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ANALYSIS OF THE STOCHASTIC NAVIER – STOKES SYSTEM WITH A MULTIPOINT INITIAL-FINAL VALUE CONDITION

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Abstract. Recently, the theory of stochastic equations has been actively developing. Here it is worth noting the classical direction of research by Ito – Stratonovich – Skorokhod.Its main problem is to overcome the difficulties associated with the differentiation of a non-differentiable (in "the usual sense") Wiener process. It is also necessary to note the approach of I.V. Melnikova, within the framework of which stochastic equations are considered in Schwarz spaces using the generalized derivative. Our research will use methods and results of the theory, which is based on the concept of the Nelson – Glicklich derivative. Most studies consider the Cauchy problem for stochastic equations. In this article, instead of the Cauchy condition, it is proposed to consider a multipoint initial-final value condition. The obtained abstract results are used to analyze the solvability of the stochastic Navier-Stokes system, which models the dynamics of the velocity and pressure of a viscous incompressible fluid. It is considered with a no-slip boundary condition and a multipoint initial-final value condition. The main result of the article is the proof of the solvability of the posed problem.

Introduction

Consider the system of equations

$$v_t = \nu\nabla^2 v - (v \cdot \nabla)v - \nabla p + f, \quad \nabla \cdot v = 0, \tag{0.1}$$

modeling the dynamics of a viscous incompressible fluid, was obtained more than a century ago. Here the vector function $v = (v_1, v_2, ..., v_m)$, $v_l = v_l(x, t)$, corresponds to the fluid velocity, the function $p = p(x, t)$ corresponds to the pressure, the parameter $\nu \in \mathbb{R}_+$ characterizes the viscosity. And $\Omega \subset \mathbb{R}^m$, $m = \{2, 3\}$, is a bounded domain with the boundary $\partial\Omega$ of class C^∞.

Over the past time, the system of equations (0.1) has been studied in various aspects. Let us note here the fundamental monographs of O.A. Ladyzhenskaya [1] and R. Temam [2]. However, the question of the existence of solutions to the

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Cauchy – Dirichlet problem for the system of equations (0.1) has not yet been resolved. The problem of the existence of solutions to this problem turned out to be so difficult that it was included in the list of the most difficult mathematical problems of the current century, and a reward of one million dollars has been appointed for its solution.

To solve the system of equations (0.1) we consider "the condition no-slip" to the boundary of the domain

$$u(x,t) = 0,\ (x,t) \in \partial\Omega \times \mathbb{R}. \tag{0.2}$$

Let us recall how the system (0.1) is obtained. As is known, Newton's rheological relation, modeling the dynamics of viscous incompressible fluids has the form [3], [4], [5]

$$\sigma = 2\nu D - p\mathbb{I}. \tag{0.3}$$

Here σ and D are stress and strain rate tensors, respectively, $\nu \in \mathbb{R}_+$ is the viscosity coefficient, $\mathbb{I}$ is the identity matrix, p characterizes the pressure. After substituting (0.3) into the equations of motion of a continuous incompressible medium in Cauchy form

$$v_t = \nabla \cdot \sigma, \quad \nabla \cdot v = 0, \tag{0.4}$$

we obtain the above famous system of Navier – Stokes equations (0.1).

Let's consider the linear abstract model

$$L\dot{u} = Mu + f, \tag{0.5}$$

in Banach spaces $\mathcal{U}$ and $\mathcal{F}$, and the operators $L \in \mathcal{L}(\mathcal{U};\mathcal{F})$ (i.e. linear and continuous), $M \in \mathcal{C}l(\mathcal{U};\mathcal{F})$ (i.e. linear, closed and densely defined).

The work is devoted to the study of the stochastic linear Sobolev type equation

$$L \overset{\circ}{\eta} = M\eta + N\omega, \tag{0.6}$$

where $\eta = \eta(t)$ is the required one, and $\omega = \omega(t)$ is a given stochastic **K**-process (**K**-"noise"), with multipoint initial-final value condition

$$\lim_{t\to 0+} P_0(\eta(t) - \xi_0) = 0, \quad P_j(\eta(\tau_j) - \xi_j) = 0, \quad j = \overline{1,m}. \tag{0.7}$$

A detailed description will be given in the second paragraph.

The article, in addition to the introduction and bibliography, contains three parts. In the first part, spaces of differentiable random processes with values in a separable Hilbert space are constructed. Moreover, by derivative we mean the Nelson – Gliklich derivative [6], [7], [8], [9]. We call random processes that have Nelson – Glicklich derivatives differentiable "noises" [10], [11], [12], [13]. The second part of the article presents results on the solvability of the stochastic problem (0.6), (0.7) under the condition that the operator M, $p \in \{0\} \cup \mathbb{N} \equiv \mathbb{N}_0$, is (L,p)-bounded, and a condition guaranteeing the existence of relative spectral projectors P_j, $j = \overline{0,n}$, [14]. These results generalize and develop the abstract results of the works [10], [11], [12], [13]. The third part contains applications of the obtained abstract results for the stochastic Navier – Stokes system. The list of references does not pretend to be complete and reflects only the tastes and preferences of the authors.

1. Spaces of differentiable "noises"

Let $\Omega \equiv (\Omega, \mathcal{A}, \mathbf{P})$ is a complete probability space with a probability measure $\mathbf{P}$ associated with the σ-algebra $\mathcal{A}$ of subsets of the set Ω, and $\mathbb{R}$ is a set of real numbers, endowed with a Borel σ-algebra. A measurable mapping $\xi : \Omega \to \mathbb{R}$ is called *random variable.* A set of random variables whose mathematical expectation is zero, and the variance is finite, forms the Hilbert space $\mathbf{L_2} = \{\xi : \mathbf{E}\xi = 0,\ \mathbf{D}\xi < +\infty\}$ with the scalar product $(\xi_1, \xi_2) = \mathbf{E}\xi_1\xi_2$ and the norm $\|\xi\|^2_{\mathbf{L_2}} = \mathbf{D}\xi$. Note that in $\mathbf{L_2}$ the orthogonality of the vectors ξ and η (i.e. $(\xi, \eta) = 0$) is equivalent to correlated random variables ξ and η. Indeed, $0 = \mathrm{cov}(\xi, \eta) = \mathbf{E}\xi\eta = (\xi, \eta) = 0$.

Let us take the set $\mathfrak{I} \subset \mathbb{R}$ and consider two mappings: $f : \mathfrak{I} \to \mathbf{L_2}$, which each $t \in \mathfrak{I}$ assigns a random variable $\xi \in \mathbf{L_2}$, and $g : \mathbf{L_2} \times \Omega \to \mathbb{R}$, which assigns to each pair (ξ, ω) point $\xi(\omega) \in \mathbb{R}$. Display $\eta : \mathfrak{I} \times \Omega \to \mathbb{R}$, which has the form $\eta = \eta(t, \omega) = g(f(t), \omega)$, we call *(one-dimensional) stochastic process.* For every fixed $t \in \mathfrak{I}$ value of the stochastic process $\eta = \eta(t, \cdot)$ is a random variable, i.e. $\eta(t, \cdot) \in \mathbf{L_2}$, which we call *cross section* of the stochastic process at point $t \in \mathfrak{I}$. For each fixed $\omega \in \Omega$ the function $\eta = \eta(\cdot, \omega)$ is called *(selective) trajectory* of a random process corresponding to the elementary outcome $\omega \in \Omega$. Trajectories are also called *realizations* or *sample functions* of a random process. Usually, when this does not lead to ambiguity, the dependence of $\eta(t, \omega)$ on ω is not indicated and the random process is simply denoted by $\eta(t)$.

Considering $\mathfrak{I} \subset \mathbb{R}$ to be an interval, we call the stochastic process $\eta = \eta(t)$, $t \in \mathfrak{I}$, *continuous,* if a.s. (almost surely) all its trajectories are continuous (i.e. for almost all $\omega \in \mathcal{A}$ trajectories $\eta(\cdot, \omega)$ are continuous functions). A set of continuous stochastic processes forms Banach space, which we denote by the symbol $\mathbf{C}(\mathfrak{I}; \mathbf{L_2})$ with the norm $\|\eta\|_{\mathbf{CL_2}} = \sup\limits_{t \in \mathfrak{I}} (\mathbf{D}\eta(t, \omega))^{1/2}$. Let $\mathcal{A}_0$ be a σ-subalgebra σ-algebras $\mathcal{A}$. Let us construct the subspace $\mathbf{L^0_2} \subset \mathbf{L_2}$ random variables measurable with respect to $\mathcal{A}_0$. Let us denote by $\Pi : \mathbf{L_2} \to \mathbf{L^0_2}$ – ortho projector. Let $\xi \in \mathbf{L_2}$, then $\Pi\xi$ is called *conditional mathematical expectation* of the random variable ξ and is denoted by the symbol $\mathbf{E}(\xi|\mathcal{A}_0)$. Let us fix $\eta \in \mathbf{C}(\mathfrak{I}; \mathbf{L_2})$ and $t \in \mathfrak{I}$, by $\mathcal{N}^\eta_t$ we denote the *σ-algebra generated by random variable* $\eta(t)$, and denote $\mathbf{E}^\eta_t = \mathbf{E}(\cdot|\mathcal{N}^\eta_t)$.

Example 1.1. Wiener process describing Brownian motion in the Einstein – Smoluchowski model (see [7])

$$\beta(t, \omega) = \sum_{k=0}^{\infty} \xi_k(\omega) \sin \frac{\pi}{2}(2k+1)t,\ \ t \in \{0\} \cup \mathbb{R}_+,$$

is a continuous stochastic process. Here the coefficients $\{\xi_k = \xi_k(\omega)\} \subset \mathrm{L}_2$ are pairwise uncorrelated Gaussian random variables such that $\mathbf{D}\xi^2_k = \left[\frac{\pi}{2}(2k+1)\right]^{-2}$, $k \in \mathbb{N}_0$.

Definition 1.2. [6], [7] Let $\eta \in \mathbf{C}(\mathfrak{I}; \mathbf{L_2})$. By the *Nelson – Glicklich derivative* $\overset{\circ}{\eta}$ *stochastic process* η *at point* $t \in \mathfrak{I}$ a random variable

$$\overset{\circ}{\eta}(t, \cdot) = \frac{1}{2} \lim_{\Delta t \to 0+} \mathbf{E}^\eta_t \left(\frac{\eta(t + \Delta t, \cdot) - \eta(t, \cdot)}{\Delta t} \right) + \frac{1}{2} \lim_{\Delta t \to 0+} \mathbf{E}^\eta_t \left(\frac{\eta(t, \cdot) - \eta(t - \Delta t, \cdot)}{\Delta t} \right),$$

is called, if the limit exists in the sense of a uniform metric on $\mathbb{R}$.

If the Nelson – Glicklich derivatives $\overset{\circ}{\eta}(t,\cdot)$ of the stochastic process $\eta(t,\cdot)$ exist in all (or almost all) points of the interval $\Im$, then we talk about the existence of the Nelson – Glicklich derivative $\overset{\circ}{\eta}(t,\cdot)$ on $\Im$ (a.s. on $\Im$.)

Set of continuous stochastic processes having continuous Nelson – Glicklich derivatives $\overset{\circ}{\eta}$ forms a Banach $\mathbf{C}^1(\Im;\mathbf{L_2})$ space with the norm

$$\|\eta\|_{\mathbf{C}^1\mathbf{L}_2} = \sup_{t\in\Im}\Big(\mathbf{D}\eta(t,\omega) + \mathbf{D}\overset{\circ}{\eta}(t,\omega)\Big)^{1/2}.$$

We further define by induction the Banach spaces $\mathbf{C}^l(\Im;\mathbf{L_2})$, $l\in\mathbb{N}$, stochastic processes whose trajectories a.s. differentiable with respect to Nelson – Gliklich on $\Im$ up to order $l\in\mathbb{N}_0$ inclusive [15]. The norms in them are given by the formulas $\|\eta\|_{\mathbf{C}^l\mathbf{L}_2} = \sup\limits_{t\in\Im}\left(\sum\limits_{k=0}^{l}\mathbf{D}\overset{\circ}{\eta}{}^{(k)}(t,\omega)\right)^{1/2}$. Here we will consider the zero-order Nelson – Gliklich derivative to be the original random process, i.e. $\overset{\circ}{\eta}{}^{(0)}\equiv\eta$, and under the Nelson – Gliklich derivative are of order k we will understand the Nelson – Gliklich derivative of the first order from the Nelson – Gliklich derivative of order $k-1$. For brevity we will call *spaces of differentiable "noises"* (see [10], [11], [12], [13]).

Example 1.3. In [7, 15] it is shown that $\beta\in\mathbf{C}^l(\mathbb{R}_+;\mathbf{L_2})$, $l\in\mathbb{N}_0$, and $\overset{\circ}{\beta}(t) = \dfrac{\beta(t)}{2t}$, $t\in\mathbb{R}_+$.

Thus, spaces of random variables $\mathbf{L}_2$ and spaces of differentiable "noises" $\mathbf{C}^l(\Im;\mathbf{L}_2)$, $l\in\mathbb{N}_0$. Let's move on to constructing a space *of random* $\mathbf{K}$*-variables.* Take $\mathfrak{H}$ is a separable Hilbert space with an orthonormal basis $\{\varphi_k\}$, a monotone sequence $\mathbf{K}=\{\lambda_k\}\subset\mathbb{R}_+$ such that that $\sum\limits_{k=1}^{\infty}\lambda_k^2<+\infty$, as well as a sequence $\{\xi_k\}=\xi_k(\omega)\subset\mathbf{L}_2$ of random variables such that that $\|\xi_k\|_{\mathbf{L}_2}\le C$, for all $C\in\mathbb{R}_+$, for all $k\in\mathbb{N}$.

Let us construct a $\mathfrak{H}$*-valued random* $\mathbf{K}$*-variable* $\xi(\omega)=\sum\limits_{k=1}^{\infty}\lambda_k\xi_k(\omega)\varphi_k$. Completion of the linear hull of the set $\{\lambda_k\xi_k\varphi_k\}$ by the norm

$$\|\eta\|^2_{\mathbf{H_K}\mathbf{L}_2} = \left(\sum_{k=1}^{\infty}\lambda_k^2\mathbf{D}\xi_k\right)^{1/2}$$

is called the *space of ($\mathfrak{H}$-valued) random* $\mathbf{K}$*-variables* and is denoted by the symbol $\mathbf{H_K L}_2$. How easy it is to see the space $\mathbf{H_K L}_2$ is the Hilbert space, and the random $\mathbf{K}$-variable constructed above $\xi=\xi(\omega)\in\mathbf{H_K L_2}$. Likewise, Banach space *($\mathfrak{H}$-valued)* $\mathbf{K}$*- "noises"* $\mathbf{C}^l(\Im;\mathbf{H_K L}_2)$, $l\in\mathbb{N}_0$, we define as the completion of the linear hull of the set $\{\lambda_k\eta_k\varphi_k\}$ by the norm

$$\|\eta\|^2_{\mathbf{C}^l\mathbf{H_K}\mathbf{L}_2} = \sup_{t\in\Im}\left(\sum_{k=1}^{\infty}\lambda_k^2\sum_{m=1}^{l}\mathbf{D}\overset{\circ}{\eta}{}_k^m\right)^{1/2},$$

where the sequence of "noises" $\{\eta_k\} \subset \mathbf{C}^l(\mathfrak{I};\mathbf{L}_2)$, $l \in \mathbb{N}_0$. As is easy to see, the vector $\eta(t,\omega) = \sum_{k=1}^{\infty} \lambda_k \eta_k(t,\omega)\varphi_k$ lies in the space $\mathbf{C}^l(\mathfrak{I};\mathbf{H_K L}_2)$, if a sequence of vectors $\{\eta_k\} \subset \mathbf{C}^l(\mathfrak{I};\mathbf{L}_2)$ and all their Nelson – Glicklich derivatives up to order $l \in \mathbb{N}_0$ inclusive are uniformly bounded by the norm $\|\cdot\|_{\mathbf{C}^l\mathbf{L}_2}$.

Example 1.4. Vector lying in all spaces $\mathbf{C}^l(\mathbb{R}_+;\mathbf{H_K L}_2)$, $l \in \mathbb{N}_0$,

$$W_{\mathbf{K}}(t,\omega) = \sum_{k=1}^{\infty} \lambda_k \beta_k(t,\omega)\varphi_k,$$

where $\{\beta_k\} \subset \mathbf{C}^l(\mathfrak{I};\mathbf{L}_2)$ is sequence of Brownian motions, called *($\mathfrak{H}$-valued) Wiener* $\mathbf{K}$*-process.*

2. The multipoint initial-final value condition

Let $\mathcal{U}$ and $\mathcal{F}$ be Banach spaces, operator $L \in \mathcal{L}(\mathcal{U};\mathcal{F})$ (i.e. linear and continuous), and the operator $M \in \mathcal{C}l(\mathcal{U};\mathcal{F})$ (i.e. a linear, closed and densely defined). Consider the L-resolvent set $\rho^L(M) = \{\mu \in \mathbb{C} : (\mu L - M)^{-1} \in \mathcal{L}(\mathcal{F};\mathcal{U})\}$ and the L-spectrum $\sigma^L(M) = \mathbb{C} \setminus \rho^L(M)$ of the operator M. Let $\rho^L(M) \neq \emptyset$, then we can consider *right* and *left*

$$R^L_{(\mu,p)}(M) = \prod_{k=0}^{p} R^L_{\mu_k}(M) \quad \text{and} \quad L^L_{(\mu,p)}(M) = \prod_{k=1}^{L} p^L_{\mu_k}(M)$$

(L,p)*-resolvents* of the operator M. Here $R^L_\mu(M) = (\mu L - M)^{-1}L$, $L^L_\mu(M) = L(\mu L - M)^{-1}$, and points $\mu_k \in \rho^L(M)$, $k = \overline{0,p}$.

Definition 2.1. ([16], chapter 3) Operator M is called *p-sectorial relatively* of operator L with the number $p \in \mathbb{N}_0$ (in short, (L,p)*-sectorial*), if there exist constants $K \in \mathbb{R}_+$, $a \in \mathbb{R}$, $\Theta \in (\pi/2,\pi)$ such, that the sector

$$S^L_{a,\Theta}(M) = \{\mu \in \mathbb{C} : |\arg(\mu - a)| < \Theta,\ \mu \neq a\}, \quad S^L_{a,\Theta}(M) \subset \rho^L(M),$$

and

$$\max\left\{\left\|R^L_{(\mu,p)}(M)\right\|_{\mathcal{L}(\mathcal{U})}, \left\|L^L_{(\mu,p)}(M)\right\|_{\mathcal{L}(\mathcal{F})}\right\} \leq \frac{K}{\prod\limits_{k=0}^{p} |\mu_k - a|} \tag{$*$}$$

for all $\mu_k \in S^L_{a,\Theta}(M)$, $k = \overline{0,p}$.

Remark 2.2. It is clear that if inequality (*) is executed when any $p \in \mathbb{N}_0$, then it will be executed and if $q \in \mathbb{N}$ such that $q > p$. In the proof this fact does not matter, and in applications we take the smallest p for which (*) is executed.

Lemma 2.3. *Let operator M be (L,p)-sectorial. Then in the sector $\Sigma = \{\tau \in \mathbb{C} : |\arg\tau| < \Theta - \pi/2,\ \tau \neq 0\}$, where Θ is taken from definition 2.1, there exists an analytic and uniformly bounded resolving semigroup $\{U^t : t > 0\}$ ($\{F^t : t > 0\}$) of the equation (0.5), $f \equiv 0$, and it is represented by Dunford – Taylor type integrals*

$$U^t = \frac{1}{2\pi i}\int_\Gamma R^L_\mu(M)e^{\mu t}d\mu \qquad \left(F^t = \frac{1}{2\pi i}\int_\Gamma L^L_\mu(M)e^{\mu t}d\mu\right),$$

where $t \in \mathbb{R}_+$, countour $\Gamma \subset S^L_{a,\Theta}(M)$ is such that $|\arg \mu| \to \Theta$ with $\mu \to \infty$, $\mu \in \Gamma$.

Lemma 2.4. *Let operator M be (L,p)-sectorial. Then $\lim\limits_{t\to 0+} U^t u = u$ for any $u \in \mathrm{im} R^L_{(\mu,p)}(M)$ and $\lim\limits_{t\to 0+} F^t f = f$ for any $f \in \mathrm{im} L^L_{(\mu,p)}(M))$.*

Consider kernels $\ker U^{\cdot} = \mathcal{U}^0$, $\ker F^{\cdot} = \mathcal{F}^0$ and images $\mathrm{im}\, U^{\cdot} = \mathcal{U}^1$, $\mathrm{im}\, F^{\cdot} = \mathcal{F}^1$ of these semigroups. Introduce the condition [17]

$$\mathcal{U}^0 \oplus \mathcal{U}^1 = \mathcal{U} \quad (\mathcal{F}^0 \oplus \mathcal{F}^1 = \mathcal{F}), \tag{A1}$$

Remark 2.5. [16] Units of semigroups $\{U^t \in \mathcal{L}(\tilde{\mathcal{U}}) : t \in \overline{\mathbb{R}}_+\}$ and $\{F^t \in \mathcal{L}(\tilde{\mathcal{F}}) : t \in \overline{\mathbb{R}}_+\}$ are projectors $P = s - \lim\limits_{t\to 0+} U^t$ and $Q = s - \lim\limits_{t\to 0+} F^t$ along $\mathcal{U}^0$ or $\mathcal{F}^0$ on subspace $\mathcal{U}^1$ or $\mathcal{F}^1$ correspondingly.

We denote by L_k (M_k) a contraction of operator L (M) on $\mathcal{U}^k$ ($\mathrm{dom} M \cap \mathcal{U}^k$), $k = 0, 1$. Let us introduce one more condition [17]

$$\text{there exists the operator} \quad L_1^{-1} \in \mathcal{L}(\mathcal{F}^1; \mathcal{U}^1). \tag{A2}$$

Lemma 2.6. *Let operator M be (L,p)-sectorial and conditions (A1), (A2) are fulfilled. Then*

(i) $L_0 \in \mathcal{L}(\mathcal{U}^0; \mathcal{F}^0)$, $M_0 \in \mathcal{C}l(\mathcal{U}^0; \mathcal{F}^0)$, and there exists the operator $M_0^{-1} \in \mathcal{L}(\mathcal{F}^0; \mathcal{U}^0)$,

(ii) operators $L_1 \in \mathcal{L}(\mathcal{U}^1; \mathcal{F}^1)$, $M_1 \in \mathcal{C}l(\mathcal{U}^1; \mathcal{F}^1)$.

Finally, we introduce another important condition on the L-spectrum of the operator M [19] in the following form

$$\begin{cases} \sigma^L(M) = \bigcup\limits_{j=0}^{n} \sigma_j^L(M),\ n \in \mathbb{N}, \text{ and } \sigma_j^L(M) \neq \emptyset \text{ is contained in bounded} \\ \text{domain } D_j \subset \mathbb{C} \text{ with piecewise smooth boundary } \partial D_j = \Gamma_j \subset \mathbb{C}. \text{ Also,} \\ \overline{D_j} \cap \sigma_0^L(M) = \emptyset \text{ and } \overline{D_k} \cap \overline{D_l} = \emptyset \text{ for all } j,k,l = \overline{1,n}, k \neq l. \end{cases} \tag{A3}$$

Now let $\mathcal{U}$ ($\mathcal{F}$) be a real separable Hilbert space with an orthonormal basis $\{\varphi_k\}$ ($\{\psi_k\}$). Let us introduce into consideration a monotone sequence $\mathbf{K} = \{\lambda_k\} \subset \{0\} \cup \mathbb{R}$ such that $\sum\limits_{k=1}^{\infty} \lambda_k^2 < +\infty$. The symbol $\mathbf{U_K L_2}$ ($\mathbf{F_K L_2}$) denotes the Hilbert space, which is the completion of the linear hull of *random* $\mathbf{K}$*-variables*

$$\xi = \sum_{k=1}^{\infty} \lambda_k \xi_k \varphi_k,\ \xi_k \in \mathbf{L_2}, \quad \left(\zeta = \sum_{k=1}^{\infty} \mu_k \zeta_k \psi_k,\ \zeta_k \in \mathbf{L_2} \right),$$

according to the norm $\|\eta\|^2_{\mathbf{U}} = \sum\limits_{k=1}^{\infty} \lambda_k^2 \mathbf{D} \xi_k \quad \left(\|\omega\,|^2_{\mathbf{F}} = \sum\limits_{k=1}^{\infty} \mu_k^2 \mathbf{D} \zeta_k \right)$. Note that in different spaces ($\mathbf{U_K L}_2$ and $\mathbf{F_K L}_2$) the sequence $\mathbf{K}$ can be different ($\mathbf{K} = \{\lambda_k\}$ in $\mathbf{U_K L}_2$ and $\mathbf{K} = \{\mu_k\}$ in $\mathbf{F_K L}_2$), however, all sequences marked with $\mathbf{K}$ must be monotonic and summable with square. All results, generally speaking, will be true for different sequences $\{\lambda_k\}$ and $\{\mu_k\}$, but for the sake of simplicity we will limit ourselves to the case $\lambda_k = \mu_k$.

Lemma 2.7. [18] *Operator* $A \in \mathcal{L}(\mathcal{U};\mathcal{F})$ *exactly when* $A \in \mathcal{L}(\mathbf{U_K L}_2;\mathbf{F_K L}_2)$.

How easy it is to see

$$\|A\xi\|_{\mathbf{F}} \le \sum_{k=1}^{\infty} \lambda_k^2 \mathbf{D}\xi_k \|A\varphi_k\|_{\mathfrak{F}}^2 \le \text{const} \sum_{k=1}^{\infty} \lambda_k^2 \mathbf{D}\xi_k = \text{const}\|\xi\|_{\mathbf{U}}.$$

Lemma 2.8. *Operator* $M \in \mathcal{L}(\mathcal{U};\mathcal{F})$ *is* (L,p)*-sectorial with respect to operator* $L \in \mathcal{L}(\mathcal{U};\mathcal{F})$ *exactly when* $M \in \mathcal{L}(\mathbf{U_K L}_2;\mathbf{F_K L}_2)$ *is* (L,p)*-sectorial with respect to the operator* $L \in \mathcal{L}(\mathbf{U_K L}_2;\mathbf{F_K L}_2)$*. Moreover, the L-spectrum of the operator M coincide in both cases. Condition (A1) and (A2) are fulfilled in spaces* $\mathcal{U}$, $\mathcal{F}$ *exactly when they are fulfilled in spaces* $\mathbf{U_K L}_2$, $\mathbf{F_K L}_2$

We construct relatively spectral projectors [19]

$$\begin{aligned} P_j &= \frac{1}{2\pi i}\int_{\Gamma_j} R_\mu^L(M)d\mu \in \mathcal{L}(\mathbf{U_K L}_2), \\ Q_j &= \frac{1}{2\pi i}\int_{\Gamma_j} L_\mu^L(M)d\mu \in \mathcal{L}(\mathbf{F_K L}_2),\ j=\overline{1,n}. \end{aligned} \tag{2.1}$$

and it turns out that when the operator M is strongly (L,p)-sectorial, then $P_jP = PP_j = P_j$ and $Q_jQ = QQ_j = Q_j$, $j=\overline{1,n}$. So, in this case, there is a projector $P_0 = P - \sum\limits_{j=1}^{n} P_j$, $P_0 \in \mathcal{L}(\mathbf{U_K L}_2)$. So, let the conditions (A1) – (A3) be fulfilled.

So we consider im $P_j = \mathbf{U_K^{1j} L}_2$, im $Q_j = \mathbf{F_K^{1j} L}_2$, $j=\overline{0,n}$. By construction $\mathbf{U_K^{1} L}_2 = \bigoplus\limits_{j=0}^{n} \mathbf{U_K^{1j} L}_2$ and $\mathbf{F_K^{1} L}_2 = \bigoplus\limits_{j=0}^{n} \mathbf{F_K^{1j} L}_2$. We denote by L_j (M_j) the narrowing of operator L (M) on $\mathbf{U_K^{1j} L}_2$ (dom$M \cap \mathbf{U_K^{1j} L}_2$), $j=\overline{0,n}$. It is easy to show that the operators $L_j \in \mathcal{L}(\mathbf{U_K^{1j} L}_2;\mathbf{F_K^{1j} L}_2)$, $M_j \in \mathcal{C}l(\mathbf{U_K^{1j} L}_2;\mathbf{F_K^{1j} L}_2)$, $j=\overline{0,n}$, moreover, due to (A2) there exists the operator $L_j^{-1} \in \mathcal{L}(\mathbf{F_K^{1j} L}_2;\mathbf{U_K^{1j} L}_2)$, $j=\overline{0,n}$. Also it is easy to show that the operator $G = M_0^{-1}L_0 \in \mathcal{L}(\mathbf{U_K^{0} L}_2)$, $S_0 = L_0^{-1}M_0 \in \mathcal{C}l(\mathbf{U_K^{0} L}_2)$ will be sectorial, and the operator $S_j = L_j^{-1}M_j : \mathbf{U_K^{1j} L}_2 \to \mathbf{U_K^{1j} L}_2$, $j=\overline{1,n}$, restricted.

Lemma 2.9. *Let the operator M be* (L,p)*-sectorial, and conditions (A1) – (A3) are fulfilled. Then* $U^t = \sum\limits_{j=0}^{n} P_jU^t = \sum\limits_{j=0}^{n} U_j^t$, $F^t = \sum\limits_{j=0}^{n} Q_jF^t = \sum\limits_{j=0}^{n} F_j^t$*, and* U_j^t *and* F_j^t *can be represented in the form*

$$\begin{aligned} U_j^t &= \frac{1}{2\pi i}\int_{\Gamma_j} (\mu L - M)^{-1} L e^{\mu t} d\mu, \\ F_j^t &= \frac{1}{2\pi i}\int_{\Gamma_j} L(\mu L - M)^{-1} e^{\mu t} d\mu,\ j=\overline{1,n}. \end{aligned} \tag{2.2}$$

Let us call the stochastic $\mathbf{K}$-process $\eta \in \mathbf{C}^1(\mathbb{R}_+;\mathbf{L}_2)$ (*classical*) *solution of the equation* (0.6), if a.s. all of it trajectories satisfy the equation (0.6) with some $\mathbf{K}$-"noise" $\omega \in \mathbf{C}(\mathbb{R}_+;\mathbf{L}_2)$ and all $t \in \mathbb{R}_+$. The solution $\eta = \eta(t)$ of the equation (0.6) will be called a *solution of the problem* (0.6), (0.7) if the condition (0.7) for some random $\mathbf{K}$-variables $\xi_k \in \mathbf{U_K L}_2$, $k=\overline{0,l}$.

Theorem 2.10. *Let the operator M be (L,p)-sectorial, $p \in \mathbb{N}_0$, moreover, the conditions (A1) – (A3) are fulfilled. Then for any $\tau_j \in \mathbb{R}_+$, $j = \overline{1,m}$, operator $N \in \mathcal{L}(\mathcal{U};\mathcal{F})$, monotonic sequence $K \subset \{\lambda_k\}$ such that $\sum\limits_{k=1}^{\infty} \lambda_k^2 < +\infty$, $\mathbf{K}$- "noise" $\omega = \omega(t)$ such that $(\mathbb{I} - Q)Nw \in \mathbf{C}^{p+1}(\mathbb{R}_+;\mathbf{U_K L}_2)$ and $QNw \in \mathbf{C}(\mathbb{R}_+;\mathbf{U_K L}_2)$, and random $\mathbf{K}$-variables $\xi_j \in \mathbf{U_K L}_2$, $j = \overline{0,m}$, independent of ω, there is a unique solution $\eta \in \mathbf{C}^1(\mathbb{R}_+;\mathbf{U_K L}_2)$, problem (0.6), (0.7), having the form*

$$\eta(t) = -\sum_{q=0}^{p} H^q M_0^{-1}(\mathbb{I} - Q) \overset{\circ}{\omega}{}^{(q)}(t) +$$

$$+\sum_{j=0}^{m} \left[U_j^{t-\tau_j} \xi_j + \int_{\tau_j}^{t} U_j^{s-\tau_j} L_{1j}^{-1} Q_j N\omega(s)ds \right], \ t \in \mathfrak{I}.$$

3. Linear stochastic Navier – Stokes system of equations

Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N} \setminus \{1\}$, be a bounded domain with boundary $\partial\Omega$ of the class C^∞. In the cylinder $\Omega \times \mathbb{R}$ consider the linear stochastic Navier – Stokes system of equations

$$v_t = \nu\nabla^2 v - \nabla p + f, \quad \nabla(\nabla \cdot v) = 0. \tag{3.1}$$

Based on the results of points 2 and 3, we will reduce the system (3.1) and the condition (0.2) to the equation (0.6). Following [20], [21], [22], we denote by $\mathbb{H}^2 = (W_2^2)^n$, $\overset{\circ}{\mathbb{H}}{}^1 = (\overset{\circ}{W}{}_2^1)^n$, $\mathbb{L}^2 = (L_2)^n$ space vector-functions $v = (v_1, v_2, \ldots, v_n)$ defined on Ω. Consider the lineal $\mathfrak{L} = \{v \in (C_n^\infty)^n : \nabla \cdot v = 0\}$ of vector-functions, solenoidal and finite in the domain Ω. We denote the closure of $\mathfrak{L}$ with respect to the norm of the space $\mathbb{L}^2$ by $\mathbb{H}_\sigma$. The space $\mathbb{H}_\sigma$ is Hilbert with the scalar product $\langle \cdot, \cdot \rangle$ inherited from $\mathbb{L}^2$; in addition, there is a splitting $\mathbb{L}^2 = \mathbb{H}_\sigma \oplus \mathbb{H}_\pi$, where $\mathbb{H}_\pi$ is the orthogonal complement of $\mathbb{H}_\sigma$. We denote by $\Sigma : \mathbb{L}^2 \to \mathbb{H}_\sigma$ the corresponding orthoprojector. The restriction of the projector Σ to $\mathbb{H}^2 \cap \overset{\circ}{\mathbb{H}}{}^1$ is a continuous operator, $\Sigma : \mathbb{H}^2 \cap \overset{\circ}{\mathbb{H}}{}^1 \to \mathbb{H}^2 \cap \overset{\circ}{\mathbb{H}}{}^1$. Let us therefore represent the space $\mathbb{H}^2 \cap \overset{\circ}{\mathbb{H}}{}^1$ as a direct sum $\mathbb{H}^2 \cap \overset{\circ}{\mathbb{H}}{}^1 = \mathbb{H}_\sigma^2 \oplus \mathbb{H}_\pi^2$, where $\mathbb{H}_\sigma^2 = \text{im}\,\Sigma$, $\mathbb{H}_\pi^2 = \ker \Sigma$. There are continuous and dense embeddings $\mathbb{H}_\sigma^2 \hookrightarrow \mathbb{H}_\sigma$ and $\mathbb{H}_\pi^2 \hookrightarrow \mathbb{H}_\pi$. The space $\mathbb{H}_\pi^2$ consists of vector functions that are equal to zero on $\partial\Omega$ and are gradients of functions from $W_2^3(\Omega)$.

Lemma 3.1. [23]

(i) By the formula $A = (-\nabla^2)^n : \mathbb{H}^2 \cap \overset{\circ}{\mathbb{H}}{}^1 \to \mathbb{L}^2$ defines a linear continuous operator with a positive discrete finite multiple spectrum $\sigma(A)$ condensing only to the point $+\infty$, and the mapping $A : \mathbb{H}_{\sigma(\pi)}^2 \to \mathbb{H}_{\sigma(\pi)}$ is bijective.

(ii) Formula $B : v \to -\nabla(\nabla \cdot v)$ a linear continuous surjective operator is given $B : \mathbb{H}^2 \cap \overset{\circ}{\mathbb{H}}{}^1 \to \mathbb{H}_\pi$, and $\ker B = \mathbb{H}_\sigma^2$.

Let $\mathcal{U} = \mathbb{H}_\sigma^2 \times \mathbb{H}_\pi^2 \times \mathbb{H}_p$, $\mathcal{F} = \mathbb{H}_\sigma \times \mathbb{H}_\pi \times \mathbb{H}_p$ are real separable Hilbert spaces with an orthonormal basis $\{\varphi_k\}$ and $\{\psi_k\}$, respectively. Moreover $\mathbb{H}_p = \mathbb{H}_\pi$, Let's construct operators [24]

$$L = \begin{pmatrix} \mathbb{I} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \end{pmatrix}, \quad M = \begin{pmatrix} -\nu A_\sigma & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & -\nu A_\pi & -\mathbb{I} \\ \mathbb{O} & B & \mathbb{O} \end{pmatrix}.$$

Obviously, $L, M \in \mathcal{L}(\mathcal{U};\mathcal{F})$, and $\operatorname{im} L = \mathbb{H}_\sigma \times \mathbb{H}_\pi \times \{0\}$, $\ker L = \{0\} \times \{0\} \times \mathbb{H}_p$.

Let us introduce into consideration a monotone sequence $\mathbf{K} = \{\lambda_k\} \subset \{0\} \cup \mathbb{R}_+$ such that $\sum\limits_{k=1}^{\infty} \lambda_k^2 < +\infty$. Let us consider Hilbert spaces, which is the completion of the linear hull of *random* **K**-*variables* [25]

$$\mathbf{U_K L_2} = \left\{ \xi = \sum_{k=1}^{\infty} \lambda_k \xi_k \varphi_k,\ \xi_k \in \mathbf{L_2},\ \varphi_k \in \mathbb{H}_\sigma^2 \times \mathbb{H}_\pi^2 \times \mathbb{H}_p \right\},$$

$$\mathbf{F_K L_2} = \left\{ \zeta = \sum_{k=1}^{\infty} \mu_k \zeta_k \psi_k,\ \zeta_k \in \mathbf{L_2},\ \psi_k \in \mathbb{H}_\sigma \times \mathbb{H}_\pi \times \mathbb{H}_p \right\}.$$

Lemma 3.2. [24] *For any $\nu \in \mathbb{R}_+$ the operator M is $(L,1)$- sectorial.*

If we set $N\omega = \operatorname{col}(\Sigma f, \Pi f, 0)$, and $f(t) = \overset{\circ}{W}_{\mathbf{K}}(t)$, then the reduction of the problem (0.2), (3.1) to the equation (0.6) finished.

Let the operators L, M, $N \in \mathcal{L}(\mathbf{U_K L_2}; \mathbf{F_K L_2})$. Consider a linear stochastic Sobolev type equation (0.6) with multipoint initial-final value condition (0.7).

Let us call the stochastic **K**-process $\eta \in \mathbf{C}^1(\mathbb{R}_+;\mathbf{L}_2)$ (*classical*) *solution of the equation* (0.6), if a.s. all of it trajectories satisfy the equation (0.6) for some **K**-"noise" $\omega \in \mathbf{C}(\mathbb{R}_+;\mathbf{L}_2)$ and all $t \in \mathbb{R}_+$. Solution $\eta = \eta(t)$ to the equation (0.6) let's call *solution to the problem* (0.6), (0.7), if the condition (0.7) is met for some random **K**-variables $\xi_k \in \mathbf{U_K L_2}$, $k = \overline{0,l}$.

It is known [16], [24] that L-spectrum $\sigma^L(M)$ of the operator M has the form $\sigma^L(M) = \{\mu_k = -\nu\lambda_k\}$. It is clear that for such a set one can select contours $\gamma_j \subset \mathbb{C}$. Let's construct

$$U_j^t = \begin{pmatrix} \sum\limits_{\lambda_k \in \sigma_j^L(M)} e^{-\lambda_k t} \langle \cdot, \varphi_k \rangle_\sigma \varphi_k & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} \end{pmatrix}, j = \overline{0,m}.$$

It follows from Lemma 3.2 that under the conditions of this lemma the condition (A1).

Theorem 3.3. *Let the operators L and M be defined as in Lemma 3.2. Then for any $\tau_j \in \mathbb{R}_+$, $j = \overline{1,m}$, operator $N \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$, monotonic sequence $\mathbf{K} \subset \{\lambda_k\}$ such that $\sum\limits_{k=1}^{\infty} \lambda_k^2 < +\infty$, **K**- "noise" $\omega = \omega(t)$ such that $(\mathbb{I} - Q)Nw \in \mathbf{C}^{p+1}(\mathbb{R}_+;\mathbf{U_K L_2})$ and $QNw \in \mathbf{C}(\mathbb{R}_+;\mathbf{U_K L_2})$, $\omega(t) = \overset{\circ}{W}_{\boldsymbol{K}}(t)$ and random **K**-variables $\xi_j \in \mathbf{U_K L_2}$, $j = \overline{0,m}$, independent of ω, there is a unique solution $\eta \in \mathbf{C}^1(\mathbb{R}_+;\mathbf{U_K L_2})$, problem*

(0.6), (0.7), *having the form*

$$\eta(t) = \sum_{j=0}^{m} \left[U_j^{t-\tau_j} \xi_j + L_{1j}^{-1} Q_j N W_{\mathbf{K}}(t) - S_j P_j \int_{\tau_j}^{t} U_j^{s-\tau_j} L_{1j}^{-1} Q_j N W_{\mathbf{K}}(s) ds \right] -$$

$$- \sum_{q=0}^{p} H^q M_0^{-1} (\mathbb{I} - Q) \overset{\circ}{W}{}_{\mathbf{K}}^{(q+1)}(t), \; t \in \overline{\mathbb{R}}_+.$$

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