

In this paper we study a Leontief-type inclusion with symmetric first-order mean derivative whose matrix pencil is regular and satisfies the rank-degree condition. We prove the existence of a perfect solution and a solution that minimizes some quality functional for such inclusion.

1. Some facts from matrix theory

Everywhere below we deal with processes, equations, etc., defined on some finite interval $[0, T]$.

We deal with an n-dimensional linear space $\mathbb{R}^n$, vectors from $\mathbb{R}^n$ and $n \times n$ matrices. Let two $n \times n$ constant matrices A and B be given, where A is singular and B is non-singular. An expression of the form $\lambda A + B$, where λ is a real parameter, is called a matrix pencil. The polynomial $\theta(\lambda) = \det(\lambda A + B)$ is called the characteristic polynomial of the pencil $\lambda A + B$. The pencil is called regular if

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its characteristic polynomial is not identically zero. If the matrix pencil $\lambda A + B$ is regular, then there exist non-degenerate linear operators P (acting from the left) and Q (acting from the right) that reduce the matrices A and B to the canonical quasi-diagonal form (see [6]).

In the canonical quasi-diagonal form, having chosen the desired order of the basis vectors, in the matrix PAQ first along the main diagonal there is the $d \times d$ identity matrix, and then along the main diagonal there are Jordan cells with zeros on the diagonal.

In PBQ in the lines corresponding to Jordan boxes, there is the unit matrix, and in the lines corresponding to the unit matrix in L there is a certain non-degenerate matrix J. Thus

$$P(\lambda A(t) + B(t))Q = \lambda \begin{pmatrix} I_d & 0 \\ 0 & N(t) \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & I_{n-d} \end{pmatrix}, \tag{1.1}$$

A non-degenerate pencil satisfies the rank-degree condition if

$$rank(A(t)) = \deg(\det(\lambda A(t) + B(t))). \tag{1.2}$$

If the pencil satisfies the rank-degree condition, then the formula (1.1) takes the form

$$P(t)(\lambda A(t) + B(t))Q(t) = \lambda \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & I_{n-d} \end{pmatrix}. \tag{1.3}$$

where J is non-singular, since B is also a non-singular matrix.

2. Preliminary information on mean derivatives

Consider a stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0, T]$, defined on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and such that $\xi(t)$ is an L_1-random variable for all t.

Each stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0, l]$, generates three families of the σ-subalgebra of the σ-algebra $\mathcal{F}$:
(i) the "past" $\mathcal{P}_t^\xi$ generated by the preimages of Borel sets from $\mathbb{R}^n$ under all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $0 \le s \le t$;
(ii) the "future" $\mathcal{F}_t^\xi$ generated by the preimages of the Borel sets from $\mathbb{R}^n$ under all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $t \le s \le T$;
(iii) the "presence"$\mathcal{N}_t^\xi$ generated by the preimages of the Borel sets from $\mathbb{R}^n$ under all mappings $\xi(t)$.

All families are assumed to be closed, i.e., containing all sets with probability 0.

Strictly speaking, almost surely (a.s.) sample trajectories of the process $\xi(t)$ are not differentiable for almost all t. Thus, the "classical" derivative exists only in the sense of generalized functions. To avoid using generalized functions, following Nelson (see, e.g., [1, 2, 3]) we define the notion of mean derivatives. Denote by E_t^ξ the conditional expectation of ξ with respect to the "presence" σ- algebra $\mathcal{N}_t^\xi$.

Definition 2.1. (i) The forward mean derivative $D\xi(t)$ of $\xi(t)$ at time $t \in [0, T)$ is an L_1-random variable of the form

$$D\xi(t) = \lim_{\Delta t \to +0} E_t^\xi\left(\frac{\xi(t + \Delta t) - \xi(t)}{\Delta t}\right) \tag{2.1}$$

where the limit is assumed to exist in $L_1(\Omega, \mathcal{F}, \mathsf{P})$ and $\Delta t \to +0$ means that Δt tends to 0, with $\Delta t > 0$.

(ii) The backward mean derivative $D_*\xi(t)$ of $\xi(t)$ at time $t \in (0, T]$ is an L_1-random variable

$$D_*\xi(t) = \lim_{\Delta t \to +0} E_t^{\xi}\left(\frac{\xi(t) - \xi(t - \Delta t)}{\Delta t}\right) \tag{2.2}$$

where the assumptions and definitions are the same as in (i).

(iii) The derivative $D_S = \frac{1}{2}(D + D_*)$ is called the symmetric mean derivative. The vector $v^{\xi}(t) = v^{\xi}(t, \xi(t)) = D_S\xi(t)$ is called the current velocity of the process $\xi(t)$

Note that the current velocity is a natural analogue of the physical velocity of a deterministic process.

Definition 2.2. [See e.g. [7]] For an L^1-stochastic process $\xi(t)$, $t \in [0, T]$, we introduce the quadratic mean derivative $D_2\xi(t)$, defined by the formula

$$D_2\xi(t) = \lim_{\triangle t \to +0} E_t^{\xi}\left(\frac{(\xi(t + \triangle t) - \xi(t))(\xi(t + \triangle t) - \xi(t))^*}{\triangle t}\right), \tag{2.3}$$

where $(\xi(t + \triangle t) - \xi(t))$ is a column vector and $(\xi(t + \triangle t) - \xi(t))^*$ is its conjugate, i.e., a row vector, and the limit is assumed to exist in $L^1(\Omega, \mathcal{F}, \mathsf{P})$.

It is easy to verify that for the Ito process $\xi(t) = \int_0^t a(s)ds + \int_0^t A(s)dw(s)$ the quadratic mean derivative takes the form $D_2\xi(t) = AA^*$.

Let $a(t, x)$ and $\alpha(t, x)$ be Borel measurable mappings from $[0, T] \times \mathbb{R}^n$ to $\mathbb{R}^n$ and to $\overline{S}_+(n)$, respectively, where $\overline{S}_+(n)$ is the set of symmetric positive-definite $n \times n$ matrices. We will call a system of the form

$$\begin{cases} D_S\xi(t) = a(t, \xi(t)), \\ D_2\xi(t) = \alpha(t, \xi(t)), \end{cases} \tag{2.4}$$

a first-order stochastic differential equation with symmetric mean derivative.

Let $\mathbf{a}(t, x)$ and $\boldsymbol{\alpha}(t, x)$ be multivalued mappings from $[0, T] \times \mathbb{R}^n$ to $\mathbb{R}^n$ and to $\overline{S}_+(n)$, respectively. We will call a system of the form

$$\begin{cases} D_S\xi(t) \in \mathbf{a}(t, \xi(t)), \\ D_2\xi(t) \in \boldsymbol{\alpha}(t, \xi(t)), \end{cases} \tag{2.5}$$

a first-order stochastic differential inclusion with symmetric mean derivative.

Definition 2.3. A differential inclusion (2.5) is said to have a solution with initial condition $\xi_0 \in \mathbb{R}^n$ if there exists a probability space and a random process $\xi(t)$ defined on it and taking values in $\mathbb{R}^n$ such that $\xi(0) = \xi_0$ and $\xi(t)$ satisfies the inclusion (2.5) a.s.

Definition 2.4 ([9])**.** A perfect solution of an inclusion is a stochastic process with continuous sample trajectories such that it is a solution in the sense of the definition (2.3), and the measure corresponding to it on the space of continuous curves is a weak limit of the measures generated by solutions of the sequence of diffusion-type Ito equations with continuous coefficients.

For an arbitrary multivalued mapping $F : X \to Y$ from one metric space to another, there are several different notions of continuity, which for single-valued mappings turn into the usual notion of continuity (see [8]). In this paper, we are interested in the notion of upper semicontinuity.

Definition 2.5. A multivalued mapping $F : X \to Y$ is called upper semicontinuous at a point $x \in X$ if for any open set $V \subset Y$ such that $F(x) \subset V$, there exists a neighborhood $U(x)$ of x such that $F(U(x)) \subset V$.

Definition 2.6. A multivalued mapping $F : X \to Y$ is called upper semicontinuous if it is upper semicontinuous at every point $x \in X$.

The so-called ε-approximations are useful for further investigation.

Definition 2.7. For a given $\varepsilon > 0$, a continuous single-valued mapping $f_\varepsilon : X \to Y$ is called an ε-approximation of a multivalued mapping $F : X \to Y$ if the graph of f as a set in $X \times Y$ lies in an ε-neighborhood of the graph of F.

It is proved that for upper semicontinuous multivalued mappings with convex closed values, there exist ε-approximations for any $\varepsilon > 0$.

3. Main result

Let A, B be arbitrary matrices, A be singular, B be non-singular. Assume that they form a pencil satisfying the rank-degree condition and let $F : [0, T] \to \mathbb{R}^n$ be an upper semicontinuous multivalued mapping with closed convex images.

Consider a stochastic algebraic-differential inclusion with symmetric mean derivatives and additional "noise" in the right-hand and left-hand sides of the following form

$$\begin{cases} D_S\left(A\xi(t) + \int_0^t w(s)ds\right) \in B\xi(t) + F(t) + w(t), \\ D_2\xi(t) = \Lambda(t). \end{cases} \tag{3.1}$$

Here $\Lambda = Q\begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix}Q^*$, where Q is the operator that reduces matrices A and B to canonical form, w is the Wiener process, ξ is the desired process.

We introduce the substitution $\eta(t) = Q^{-1}\xi(t)$, then the system (3.1) takes the form

$$\begin{cases} D_S\left(AQ\eta(t) + \int_0^t w(s)ds\right) \in BQ\eta(t) + F(t) + w(t), \\ D_2\eta(t) = \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix}. \end{cases} \tag{3.2}$$

We multiply both parts of the first line of inclusion by P on the left:

$$\begin{cases} PAQD_S\eta(t) + Pw(t) \in PBQ\eta(t) + PF(t) + Pw(t), \\ D_2\eta(t) = \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix}. \end{cases} \tag{3.3}$$

Now from (1.3) it follows that (3.3) can be written as

$$\begin{cases} \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} D_S\eta(t) \in \begin{pmatrix} J & 0 \\ 0 & I_{n-d} \end{pmatrix} \eta(t) + PF(t), \\ D_2\eta(t) = \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix}. \end{cases} \tag{3.4}$$

Denote the multivalued mapping $PF(t)$ by $\tilde{F}$ and note that (3.4) can be split into two independent systems

$$\begin{cases} D_S\eta^{(1)}(t) \in J\eta^{(1)}(t) + \tilde{F}^{(1)}(t) \\ D_2\eta^{(1)}(t) = I_d \end{cases} \tag{3.5}$$

in $\mathbb{R}^d$ and

$$\begin{cases} \eta^{(2)}(t) \in -\tilde{F}^{(2)}(t) \\ D_2\eta^{(2)}(t) = 0 \end{cases} \tag{3.6}$$

in $\mathbb{R}^{n-d}$.

The second inclusion obviously has a perfect solution, this is any deterministic process $f(t)$, where $f(t) \in -\tilde{F}^2(t)$ for any t.

In the first inclusion, we note that $\tilde{F}^{(1)}(t)$ has ε-approximations for any $\varepsilon > 0$, since it is an upper semicontinuous set-valued mapping with closed convex values. Let $Jx + \tilde{F}^{(1)}(t)$ be a uniform set-valued bounded vector field on a flat d-dimensional torus $\mathcal{T}^d$, which can be denoted by $\mathbf{v}(t,x)$. Then this inclusion satisfies the condition of [10, Theorem 17.1] and hence has a perfect solution.

Thus, the following theorem is proved

Theorem 3.1. *Under all the conditions described above, any sequence of ε-approximations $\varepsilon > 0$ of the multivalued mapping $F(t)$ generates a perfect solution of the algebraic-differential stochastic inclusion (3.1) with any initial condition $\xi(0) = \xi_0$.*

Let g be a continuous bounded numerical function defined on $\mathbb{R} \times \mathbb{R}^n$. For solutions of equation (3.1), we consider the cost criterion

$$G(\xi(\cdot)) = \int_0^T \mathbb{E}[g(t, \xi(t))]dt \tag{3.7}$$

Theorem 3.2. *Among the perfect solutions of the inclusion (3.1), there is a solution ξ on which the value of G is minimal.*

The proof is quite analogous to that of [10, Theorems 14.2 and 17.2].

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