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A STOCHASTIC ALGEBRAIC-DIFFERENTIAL EQUATION OF GEOMETRIC BROWNIAN MOTION TYPE WITH SYMMETRIC MEAN DERIVATIVES

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Abstract. In this paper we try to combine the machinery of mean derivatives and Leontiev type equations with the processes of the so called geometric Brownian motion that are in use in mathematical model of economy and some other applications.Namely we want to find what sort of equations arise in this combination.

Introduction

The notion of mean derivatives (forward, backward, symmetric and antisymmetric) was introduced by Edward Nelson in 60-th in his construction of the so-called Stochastic Mechanics, a version of Quantum Mechanics ([1, 2, 3]). After that, in [8], as a slight modification of some Nelson's constructions, a new sort of mean derivative called quadratic (it is responsible for the diffusion term of a process) was introduced so that, strictly speaking, it became possible to find processes having given mean derivatives. A lot of physical, economical and some other problems (besides Quantum mechanics) that are described by equations with mean derivatives (see, e.g., [7]), have been found.

In [4, 5], a new method for studying dynamically distorted signals in electronic devices was developed based on algebraic differential equations called Leontief-type equations. Later, in the works of G.A. Sviridyuk and his school, and some other researchers (including the author of this paper) the noise was taken into account, which was represented in terms of symmetric Nelson's mean derivatives (current velocities).

In this paper we try to combine the machinery of mean derivatives and Leontiev type equations with the processes of the so called geometric Brownian motion that are in use in mathematical model of economy and some other applications.Namely we want to find what sort of equations arise in this combination.

We use Einstein's summation convention on the sum by identical upper and lower indexes: If some term has lower and upper indexes denoted by the same letter, this means that the sum is conducted by this index from 1 to n equal to

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the dimension of the space, although the sum symbol is omitted. Let us illustrate this by examples. The notation $B_k^j = A_{ki}^{ij}$ means $B_K^J = \sum_{i=1}^{n} A_{ki}^{ij}$ and $R_k^j = a_i b^{ijs} c_s$ means $R_k^j = \sum_{i=1}^{n} \sum_{s=1}^{n} a_i b^{ijs} c_s$.

1. Some facts from matrix theory

Everywhere below we deal with processes, equations, etc., defined on some finite interval $[0, T]$.

We deal with an n-dimensional linear space $\mathbb{R}^n$, vectors from $\mathbb{R}^n$ and $n \times n$ matrices. Let two $n \times n$ constant matrices L and M be given, where L is singular and M is non-singular. An expression of the form $\lambda L + M$, where λ is a real parameter, is called a matrix pencil. The polynomial $\theta(\lambda) = \det(\lambda L + M)$ is called the characteristic polynomial of the pencil $\lambda L + M$. The pencil is called regular if its characteristic polynomial is not identically zero. If the matrix pencil $\lambda L + M$ is regular, then there exist non-degenerate linear operators P (acting from the left) and Q (acting from the right) that reduce the matrices L and M to the canonical quasi-diagonal form (see [6]).

In the canonical quasi-diagonal form, having chosen the desired order of the basis vectors, in the matrix PLQ first along the main diagonal there is the $d \times d$ identity matrix, and then along the main diagonal there are Jordan cells with zeros on the diagonal. We denote the $(n-d) \times (n-d)$ matrix with Jordan cells by N .

In PMQ in the lines, corresponding to the unit matrix in L there is a certain non-degenerate matrix J, and in lines, corresponding to Jordan boxes, there is the unit matrix. Thus

$$P(\lambda L + M)Q = \lambda \begin{pmatrix} I_d & 0 \\ 0 & N \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & I_{n-d} \end{pmatrix}, \tag{1.1}$$

A non-degenerate pencil satisfies the rank-degree condition if

$$rank(L) = \deg(\det(\lambda L + M(t))). \tag{1.2}$$

If the pencil satisfies the rank-degree condition, then formula (1.1) takes the form

$$P(\lambda L + M))Q = \lambda \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & I_{n-d} \end{pmatrix}. \tag{1.3}$$

where J is non-singular, since M is also a non-singular matrix.

2. A survey of mean derivatives

Consider a stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0, T]$, defined on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and such that $\xi(t)$ is an L_1-random variable for all t.

Each stochastic process $\xi(t)$ in $\mathbb{R}^n$, $t \in [0, l]$, generates three families of the σ-subalgebra of the σ-algebra $\mathcal{F}$:
(i) the "past" $\mathcal{P}_t^\xi$ generated by the preimages of Borel sets from $\mathbb{R}^n$ under all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $0 \le s \le t$;
(ii) the "future" $\mathcal{F}_t^\xi$ generated by the preimages of the Borel sets from $\mathbb{R}^n$ under all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $t \le s \le T$;

(iii) the "presence"$\mathcal{N}_t^\xi$ generated by the preimages of the Borel sets from $\mathbb{R}^n$ under all mappings $\xi(t)$.

All families are assumed to be closed, i.e., containing all sets with probability 0.

Strictly speaking, almost surely (a.s.) sample trajectories of the process $\xi(t)$ are not differentiable for almost all t. Thus, the "classical" derivative exists only in the sense of generalized functions. To avoid using generalized functions, following Nelson (see, e.g., [1, 2, 3]) we define the notion of mean derivatives. Denote by E_t^ξ the conditional expectation of ξ with respect to the "presence" σ- algebra $\mathcal{N}_t^\xi$.

Definition 2.1. (i) The forward mean derivative $D\xi(t)$ of $\xi(t)$ at time $t \in [0, T)$ is an L_1-random variable of the form

$$D\xi(t) = \lim_{\Delta t \to +0} E_t^\xi \left(\frac{\xi(t+\Delta t) - \xi(t)}{\Delta t}\right) \tag{2.1}$$

where the limit is assumed to exist in $L_1(\Omega, \mathcal{F}, \mathsf{P})$ and $\Delta t \to +0$ means that Δt tends to 0, with $\Delta t > 0$.

(ii) The backward mean derivative $D_*\xi(t)$ of $\xi(t)$ at time $t \in (0, T]$ is an L_1-random variable

$$D_*\xi(t) = \lim_{\Delta t \to +0} E_t^\xi \left(\frac{\xi(t) - \xi(t-\Delta t)}{\Delta t}\right) \tag{2.2}$$

where the assumptions and definitions are the same as in (i).

(iii) The derivative $D_S = \frac{1}{2}(D + D_*)$ is called the symmetric mean derivative. The vector $v^\xi(t) = v^\xi(t, \xi(t)) = D_S\xi(t)$ is called the current velocity of the process $\xi(t)$

Note that the current velocity is a natural analogue of the physical velocity of a deterministic process.

Definitions of mean derivatives (2.1) and (2.2) have important generalizations where we differentiate a process $\xi(t)$ but use the present E_t^η of another process $\eta(t)$. In this case wse apply the notation $D^\eta\xi(t)$, $D_*^\eta\xi(t)$ and $D_S^\eta\xi(t)$. In particular, below in Sect. 3 and 4 we use the present of the Wiener process $w(t)$ and so apply D_S^w.

Since $w(t)$ is a martingale, $Dw(t) = 0$.

Lemma 2.2. . *For $t \in (0, T]$ we obtain $D_*w(t) = \frac{w(t)}{t}$ and $D_*\frac{w(t)}{t} = 0$.*

*P*roof. In this case, from the definition of osmotic velocity $u^w(t, w(t))$ it follows that $D_*w(t) = -2u^w(t, w(t))$. Since the distribution density $\rho^w(t, x)$ of Wiener process is given by the formula

$$\rho^w(t, x) = \frac{1}{(2\pi t)^{\frac{n}{2}}}\, e^{-\frac{\|x\|^2}{2t}}, \tag{2.3}$$

according to formula (**??**) we have

$$u^w(t, x) = -\frac{1}{2} \cdot \frac{x}{t} \text{ i.e. } D_*w(t) = \frac{w(t)}{t}.$$

It is easy to see that $D_*\frac{w(t)}{t} = (\frac{d}{dt}\frac{1}{t})w(t) + \frac{1}{t}D_*w(t) = 0.$ $\square$

Corollary 2.3. $D_S w(t) = \frac{w(t)}{2t}$.

Definition 2.4. [See e.g. [7]] For an L^1-stochastic process $\xi(t)$, $t \in [0,T]$, we introduce the quadratic mean derivative $D_2\xi(t)$, defined by the formula

$$D_2\xi(t) = \lim_{\triangle t \to +0} E^{\xi}_t\Big(\frac{(\xi(t+\triangle t) - \xi(t))(\xi(t+\triangle t) - \xi(t))^*}{\triangle t}\Big), \tag{2.4}$$

where $(\xi(t+\triangle t) - \xi(t))$ is a column vector and $(\xi(t+\triangle t) - \xi(t))^*$ is its conjugate, i.e., a row vector, and the limit is assumed to exist in $L^1(\Omega, \mathcal{F}, \mathsf{P})$.

It is easy to verify that for the Ito process $\xi(t) = \int_0^t a(s)ds + \int_0^t A(s)dw(s)$ the quadratic mean derivative takes the form $D_2\xi(t) = AA^*$.

Let $a(t,x)$ and $\alpha(t,x)$ be Borel measurable mappings from $[0,T] \times \mathbb{R}^n$ to $\mathbb{R}^n$ and to $\overline{S}_+(n)$, respectively, where $\overline{S}_+(n)$ is the set of symmetric positive-definite $n \times n$ matrices. We will call a system of the form

$$\begin{cases} D_S\xi(t) = a(t,\xi(t)), \\ D_2\xi(t) = \alpha(t,\xi(t)), \end{cases} \tag{2.5}$$

a first-order stochastic differential equation with current velocity..

3. Processes of geometric Brownian motion type

We deal with the following generalization of the so-called geometric Brownian motion, namely with a process S(t) that satisfies the system of stochastic differential equations

$$S^\alpha(t) = \int_0^t S^\alpha(t)a^\alpha(t)dt + \int_0^t S^\alpha(t)A^\alpha_\beta(t)dw^\beta(t) \tag{3.1}$$

where $w^\beta(t)$ are independent Wiener processes in $\mathbb{R}^n$ that together form a Wiener process $w(t)$ in $\mathbb{R}^n$, $a(t)$ is a vector in $\mathbb{R}^n$, $A(t)$ is a mapping from $[0,T]$ to the space of linear operators $L(\mathbb{R}^n, \mathbb{R}^n)$ and $(A^\alpha_\beta(t))$ denotes the matrix of operator $A(t)$.

The processes satisfying (3.1), arise in various stochastic models (e.g., in economy).

Suppose that the coordinates S^α of the solution of (3.1) are positive for all t. Thus by the Ito formula the process $\xi(t) = logS(t) = (logS^1(t), \dots logS^n(t))$ satisfies the equation

$$\xi(t) = \int_0^t \big(a^\alpha - \frac{1}{2}(A^\alpha_\beta\delta^{\beta\gamma}A^\alpha_\gamma\big)(t)dt + \int_0^t A^\alpha_\beta(t)dw^\beta(t) \tag{3.2}$$

since $dw^\alpha dw^\beta = \delta^{\alpha\beta}dt$ (here $\delta^{\alpha\beta}$ is Kronecker's symbol: $\delta^{\alpha\alpha} = 1$, $\delta^{\alpha\beta} = 0$ for $\alpha \neq \beta$).

Analogously, from the Ito formula we derive that if a process $\xi(t)$ satisfies (3.2), the process $S(t) = exp\xi(t) = (exp\xi^1(t), \dots, exp\xi^n(t))$ satisfies (3.1). Note that in this case the coordinates S^α are positive.

Denote by B the symmetric positive semi-definite matrix AA^* (where A^* is the operator conjugate to A as above) and by $diagB$ the vector constructed from the diagonal elements of matrix B. Note that $A^\alpha_\beta\delta^{\beta\gamma}A^\alpha_\gamma$ is the α-th element $B^{\alpha\alpha}$ of $diagB$.

Consider the equation of the form

$$\left\{\begin{array}{l} D^w_S\xi(t) = D^w_S\int_0^t\big(a-\frac{1}{2}diagB\big)(t)dt + D^w_S\int_0^t A(t)dw(t) = \\ = \big(a-\frac{1}{2}diagB)(t) + A\frac{w(t)}{2t}, \\ D^w_2\xi(t) = B \end{array}\right. \tag{3.3}$$

We call (3.3) an equation of geometric Brownian motion type.

4. Main result

Lemma 4.1. *Let $B(t,x)$ be a jointly continuous (measurable, smooth) mapping from $[0,T]\mathbb{R}^n$ to to the space of symmetric positive definite $n\times n$ matrices $S_+(n)$). Then there exists a jointly continuous (measurable, smooth, respectively) mapping $A(t,x)$ from $[0,T]\mathbb{R}^n$ to the space of $n\times n$ matrices $L(\mathbb{R}^n,\mathbb{R}^n)$ such that for all $t\in[0,T], xin\mathbb{R}^n$ the equality $A(t,x)A^*(t,x)=B(t,x)$ holds.*

The proof is available in [8, Lemma 2.2].

Here we use the material and notation from Section 1

Let L be a constant degenerate $n\times n$ matrix and M be a constant non-degenerate matrix such that the characteristic polynomial of the pencil $\lambda L+M$ is regular and satisfies the rank-degree condition.

Let $B(t)$ be a smooth positive definite matrix in $\mathbb{R}^n$ of the form

$$\begin{pmatrix} B^{(1)}(t) & 0 \\ 0 & B^{(2)}(t) \end{pmatrix} \tag{4.1}$$

where $B^{(1)}(t)$ is a smooth symmetric positive definite matrix in $\mathbb{R}^d$ and $B^{(2)}(t)$ is a smooth symmetric positive definite matrix in $\mathbb{R}^{n-d}$. Let a smooth vector $a(t)$ in $\mathbb{R}^n$ be a sum of vectors $a^{(1)}(t)$ in $\mathbb{R}^d$ and $a^{(2)}(t)$ in $\mathbb{R}^{n-d}$. Recall that $a(t)$ as all other processes are given on closed finite interval $[0,T]$ and so $a(t)$, $a^{(1)}(t)$ and $a^{(2)}$ are bounded.

By Lemma 4.1 there exist $A^{(1)}(t)$ such that $B^{(1)} = A^{(1)}A^{*(1)}$ and $A^{(2)}(t)$ such that $B^{(2)} = A^{(2)}A^{*(2)}$. Introduce the matrix A by the formula

$$\begin{pmatrix} A^{(1)}(t) & 0 \\ 0 & A^{(2)}(t) \end{pmatrix} \tag{4.2}$$

. Evidently $B=AA^*$.

We suppose that matrices L and M are translated to canonical form. Consider the following stochastic algebraic-differential equation with symmetric mean derivative that is obtained from the geometric Brownian motion type one in $\mathbb{R}^n$

(4.3)
$$\begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} D^w_S\xi(t) = \begin{pmatrix} J & 0 \\ 0 & I_{n-d} \end{pmatrix}\xi(t) + \left((a(t)-\frac{1}{2}diagB(t))\right) + A(t)\frac{w(t)}{2t}$$
$$D^w_2\xi(t) = B(t).$$

One can easily see that (4.3) is split into two independent equations

$$D^w_S\xi^{(1)}(t) = J\xi^{(1)}(t) + \left(a^{(1)}(t) - \frac{1}{2}diagB^{(1)}(t))\right) + A^{(1)}(t)\frac{w(t)}{2t} \tag{4.4}$$

$$D_2^w \xi(t) = B^{(1)}(t)$$

in $\mathbb{R}^d$ and

$$\xi^{(2)}(t) = -\left(a^{(2)}(t) - \frac{1}{2} diag B^{(2)}(t))\right) - A^{(2)}(t)\frac{w(t)}{2t} \quad (4.5)$$

$$D_2^w \xi^{(2)}(t) = B^{(2)}(t)$$

in R^{n-d}.

Note that in (4.5) we have found the process $\xi^{(2)}(t)$.

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