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A FORMULA FOR THE SECOND ORDER BACKWARD MEAN DERIVAVES

SVETLANA AZARINA

ABSTRACT. We study second order backward mean derivatives and compute them for some Itô diffusion processes. It is also possible to do using backward Itô-Ventzell formula. We compute these derivatives along the Wiener process.

1. Introduction

This paper is devoted to computing second order backward mean derivatives along the Wiener process. It is needed for the investigation of the motion of incompressible viscous fluid.

We can write the equation of the motion of such fluid in Lagrange coordinates. It will look like a second order backward mean derivative of a process with some force. Simultaneously these motion could be described in Euler coordintes with some analog of the Navie-Stockes equation.

In the second section we introduce the backward mean derivative for a stochastic process with respect to another one. Then we construct the backward mean derivative of vector field along a stochastic process and the second order backward mean derivative.

In the next section we suppose that a stochastic process is an Itô diffusion process. So it is possible to compute the second order mean derivative along the Wiener process in some special cases.

2. Backward mean derivatives

Let us consider stochastic processes $\xi(t)$, $\eta(t)$ given on a certain probability space $(\Omega,\mathscr{F},\mathsf{P})$ with values in $\mathbb{R}^n$. These processes are L_1-variables for all $t\in[0,T]$. We denote by $E_t^{\eta}(\xi(t))$ the conditional expectation of the process $\xi(t)$ for the σ-algebra, generated by preimages of the borel sets for the $\eta(t):\Omega\to\mathbb{R}^n$.

We call the backward mean derivative of the process $\xi(t)$ at the instant t the L_1 random variable

$$D_*^{\eta}\xi(t)=\lim_{\triangle t\to+0}E_t^{\eta}\left(\frac{\xi(t)-\xi(t-\triangle t)}{\triangle t}\right). \tag{2.1}$$

These derivative can be considered as the composition of $\xi(t)$ and the borel vector field $Z(t,x)$:

$$Z(t,x)=\lim_{\triangle t\to+0}E\left(\frac{\xi(t)-\xi(t-\triangle t)}{\triangle t}\Big|\eta(t)=x\right). \tag{2.2}$$

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Let $Z(t,x)$ be a C^2-vector field of $\mathbb{R}^n$. The backward mean derivative of $Z(t,x)$ along $\xi(t)$ is the L_1-random process

$$D_*^{\eta} Z(t,\xi(t)) = \lim_{\triangle t \to +0} E_t^{\eta} \left(\frac{Z(t,\xi(t)) - Z(t-\triangle t, \xi(t-\triangle t))}{\triangle t} \right). \tag{2.3}$$

Consider a Wiener process $w(t)$ in $\mathbb{R}^n$ as a process $\eta(t)$ and consider $\xi(t)$ as a solution of the stochastic differential Itô equation

$$\mathrm{d}\xi(t) = a(t,\xi(t))\mathrm{d}t + A(t,\xi(t))\mathrm{d}w(t), \tag{2.4}$$

where the drift $a(t,x)$ and the diffusion summand $A(t,x)$ are a vector field and a field of linear operators in $\mathbb{R}^n$ respectively, denote $A(t,x)A^*(t,x)$ by $\sigma(t,x)$.

Let $f(t,x)$ be a $\mathbb{R}^n$-valued function, then for process (2.4) as in [4] we get

$$D_*^{\eta} f(t,\xi(t)) = \left(\frac{\partial f}{\partial t} + D_*\xi(t)\nabla f - \frac{1}{2}\sigma\nabla^2 f \right)(t,\xi(t)), \tag{2.5}$$

where ∇f is the gradient of f and $\nabla^2 f$ is the Hessian of the f.

We need some properties of the backward mean derivatives. For the L_1-stochastic processes $\xi(t)$, $\eta(t)$, a function $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ and a constant linear operator σ in $\mathbb{R}^n$ we have

(1) $D_*(\xi(t)+\eta(t)) = D_*(\xi(t)) + D_*(\eta(t))$,
(2) $D_*(\sigma\xi(t)) = \sigma D_*(\xi(t))$,
(3) For the differentiable function f of t and the process $\xi(t)$ like above it would be useful for us the following Leibnitz rule.

$$D_*^w(f(t)\xi(t)) = \frac{\mathrm{d}f(t)}{\mathrm{d}t} E_t^w \xi(t) + f(t) D_*^w \xi(t). \tag{2.6}$$

For the Markov diffusion process of the form (2.4) as in [4] we can write

$$D_*\xi(t) = a_*(t,\xi(t)), \quad a_*^i(t,x) = a^i(t,x) - \frac{1}{p_t(x)}\partial_j(\sigma^{ij}(t,x)p_t(x)), \tag{2.7}$$

where $p_t(x)$ is the density of process $\xi(t)$, ∂_j means the partial derivative w.r.t. x_j.

3. Second order backward mean derivatives

It is possible to use second order backward mean devivatives to investigate the equation of motion of viscous fluid. The first step on this way is calculating such derivatives for the most useful processes. Another application of mean derivatives is connected with the financial mathematics. Let's consider Wiener processes, a martingales, geometric brownian motions and Ornstein-Uhlenbeck processes.

Theorem 3.1. *Let $A(t)$ be a C^2-field of linear operators in $\mathbb{R}^n$ such that*

$$\int_0^T (AA^*)(s)\mathrm{d}s < \infty.$$

Let $a(t)$, $B(t)$ be a C^2 function and a field of linear operators in $\mathbb{R}$, a and B are the constants. For the processes $\xi_1(t)$, $\xi_2(t)$, $t \in [0,T]$ with values in $\mathbb{R}^n$ and $\xi_3(t)$, $\xi_4(t)$ with values in $\mathbb{R}$ of the form

$$d\xi_1(t) = A dw(t), \quad d\xi_2(t) = A(t) dw(t),$$
$$d\xi_3(t) = a(t)\xi_3(t)dt + B(t)\xi_3(t)dw(t), \quad d\xi_4(t) = -a\xi_4(t)dt + Bdw(t),$$

the second order backward mean derivative $D^w_* D^w_*$ *is equal to*

$$D^w_* D^w_* \xi_1(t) = 0, \quad D^w_* D^w_* \xi_2(t) = C'(t)w(t) + \frac{C(t)}{t} w(t),$$
$$D^w_* D^w_* \xi_3(t) = \left(a'(t) + C'(t)w(t) + \frac{C'(t)}{t} w(t)\right) \xi_3(t) + (a(t) + C(t)w(t) - \sigma(t))\, \xi_3(t),$$
$$D^w_* D^w_* \xi_4 = a^2 \xi_3 - a \frac{2aAe^{at}}{e^{2at} - 2a} w(t) - 2a^2 A \frac{e^{3at} + 2ae^{at}}{(e^{2at} - 2a)^2} \frac{w(t)}{t}.$$

Proof. Notice that the Wiener process $w(t)$ has the density of the form

$$p_t(x) = \frac{1}{\sqrt{(2\pi t)^n}} \exp\{-\frac{1}{2t} x^* x\},$$

where x^* is conjugate to x.

The result for the process ξ_1 is a consequence of the result from [5] and formula (2.7). Note that the process $\xi_1(t) = Aw(t)$ with a constant matrix A is again a Wiener process. By the definition of the backward mean derivative we obtain

$$D^w_* \xi_1(t) = \lim_{\triangle t \to +0} E^w_t \left(\frac{Aw(t) - Aw(t - \triangle t)}{\triangle t} \right) = A \lim_{\triangle t \to +0} E^w_t \left(\frac{w(t) - w(t - \triangle t)}{\triangle t} \right) =$$
$$= AD_* w = \frac{A}{t} w(t).$$

Applying the Leibnitz rule (2.6) we get

$$D^w_* D^w_* \xi_1(t) = D^w_* \left(\frac{Aw(t)}{t} \right) = -\frac{A}{t^2} w(t) + \frac{A}{t} D^w_* w(t) = 0.$$

The Itô process $\xi_2 = \xi_2(0) + \int\limits_0^t A(s)dw(s)$ is a martingale and has the normal distribution with parameters 0 and $\Sigma(t) = \int\limits_0^t \sigma(s)ds$, where $\sigma(t) = (AA^*)(t)$.

The density of this distribution could be written in the form

$$p_t(x) = \frac{1}{\sqrt{(2\pi)^n |\Sigma(t)|}} \exp\{-\frac{1}{2t} x^* \Sigma^{-1}(t) x\},$$

By the definition we have $\Sigma'(t) = \sigma(t)$. In one-dimensional case we get

$$D^w_* \xi_2(t) = \sigma(t) \frac{w(t)}{\Sigma(t)} = (\ln \Sigma(t))' w(t).$$

Denote $(\ln \Sigma(t))'$ by $C(t)$. Applying the Leibnitz rule (2.6) we get

$$D^w_* D^w_* \xi_2(t) = D^w_* (C(t)w(t)) = C'(t)w(t) + \frac{C(t)}{t} w(t). \tag{3.1}$$

Since the coordinates of $w(t)$ are independent Wiener processes we need additional notation, $\tilde{\sigma}(t) = diag\sigma(t)$. In $\mathbb{R}^n$ denote $(\sigma(t)\tilde{B}^{-1}(t))$ by $C(t)$ and $\int_0^t \tilde{\sigma}(s)\mathrm{d}s$ by $\tilde{B}(t)$. With these notation we get the same formula (3.1) in $\mathbb{R}^n$.

Now consider the geometric brownian motion $\xi_3(t)$. It is well known that the solution of such equation can be written using the exponent, then $\ln\xi_3(t)$ satisfies the linear stochastic equation

$$\mathrm{d}\eta(t) = \mathrm{d}\ln\xi_3(t) = \tilde{a}(t)\mathrm{d}t + B(t)\mathrm{d}w(t), \tag{3.2}$$

where $\tilde{a}(t) = a(t) - \frac{1}{2}\sigma(t)$. Then

$$D_*^w\eta(t) = \tilde{a}(t) + C(t)w(t),$$

So

$$D_*^w D_*^w \eta(t) = \tilde{a}'(t) + C'(t)w(t) + \frac{C(t)}{t}w(t).$$

Now apply formula (2.5) to the function $f(x) = e^x$ thus $\xi_3(t) = e^{\eta(t)}$:

$$D_*^w\xi(t) = \left(D_*\eta(t) - \frac{1}{2}\sigma(t)\right)\xi_3(t).$$

$$D_*^w\xi(t) = \left(D_*\eta(t) - \frac{1}{2}\sigma(t)\right)\xi_3(t).$$

And one more time taking D_*^w:

$$D_*^w D_*^w \xi_3(t) = \left(D_*^w D_*^w \eta(t) - \frac{1}{2}\sigma'(t)\right)\xi_3(t) + \left(D_*^w\eta(t) - \frac{1}{2}\sigma(t)\right)^2 \xi_3(t) =$$

$$= \left(\tilde{a}'(t) + C'(t)w(t) + \frac{C'(t)}{t}w(t)\right)\xi_3(t) + (\tilde{a}(t) + C(t)w(t) - \sigma(t))\,\xi_3(t).$$

Let's write the equation of the Ornstein-Uhlenbeck process ξ_4 (see [7]) more precisely:

$$\xi_4(t) = v_0 - \int_0^t a\xi_4(t)\mathrm{d}t + \int_0^t B\mathrm{d}w(t). \tag{3.3}$$

Solution of this equation can be written in the form

$$\xi_4(t) = e^{-at}\left(v_0 + A\int_0^t e^{as}\mathrm{d}w(s)\right)$$

and have normal distribution with parameters v_0e^{-at} and $\frac{B^2}{2a}\left(1 - e^{-2at}\right)$. If v_0 has normal distribution $N(0, \frac{B^2}{2a})$, then the process ξ_3 is Markovian and its distribution is $N(0, \frac{B^2}{2a})$.

Let's compute $D_*^w\xi_3$ using previous results.

$$D_*^w\xi_3(t) = -a\xi_3(t) + e^{-at}D_*^w(v_0 + A\int_0^t e^{as}\mathrm{d}w(s)) = -a\xi_3(t) + Ae^{-at}D_*^w(\int_0^t e^{as}\mathrm{d}w(s)) =$$

$$= -a\xi_3(t) + \frac{2aAe^{at}}{e^{2at} - 2a}w(t).$$

It is easy to check that

$$D^w_* D^w_* \xi_4 = a^2 \xi_4 - a\frac{2aAe^{at}}{e^{2at} - 2a} w(t) - 2a^2 A \frac{e^{3at} + 2ae^{at}}{(e^{2at} - 2a)^2} \frac{w(t)}{t}.$$

□

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KUBAN STATE UNIVERSITY, RUSSIAN FEDERATION
Email address: azarinas@mail.ru