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SOME QUASILINEAR MODELS OF COMPLEX SYSTEMS WITH RANDOM PRIORITIES. THE CASE OF TWO DEPENDENT PRIORITIES

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Abstract. There is an economic system (for example, a budget organization). The internal requirements of the system are expressed by a function. The economic system is not independent and is affected by external "optimizers" (for example, various ministries). We will write the target function of the arbitrator as $F = E(F_1^{\alpha_1} F_2^{\alpha_2})$. Function $F(2)$ expresses the internal requirements of the system, and function $F(1)$ the requirements of some external "optimizers" to this system. The internal goals of the system and the goals of the "optimizers" do not coincide in most cases. There is a certain arbitrator (regulator) that can influence both the development of the system itself and the "optimizers". The main goal of the arbitrator is to set priorities that will ensure the successful functioning of all divisions of the economic system as a whole. Within the framework of the proposed model, the necessary conditions for the existence of stationary points of the objective function are obtained and its theorem of the existence and uniqueness of the global maximum of the objective function F is proven.

1. Introduction

This article is a continuation of works [**?**]–[**?**], which were devoted to optimization of management of an economic system with a finite number of interconnected institutions. Management is carried out by an external arbitrator, who sets priorities based on various expert assessments. The main goal of the arbitrator is to set priorities that will ensure the successful functioning of all units of the economic system as a whole. This work is devoted to finding the optimal solution to the problem of managing an economic system with several random priorities.

Basic designations. In space R^n, we consider non-negative non-zero continuous functions $F_i(x_1, x_2, ...x_n)$, twice continuously differentiable on open sets $B_i = \{F_i > 0\}$, respectively, where $i = 1, 2, ..., m$. In this case $\bigcap_{i=1}^{m} B_i \neq \emptyset$, and through $B_i^c = \{F_i \leq 0\}$ we denote their complements in what follows. Functions $F_i (i = 1, 2, ..., m)$ will be interpreted as multidirectional objective functions. The objective function F_m expresses the internal requirements of the system, and the

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functions $F_1, F_2, ..., F_{m-1}$ formulate the requirements of some external optimizers to this system. We will construct the arbitrators objective function as follows. Let $\alpha_i = \alpha_i(\omega)$ be random variables defined on the probability space $(\Omega, \mathcal{F}, P)$. It is natural to assume that $P(0 < \alpha_i < 1) > 0, i = 1, 2, ..., m$. The objective function of the arbitrator can be written as follows:

$$F = E(F_1^{\alpha_1} F_2^{\alpha_2} ... F_m^{\alpha_m}), \tag{1.1}$$

where E is the symbol of mathematical expectation for probability. The indicators $\alpha_1, \alpha_2, .., \alpha_m$ are called priorities. Multiplication in formula **??** assumes that the arbiter, influencing the internal structure of the system and external optimizers, strives to ensure the efficient operation of the entire system, that is, to maximize the objective function F. It is natural to assume that the expression $F_i^0 = 1$. This means that at zero priority, the objective function of the arbitrator does not depend on the objective function of the j -th participant of the economic system at moments $\omega \in \Omega/$. And the trivial situation when $\alpha_i(\omega) = 0, \forall \omega \in \Omega$ is not considered by us, since economically this means that the i -th participant leaves the system altogether. In these models, we will represent as functions of the "quasi-linear" type:

$$F_i(x) = \left(\sum_{k=1}^{n} a_k^i x_k + b_i \right) \cdot I_{\left\{ \sum\limits_{k=1}^{n} a_k^i x_k + b_i > 0 \right\}}, i = 1, 2, ..., m, \tag{1.2}$$

where I_A there is a set indicator A. It is natural to assume that the target functions of the participants in the economic system do not coincide. Otherwise, participants with the same target functions can be combined into one and a simpler model can be obtained.

2. Model with two dependent priorities

In this paper we will consider in more detail the model of a system with two priorities. Let $\alpha_i = \alpha_i(\omega), i = 1, 2$ dependent random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Then the objective function of the arbitrator is:

$$F = E(F_1^{\alpha_1} F_2^{\alpha_2}), \tag{2.1}$$

The first question that interests us is the conditions for the existence of stationary points of the objective function.

Proposition 2.1. *In order for the function $F(x)$ defined by equalities* **??-??** *to have stationary points, the following conditions must be met: vectors $\overline{a_i} = (a_1^i, a_2^i, ..., a_m^i)$, $i = 1, 2$ must be linearly dependent: $\overline{a_2} = -c\overline{a_1}$, where $c > 0$ and there must be points $\boldsymbol{x} \in (B_1 \cap B_2)$ which the equality is satisfied $\frac{E[\alpha_1 F_1^{\alpha_1 - 1} F_2^{\alpha_2}]}{E[\alpha_2 F_1^{\alpha_1} F_2^{\alpha_2 - 1}]} = c$.*

The proof is trivial.

Let's $\mathbf{x} \in (B_1 \cap B_2)$ consider the function $g(\mathbf{x}) = E[\alpha_1 {F_1}^{\alpha_1 - 1} F_2^{\alpha_2}] - cE[\alpha_2 F_1^{\alpha_1} F_2^{\alpha_2 - 1}]$.

From this point on, we will assume that the conditions of Proposition **??** are satisfied. Let us introduce new notations $t = \sum\limits_{k=1}^{n} a_k^i x_k$ Then the objective function will take the form:

$F(t)=E\left[\left(\sum\limits_{k=1}^{n} a_k^1 x_k + b_1\right)^{\alpha_1}\left(\sum\limits_{k=1}^{n} a_k^2 x_k + b_2\right)^{\alpha_2}\right]=E\left[\left(\sum\limits_{k=1}^{n} t + b_1\right)^{\alpha_1}\left(\sum\limits_{k=1}^{n} -ct + b_2\right.\right.$.

$g(t) = E[\alpha_1(F_1(t))^{\alpha_1-1}(F_2(t))^{\alpha_2}] - cE[\alpha_2(F_1(t))^{\alpha_1}(F_2(t))^{\alpha_2-1}]$.

Since $\mathbf{x} \in B_1 \cap B_2$ the inequalities are satisfied for $\sum\limits_{k=1}^{n} a_k^i x_k + b_i > 0, i = 1, 2$, the domain of definition of the functions $F(t)$ and $g(t)$ is described by the inequalities $\begin{cases} t + b_1 > 0, \\ -ct + b_2 > 0; \end{cases} \Leftrightarrow -b_1 < t < \dfrac{b_2}{c}$. The condition that the domain is non-empty $B_1 \cap B_2$ is equivalent to the condition $-b_1 < \dfrac{b_2}{c}$.

3. Main result

Let us move on to the question of the existence and uniqueness of the global maximum of the arbitrator's objective function. Let's introduce the notation:

$$A_{00} = \{\alpha_1 = 0\} \cap \{\alpha_2 = 0\},\ A_{01} = \{\alpha_1 = 0\} \cap \{\alpha_2 = 1\},$$
$$A_{10} = \{\alpha_1 = 1\} \cap \{\alpha_2 = 0\},\ A_{11} = \{\alpha_1 = 1\} \cap \{\alpha_2 = 1\},$$
$$A_{02} = \{\alpha_1 = 0\} \cap \{0 < \alpha_2 < 1\},\ A_{20} = \{0 < \alpha_1 < 1\} \cap \{\alpha_2 = 0\},$$
$$A_{12} = \{\alpha_1 = 1\} \cap \{0 < \alpha_2 < 1\},\ A_{21} = \{0 < \alpha_1 < 1\} \cap \{\alpha_2 = 1\},$$
$$A_{22} = \{0 < \alpha_1 < 1\} \cap \{0 < \alpha_2 < 1\},$$
$$p_{ij} = P(A_{ij}), i, j = 0, 1, 2.$$

Theorem 3.1. *Let the objective function $F(\boldsymbol{x})$ have stationary points and one of the conditions is satisfied $p_{22} \neq 0$ or $\begin{cases} p_{22} = 0, \\ p_{20} + p_{21} > 0, \\ p_{02} + p_{12} > 0. \end{cases}$ Then the equation $g(t) = 0$ has a single root $t = t^*$ $(-b_1 < t^* < \frac{b_2}{c})$ and all points of the hyperplane $\sum\limits_{k=1}^{n} a_k^i x_k = t^*$ are points of global maximum of the function $F(\boldsymbol{x})$.*

Proof. Let us prove that the function is continuous $g(t) = F'(t)$ strictly monotonically decreases on the interval $\left(-b_1; \dfrac{b_2}{c}\right)$. Differentiating under the integral sign, we obtain

$$g'(t) = E[\alpha_1(\alpha_1 - 1)(t + b_1)^{\alpha_1-2}(-ct + b_2)^{\alpha_2} -$$
$$-2c\alpha_1\alpha_2(t + b_1)^{\alpha_1-1}(-ct + b_2)^{\alpha_2-1} + +c^2\alpha_2(\alpha_2 - 1)(t + b_1)^{\alpha_1}(-ct + b_2)^{\alpha_2-2}].$$

Since $t + b_1 > 0$ and $-ct + b_2 > 0$, the expression under the sign of mathematical expectation is less than zero. Therefore, $g'(t) < 0$ when $-b_1 < t < \dfrac{b_2}{c}$, then $g(t)$ strictly monotonically decreases on the interval $\left(-b_1; \dfrac{b_2}{c}\right)$. Let's write this function in the form

$$g(t) = E\left(\alpha_1(t+b_1)^{\alpha_1-1}(-ct+b_2)^{\alpha_2}I_{A_{22}}\right) - cE\left((t+b_1)^{\alpha_1}\alpha_2(-ct+b_2)^{\alpha_2-1}I_{A_{22}}\right) + \\ +p_{10} + (b_2 - b_1c - 2ct)p_{11} - cp_{01} - cE\left(\alpha_2(-ct+b_2)^{\alpha_2-1}I_{A_{02}}\right) + \\ +E\left((-ct+b_2)^{\alpha_2}I_{A_{12}}\right) - c(t+b_1)E\left(\alpha_2(-ct+b_2)^{\alpha_2-1}I_{A_{12}}\right) + \\ +E\left(\alpha_1(t+b_1)^{\alpha_1-1}I_{A_{20}}\right) + E\left(\alpha_1(t+b_1)^{\alpha_1-1}I_{A_{21}}\right)(-ct+b_2) - cE\left((t+b_1)^{\alpha_1}I_{A_{21}}\right)$$

1) Let's consider the case when $p_{22} \neq 0$. Then the function $g(t)$ will look like:

$$g(t) = E\left(\alpha_1(t+b_1)^{\alpha_1-1}(-ct+b_2)^{\alpha_2}I_{A_{22}}\right) + E\left(\alpha_1(t+b_1)^{\alpha_1-1}I_{A_{20}}\right) + \\ +(-ct+b_2)E\left(\alpha_1(t+b_1)^{\alpha_1-1}I_{A_{21}}\right) - c(E\left((t+b_1)^{\alpha_1}\alpha_2(-ct+b_2)^{\alpha_2-1}I_{A_{22}}\right) + \\ +(t+b_1)E\left(\alpha_2(-ct+b_2)^{\alpha_2-1}I_{A_{12}}\right) + E\left((t+b_1)^{\alpha_1}I_{A_{21}}\right)) + p_{10} + (b_2 - b_1c - 2ct)p_{11} - \\ -cp_{01} - cE\left(\alpha_2(-ct+b_2)^{\alpha_2-1}I_{A_{02}}\right) + E\left((-ct+b_2)^{\alpha_2}I_{A_{12}}\right)$$

Since $(t+b_1)^{\alpha_1}\alpha \downarrow +0$ for $t \downarrow -b_1$ then, according to Levi's theorem, we obtain that for $t \downarrow -b_1$: $E\left(\alpha_1(t+b_1)^{\alpha_1-1}(-ct+b_2)^{\alpha_2}I_{A_{22}}\right) + E\left(\alpha_1(t+b_1)^{\alpha_1-1}I_{A_{20}}\right) + +(-ct+b_2)E\left(\alpha_1(t+b_1)^{\alpha_1-1}I_{A_{21}}\right) \uparrow +\infty$ and $E\left((t+b_1)^{\alpha_1}\alpha_2(-ct+b_2)^{\alpha_2-1}I_{A_{22}}\right) +$

$+(t+b_1)E\left(\alpha_2(-ct+b_2)^{\alpha_2-1}I_{A_{12}}\right) + E\left((t+b_1)^{\alpha_1}I_{A_{21}}\right) \downarrow 0$. We got that $g(t) \uparrow +\infty$ at $t \downarrow -b_2$.

Similarly proved that $g(t) \downarrow -\infty$ for $t \uparrow \frac{b_2}{c}$.

2) Now let's consider the case when $\begin{cases} p_{22} = 0, \\ p_{20} + p_{21} > 0, \\ p_{02} + p_{12} > 0. \end{cases}$

Then the function $g(t)$ will look like:

$$g(t) = E\left(\alpha_1(t+b_1)^{\alpha_1-1}I_{A_{20}}\right) + (-ct+b_2)E\left(\alpha_1(t+b_1)^{\alpha_1-1}I_{A_{21}}\right) - \\ -c\left((t+b_1)E\left(\alpha_2(-ct+b_2)^{\alpha_2-1}I_{A_{12}}\right) + E\left((t+b_1)^{\alpha_1}I_{A_{21}}\right)\right) + p_{10} + \\ +(b_2 - b_1c - 2ct)p_{11} - cp_{01} - cE\left(\alpha_2(-ct+b_2)^{\alpha_2-1}I_{A_{02}}\right) + E\left((-ct+b_2)^{\alpha_2}I_{A_{12}}\right)$$

Since $(t+b_1)^{\alpha_1}\alpha \downarrow +0$ for $t \downarrow -b_1$ then, according to Levi's theorem, we obtain that for $t \downarrow -b_1$: $E\left(\alpha_1(t+b_1)^{\alpha_1-1}I_{A_{20}}\right) + (-ct+b_2)E\left(\alpha_1(t+b_1)^{\alpha_1-1}I_{A_{21}}\right) \uparrow +\infty$ and $(t+b_1)E\left(\alpha_2(-ct+b_2)^{\alpha_2-1}I_{A_{12}}\right) + E\left((t+b_1)^{\alpha_1}I_{A_{21}}\right) \downarrow 0$. Similarly proved that $g(t) \downarrow -\infty$ for $t \uparrow \frac{b_2}{c}$. Thus $g(t)$, is a continuous strictly decreasing function on the interval $\left(-b_1; \frac{b_2}{c}\right)$, taking all real values on it. Consequently, $g(t) = 0$ the equation has a unique root $t = t^*$. Consequently, $F"(t) = g'(t) < 0$, that is, $F(t)$ is a continuous strictly convex upward function on the interval $\left(-b_1; \frac{b_2}{c}\right)$. We obtain that $t^*$is the only local maximum point of the function $F(t)$, and, therefore, the only global maximum point on $\left(-b_1; \frac{b_2}{c}\right)$. It follows that all points of the hyperplane $\sum\limits_{k=1}^{n} a_k^i x_k = t^*$ are global maximum points of the function $F(x)$.

4. Special cases of the model

We also investigated situations where the conditions of Theorem **??** are not met. The results of the study are presented in tabular:

$\begin{cases} p_{22} = 0, \\ p_{20} + p_{21} = 0, \\ p_{02} + p_{12} = 0. \end{cases}$	$F(t)$ has a global max $t^* \Leftrightarrow \begin{cases} p_{11} > 0, \\ \lvert c\dfrac{p_{01} - p_{10}}{p_{11}}\rvert < b_1 c + b_2. \end{cases}$
$\begin{cases} p_{22} = 0, \\ p_{20} + p_{21} = 0, \\ p_{02} + p_{12} > 0. \end{cases}$	$F(t)$ has a global max $t^* \Leftrightarrow p_{10} + p_{11}(b_2 + b_1 c) - cp_{01} -$ $-cE\left(\alpha_2 (b_2 + b_1 c)^{\alpha_2 - 1} I_{A02}\right) + E\left((b_2 + b_1 c)^{\alpha_2} I_{A12}\right) > 0$
$\begin{cases} p_{22} = 0, \\ p_{20} + p_{21} > 0, \\ p_{02} + p_{12} = 0. \end{cases}$	$F(t)$ has a global max $t^* \Leftrightarrow p_{10} - p_{11}(b_2 + b_1 c) - cp_{01} +$ $+E\left(\alpha_1 \left(\dfrac{b_2}{c} + b_1\right)^{\alpha_1 - 1} I_{A_{20}}\right) - cE\left(\left(\dfrac{b_2}{c} + b_1\right)^{\alpha_1} I_{A21}\right) > 0$

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